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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXVII.

NOVEMBER, 1895, TO NOVEMBER, 1896.

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VOL. XXVII.

THIRTY-SECOND SESSION, 1895–96

(since the Formation of the Society, January 16th, 1865).

November 14th, 1895.

THE SECOND ANNUAL GENERAL MEETING OF THE LONDON MATHEMATICAL SOCIETY, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

The Treasurer read his report. Its reception was moved by Mr. Basset, seconded by Mr. S. Roberts, and supported by Mr. Kempe, who also moved a vote of thanks to Dr. Larmor for the trouble he had taken in connexion with the duties of his office. Both votes were carried unanimously.

The Rev. T. R. Terry, having signified his willingness to act again as Auditor of the report, was, on the nomination of the President, appointed to that office.

The President announced the death of Mr. E. H. Rhodes, elected a member June 10th, 1875, which took place on the 1st instant.

Mr. Tucker stated that the number of names on the roll of members was 220, of whom 107 were life compounders.

The Society's losses by deaths during the past session had been exceptionally severe, as will be seen by the names which follow:— Prof. Cayley, elected June 19th, 1865; Sir James Cockle, F.R.S., elected June 9th, 1870; Prof. A. M. Nash, M.A., elected November 10th, 1887; Mr. E. H. Rhodes, B.A., June 10th, 1875, and Mr. A. Cowper Ranyard, M.A., who was an original member and joint founder of the old Society.

The following communications had been made or received:—

Mathematics (the Presidential Address): Mr. A. B. Kempe.

Third Memoir on the Expansion of certain Infinite Products: Prof. L. J. Rogers.

On the Kinematics of non-Euclidian Space: Prof. W. Burnside.

On a Class of Groups defined by Congruences: Prof. W. Burnside.

On Fundamental Systems for Algebraic Functions: Mr. H. F. Baker.

Electric Vibrations in Condensing Systems: Dr. J. Larmor.

On certain Definite ϑ-Function Integrals: Prof. L. J. Rogers.

The Electrical Distribution on a Conductor bounded by Two Spherical Surfaces cutting at any angle: Mr. H. M. Macdonald.

Note on some Properties of a Generalized Brocard Circle: Mr. J. Griffiths.

On certain Differential Operators, and their use to form a Complete System of Seminvariants of any Degree or any Weight: Prof. Elliott.

Notes on the Theory of Groups of Finite Order: Prof. W. Burnside.

The Dynamics of a Top: Prof. Greenhill.

The Electrical Distribution induced on a Circular Disc placed in any Field of Force: Mr. H. M. Macdonald.

The Perpetuant Invariants of Binary Quantics: the President.

An Extension of Vandermonde's Theorem: Mr. F. H. Jackson.

A new Theorem in Probability: Rev. T. C. Simmons.

Notes on the Theory of Groups of Finite Order (continued): Prof. W. Burnside.

On the Geometrical meaning of a Form of the Orthogonal Transformation: Prof. M. J. M. Hill.

A Property of Skew Determinants: Prof. M. J. M. Hill.

Researches in the Calculus of Variations, Part VI.: Mr. E. P. Culverwell.

On those Orthogonal Substitutions that can be generated by the Repetition of an Infinitesimal Orthogonal Substitution: Dr. H. Taber.

On Elliptic and Hyper-Elliptic Systems of Differential Equations and their Rational and Integral Algebraic Integrals, with a Discussion of the Periodicity of Elliptic and Hyper-Elliptic Functions: Rev. W. R. W. Roberts.

An Extension of Boltzmann's Minimum Theorem: Mr. S. H. Burbury.

Applications of Trigraphy: Mr. J. W. Russell.

The Reciprocators of Two Conics discussed Geometrically: Mr. J. W. Russell.

The Reciprocators of Two Conics discussed Analytically: Mr. A. E. Jolliffe.

On the Form of the Energy Integral in the Varying Motion of a Viscous Incompressible Fluid: Mr. J. Brill.

On an Expansion of the Exponential Function $1/R^{\kappa-1}$ in Legendre's Functions: Dr. Routh.

On the most General Solution of given degree of Laplace's Equation: Dr. E. W. Hobson.

Point-Groups in relation to Curves: Mr. F. S. Macaulay.

On Maxwell's Law of Partition of Energy: Mr. G. H. Bryan.

Proof that $2^{197}-1$ is Divisible by 7487: Lt.-Col. Cunningham.

On the Integration of Allégret's Integral: Mr. A. E. Daniels.

On the Expansion of Functions: Mr. E. T. Dixon.

The Spherical Catenary: Prof. Greenhill and Mr. I. Dewar.

The Transformation of Elliptic Functions: Prof. Greenhill.

A Generalized Form of the Hypergeometric Series, and the Differential Equation which is satisfied by the Series: Mr. F. H. Jackson.

The Linear Equations that present themselves in the Method of Least Squares: the President.

On the Complex Number formed by Two Quaternary Matrices: Dr. G. G. Morrice.

The same journals had been subscribed for and the same exchanges of *Proceedings* made as in the preceding Session.

The Rev. T. R. Terry and Mr. W. W. Taylor having been appointed Scrutators, the ballot was taken, with the result that the following gentlemen, nominated by the Council, were elected to serve as the Council for the ensuing Session:—Major MacMahon, R.A., F.R.S., President; Prof. M. J. M. Hill, F.R.S., Mr. M. Jenkins, and Mr. A. B. Kempe, F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Mr. R. Tucker and Mr. A. E. H. Love, F.R.S., Hon. Secs. Other Members of the Council:—Mr. H. F. Baker, Dr. G. H. Bryan, F.R.S., Lt.-Col. Cunningham, R.E., Prof. Elliott, F.R.S., Dr. Glaisher, F.R.S., Prof. Greenhill, F.R.S., Dr. Hobson, F.R.S., Prof. W. H. H. Hudson, and Mr. F. S. Macaulay.

The President then made a statement of the reasons which had led Mr. Jenkins, after thirty years' tenure of the office, to resign his position of Secretary, and proposed a vote of thanks to that gentleman for his devoted services of thirty years to the Society, coupling with it the hope that his health might be restored by his retirement to the country. This vote, which was seconded by Mr. Kempe and supported by Mr. S. Roberts, who had been connected with the old Society almost from its inception, was unanimously carried, and Mr. Jenkins suitably acknowledged the compliment.

The following papers were read, or communicated as read:—

On the Stability and Instability of certain Fluid Motions (iii.), and on the Propagation of Waves upon the Plane Surface separating Two Portions of Fluid of Different Vorticities: Lord Rayleigh.

Note on Matrices: Mr. J. Brill.

Determination of the Volumes of certain Species of Tetrahedra without Employment of the Method of Limits: Prof. M. J. M. Hill.

Some Algebraical Theorems connected with the Theory of Partitions: Prof. Forsyth.

Certain General Series: Mr. F. H. Jackson.

An Extension of Sylvester's Constructive Theory of Partitions: the President.

Note on the Representation of a Conic by a Linear Equation: Mr. J. Griffiths.

On the Representation of a Number as a Sum of Squares: Prof. G. B. Mathews.

Researches in the Calculus of Variations: Part VII., Limiting Conditions in Multiple Integrals; Part VIII., Reduction of the Problem of the Discrimination of Maxima and Minima Values in Double Integrals with Variable Limits to a new Problem in Single Integrals: Mr. E. P. Culverwell.

A Note on certain Forms of the Equation of Normals to Conic Sections: Mr. J. S. L. Hatton.

On the Evaluation of a certain Dialytic Determinant: Mr. W. W. Taylor.

Criterion of 2 as a 16-ic Residue, with Remarks upon some of Mersenne's Numbers: Lt.-Col. Cunningham.

The following presents were made to the Library:—

Elliott, E. B.—"Introduction to the Algebra of Quantics," 8vo; Oxford, 1895.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 10; Leipzig, 1895.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. IX., No. 6; 1894–95.

"Proceedings of the Physical Society of London," Vol. XIII., Pts. 11, 12; 1895.

"Berichte über die Verhandlungen der Königl. Sachsischen Gesellschaft der Wissenschaften zu Leipzig," IV.; 1895.

"Bulletin de la Société Mathématique de France," Tome XXIII., No. 8; Paris, 1895.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 1; New York, 1895.

"Bulletin des Sciences Mathématiques," Tome XIX., Sep., Oct., 1895; Paris.

"Nyt Tidsskrift for Matematik," Aargang 7, Nr. 4, A., and Nr. 3, B.; Copenhagen, 1895.

"Atti della Reale Accademia dei Lincei," Sem. 2, Vol. IV., Fasc. 7, 8; Roma, 1895.

"Annali di Matematica," Serie 2, Tome XXIII., Fasc. 4; Milano, 1895.

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"Journal fur die reine und angewandte Mathematik," Bd. cxv., Heft 3; Berlin, 1895.

"Annales de la Faculté des Sciences de Toulouse," Tome ix., Fasc. 3; 1895.

"Educational Times," November, 1895.

"Atti della Reale Accademia delle Scienze Fisiche e Matematiche," Vol. vii.; Naples, 1895.

"Memorie della Regia Accademia di Scienze, Lettere ed Arte in Modena," Vol. x.; 1894.

"Indian Engineering," Vol. xviii., Nos. 12–16.

Cayley, A.—"Collected Mathematical Papers," Vol. viii., 4to; Cambridge, 1895.

"Bulletins de l'Académie Royale de Belgique," 65me Année, 3me Série, Tomes xxvi.–xxix.; Bruxelles, 1893–95.

"Annuaire de l'Académie Royale de Belgique," Années 60 and 61; Bruxelles, 1894–5.

On the Stability or Instability of certain Fluid Motions (III.).*

By Lord Rayleigh, Sec. R. S. Received October 4th, 1895. Read November 14th, 1895.

The steady motions in question are those in which the velocity is parallel to a fixed line (x), and such that U is a function of y only. In the disturbed motion $U+u$, v, the infinitely small quantities u, v are supposed to be periodic functions of x, proportional to e^{ikx}, and, as dependent upon the time, to be proportional to e^{int}, where n is a constant, real or imaginary. Under these circumstances the equation determining v is

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right)-\frac{d^2U}{dy^2}v=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The vorticity (Z) of the steady motion is $\tfrac{1}{2}dU/dy$. If throughout any layer Z be constant, d^2U/dy^2 vanishes, and, whenever $n+kU$ does not also vanish,

$$d^2v/dy^2-k^2v=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

or

$$v=Ae^{ky}+Be^{-ky} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

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* The two earlier papers upon this subject are to be found in *Proc. Lond. Math. Soc.*, Vol. xi., p. 57, 1880; Vol. xix., p. 67, 1887. The fluid is supposed to be destitute of viscosity.

If there are several layers in each of which Z is constant, the various solutions of the form (3) are to be fitted together, the arbitrary constants being so chosen as to satisfy certain boundary conditions. The first of these conditions is evidently

$$\Delta v = 0 \qquad \text{.................................(4).}$$

The second may be obtained by integrating (1) across the boundary. Thus

$$\left(\frac{n}{k}+U\right).\Delta\left(\frac{dv}{dy}\right)-\Delta\left(\frac{dU}{dy}\right).v=0 \qquad \text{..............(5).}$$

At a fixed wall $v = 0$.

Equation (2) secures that the vorticity shall remain constant in each layer, and equation (3) that there shall be no slipping at the surface of transition. Equations (2) and (3) together may be regarded as expressing the continuity of the motion at the surface between the layers.

In the first of the papers above referred to, I have applied equation (1) to prove that, if d^2U/dy^2 be of one sign throughout the whole interval between two fixed walls, n can have no imaginary part. It is true that, if $n+kU$ vanishes anywhere, the expression for d^2v/dy^2-k^2v in (1) becomes infinite, unless indeed $v=0$ at the place in question; and Lord Kelvin* considers that the "disturbing infinity" thus introduced vitiates the proof of stability. To this criticism it may be replied† that, "if n be complex, there is no disturbing infinity, and that, therefore, the argument does not fail, regarded as one for excluding complex values of n. What happens when n has a real value, such that $n+kU$ vanishes at an interior point, is a subject for further consideration."

In embarking upon this it will be convenient to take first the case of (2), (3), (4), (5), where the vorticity of the steady motion is uniform through layers of finite thickness. Any general conclusions arrived at in this way should at least throw light upon the extreme case where the number of the layers is infinitely great, and their thickness is infinitely small.

Starting from the first wall at $y = 0$, let the surfaces between the layers occur at $y = y_1$, $y = y_2$, &c., and let the values of U at these

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* *Phil. Mag.*, Vol. xxiv., p. 275, 1887. † *Phil. Mag.*, Vol. xxxiv., p. 66, 1892.

places be U_1, U_2, &c. In conformity with (4) and with the condition that $v = 0$, when $y = 0$, we may take in the first layer

$$v = v_1 = M_1 \sinh ky \quad \text{.......................(6)};$$

in the second layer

$$v = v_2 = v_1 + M_2 \sinh k(y - y_1) \text{....................(7)};$$

in the third layer

$$v = v_3 = v_2 + M_3 \sinh k(y - y_2) \text{....................(8)};$$

and so on.*

If the second fixed wall be in the r^{th} layer at $y = y'$, then

$$M_1 \sinh ky' + M_2 \sinh k(y' - y_1) + \ldots + M_r \sinh k(y' - y_{r-1}) = 0 \text{...(9)}.$$

We have still to express the conditions (5) at the various surfaces of transition. At the first surface

$$v = M_1 \sinh ky_1, \qquad \Delta\left(\frac{dv}{dy}\right) = kM_2;$$

at the second surface

$$v = M_1 \sinh ky_2 + M_2 \sinh k(y_2 - y_1), \qquad \Delta\left(\frac{dv}{dy}\right) = kM_3;$$

and so on. If we denote the values of $\Delta(dU/dy)$ at the various surfaces by Δ_1, Δ_2, &c., the conditions may be written

$$\left.\begin{array}{l} (n + kU_1)\,M_2 - \Delta_1 . M_1 \sinh ky_1 = 0 \\ (n + kU_2)\,M_3 - \Delta_2 . \{M_1 \sinh ky_2 + M_2 \sinh k(y_2 - y_1)\} = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \text{...(10)}.$$

The $r-1$ equations (10) together with (9) suffice to determine n, and the $r-1$ ratios $M_1 : M_2 : M_3 : \ldots : M_r$. The determinantal equation in n is of degree $r-1$, the number of the surfaces of transition; and corresponding to each root there is an expression for v, definite except as regards a constant multiplier.

It is important to note that the disturbances thus expressed are such as leave the vorticity unaltered in the interior of every layer; that they relate, in fact, merely to waves upon the surfaces of transition. The additional vorticity due to the disturbance is proportional to $d^2v/dy^2 - k^2v$, and is equated to zero in (2). If we wish to consider

* This is the process followed in the second of the papers cited, with a slight difference of notation.

the most general disturbance possible, we must provide for an arbitrary vorticity at every point.

The nature of the normal modes of disturbance not yet considered will be apparent from a comparison between (1) and (2). Even though $d^2U/dy^2 = 0$, the latter does not follow from the former, unless it be assumed that $n + kU$ is finite. Wherever $n + kU$ vanishes, that is, at the places where the wave velocity is equal to the stream velocity, (1) is satisfied, even though (2) be violated. Thus any value of $-kU$ to be found anywhere in the fluid is an admissible value of n, and the corresponding normal function (v) is obtained by allowing the arbitrary constants in (3) to be discontinuous at this place as well as at the surfaces of transition, subject of course to the condition that v itself shall be continuous. The new arbitrary constant thus disposable allows all the conditions to be satisfied with the value of n already prescribed.

The equations (9), (10) already found suffice for the present purpose if we introduce a fictitious surface of transition at the place in question. Suppose, for example, that $\Delta_3 = 0$ in the third of equations (10). It will follow either that $M_4 = 0$, or that $n + kU_3 = 0$. In the first alternative the constants A and B are continuous, and all local peculiarity disappears. The second alternative is the one with which we are now concerned. The equations suffice, as usual, to determine n (equal to $-kU_3$), as well as the ratios of the M's which give the form of the normal function. The mode of disturbance is such that a new vorticity is introduced at the place, or rather at the plane in question. In one sense this is the only new vorticity; but the waves upon the surfaces of transition involve changes of vorticity as regards given positions in space, though not as regards given portions of fluid.

We have now to consider what occurs at a second place in the fluid where the velocity happens to be the same as at the first place. The second place may be either within a layer of originally uniform vorticity or upon a surface of transition. In the first case nothing very special presents itself. If there be no new vorticity at the second place, the value of v is definite as usual, save as to an arbitrary multiplier. But, consistently with the given value of n, there may be new vorticity at the second as well as at the first place, and then the complete value of v for the given n may be regarded as composed of two parts, each proportional to one of the new vorticities, and each affected by an arbitrary multiplier.

If the second place lie upon a surface of transition, we have a state of things corresponding to the "disturbing infinity" in (1). In the above example, where $\Delta_2 = 0$, $n + kU_2 = 0$, we have now further to suppose that U_1, the velocity at the first surface of transition, coincides with U_2. From the first of equations (10), since $n + kU_1 = 0$, while Δ_1 and $\sinh ky_1$ are finite, we see that M_1 must vanish. Hence $v = 0$ throughout the entire layer from the wall $y = 0$ to $y = y_1$. The remainder of the motion from $y = y_1$ to $y = y'$ is to be determined us usual.

From the fact that $v = 0$, we might be tempted to infer that the surface in question behaves like a fixed wall. But a closer examination shows that the inference would be unwarranted. In order to understand this it may be well to investigate the relation between v and the displacement of the surface, supposed also to be proportional to $e^{int} . e^{ikx}$. Thus, if the equation of the surface be

$$F = y - he^{int+ikx} = 0 \quad \text{.......................(11)},$$

the condition to be satisfied is*

$$\frac{dF}{dt} + U_1 \frac{dF}{dx} + v \frac{dF}{dy} = 0 \quad \text{.......................(12)},$$

so that

$$-ih\,(n + kU_1) + v = 0 \quad \text{....................... (13)}$$

is the required relation. Using this, we see from the first of equations (10) that h does not vanish, but is given by

$$h = \frac{v}{i\,(n + kU_1)} = \frac{M_2}{i\Delta_1} \quad \text{.......................(14)}.$$

The propagation of a wave at the same velocity as that at which the fluid moves does not entail the existence of a finite velocity v.

That v vanishes at a surface of transition where $n + kU = 0$ follows in general from (5), seeing that the value of $\Delta\,(dU/dy)$ is finite. That region of the fluid, bounded by this surface and one of the fixed walls, which does not include the added vorticity, will in general remain undisturbed, but there may be exceptions when one of the values of n proper to this region (regarded as bounded by fixed walls) happens to coincide with that prescribed. It does not appear that the infinity which enters when $n + kU = 0$ disturbs any general conclusions as to the conditions of stability, or even seriously modifies the character of the solutions themselves.

When d^2U/dy^2 is finite, we must fall back upon equation (1). The

* Lamb's *Hydrodynamics*, § 10.

character of the disturbing infinity at a place (say, $y=0$) where $n+kU$ vanishes would be most satisfactorily investigated by means of the complete solution of some particular case. It is, however, sufficient to examine the form of solution in the neighbourhood of $y=0$, and for this purpose the differential equation may be simplified. Thus, when y is small, $n+kU$ may be regarded as proportional to y, and d^2U/dy^2 as approximately constant. In comparison with the large term, k^2v may be neglected, and it suffices to consider

$$d^2v/dy^2+y^{-1}v=0 \quad\text{..........................(15)},$$

a known constant, multiplying y being omitted for brevity. This falls under the head of Riccati's equation

$$d^2v/dy^2+y^{\mu}v=0 \quad\text{..........................(16)},$$

of which the solution* is in general (m fractional)

$$v=\sqrt{y}\,\{A\,J_m(\xi)+B\,J_{-m}(\xi)\} \quad\text{..................(17)},$$

where $$m=1/(\mu+2),\quad \xi=2my^{1/2m} \text{....................(18)}.$$

When, as in the present case, m is integral, $J_{-m}(\xi)$ is to be replaced by the function of the second kind $Y_m(\xi)$. The general solution of (15) is accordingly

$$v=\sqrt{y}\,\{A\,J_1(2\sqrt{y})+BY_1(2\sqrt{y})\} \quad\text{..............(19)}.$$

In passing through zero y changes sign, and with it the character of the functions. If we regard (19) as applicable on the positive side, then on the negative side we may write

$$v=\sqrt{y}\,\{C\,J_1(2\sqrt{y})+DY_1(2\sqrt{y})\} \quad\text{..............(20)},$$

the argument of the functions in (20) being pure imaginaries.

The functions $J_1(z)$, $Y_1(z)$ are given by

$$J_1(z)=\frac{z}{2}-\frac{z^3}{2^2.4}+\frac{z^5}{2^2.4^2.6}-\dots \quad\text{...........(21)},$$

$$Y_1(z)=\frac{1}{z}\left\{1-\frac{z^2}{2^2}+\frac{z^4}{2^2.4^2}-\frac{z^6}{2^2.4^2.6^2}+\dots\right\}$$
$$-\log z\left\{\frac{z}{2}-\frac{z^3}{2^2.4}+\frac{z^5}{2^2.4^2.6}-\dots\right\}$$
$$+\frac{z}{2}S_1-\frac{z^3}{2^2.4}S_2+\frac{z^5}{2^2.4^2.6}S_3-\dots \quad\text{..............(22)},$$

* Lommel, *Studien über die Bessel'schen Functionen*, § 31, Leipzig, 1868; Gray and Mathews' *Bessel's Functions*, p. 233, 1895.

where $$S_m = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{m} \qquad \text{.......................(23).}$$

When y is small, (19) gives

$$v = A\left\{y - \tfrac{1}{2}y^2\right\}$$
$$+B\left\{\tfrac{1}{2}\left(1 - y + \tfrac{1}{4}y^2\right) - \log\left(2\sqrt{y}\right)\left(y - \tfrac{1}{2}y^2\right) + yS_1 - \tfrac{1}{2}y^2S_2\right\} \ldots(24);$$

so that ultimately

$$v = \tfrac{1}{2}B, \quad \frac{dv}{dy} = A - \tfrac{1}{2}B\log y, \quad \frac{d^2v}{dy^2} = -A - \tfrac{1}{2}By^{-1} \quad \text{......(25),}$$

v remaining finite in any case.

We will now show that any value of $-kU$ is an admissible value of n in (1). The place where $n + kU = 0$ is taken as origin of y; and in the first instance we will suppose that $n + kU$ vanishes nowhere else. In the immediate neighbourhood of $y = 0$, the solutions applicable on the two sides are (19), (20), and they are subject to the condition that v shall be continuous. Hence, by (25), $B = D$, leaving three constants arbitrary. The manner in which the functions start from $y = 0$ being thus ascertained, their further progress is subject to the original equation (1), which completely defines them when the three arbitraries are known. In the present case two relations are given by the conditions to be satisfied at the fixed walls or other boundaries of the fluid, and thus is determined the entire form of v, save as to a constant multiplier. If, as must usually be the case, B and D are finite, there is infinite vorticity at the origin, but this is no more than occurs even when d^2U/dy^2 is zero throughout the region surrounding the origin.

Any other places at which $n + kU = 0$ may be treated in a similar manner, and the most general solution will contain as many arbitrary constants as there are places of infinite vorticity. But the vorticity need not be infinite merely because $n + kU = 0$; and, in fact, a particular solution may be obtained with only one infinite vorticity. At any other of the critical places, such, for example, as we may now suppose the origin to be, B and D may vanish, so that

$$v = 0, \quad d^2v/dy^2 = A, \text{ or } C.$$

From this discussion it would seem that the infinities which present themselves when $n + kU = 0$ do not seriously interfere with the application of the general theory, so long as the square of the disturbance from steady motion is neglected. The value of conclusions relating only to infinitely small disturbances is another question.

When regard is paid to viscosity, the difficulties are of course much increased. In the particular case where the original vorticity is uniform, the problem of small disturbances has been solved by Lord Kelvin,* who shows that the motion is stable by the aid of a special solution not proportional to a simple exponential function of the time. If we retain the supposition of the present paper that the disturbance as a function of the time is proportional to e^{int}, we obtain an equation [(52) in Lord Kelvin's paper] which has been discussed by Stokes.† From his results it appears that it is not possible to find a solution applicable to an unlimited fluid which shall be periodic with respect to x, and remain finite when $y = \pm \infty$, and this whether n be real or complex. The cause of the failure would appear to lie in the fact, indicated by Lord Kelvin's solution, that the stability is ultimately of a higher order than can be expressed by any simple exponential function of the time.

[*Addendum, January*, 1896.—It may be well to emphasise more fully that the solutions of this paper only profess to apply in the limit, when the disturbances are infinitely small. The constant factor which represents the scale of the disturbance must be imagined to be so small that the actual disturbance *nowhere* rises to such a magnitude as to interfere with the approximations upon which (1) is founded. For example, in (25), although dv/dy is infinite at $y = 0$ relatively to its value at other places, it must still be regarded as infinitely small throughout in comparison with the quantities which define the steady motion.]

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* *Phil. Mag.*, Vol. xxiv., p. 191, 1887.
† *Camb. Phil. Trans.*, Vol. x., p. 105, 1857.

On the Propagation of Waves upon the Plane Surface separating Two Portions of Fluid of Different Vorticities. *By* Lord Rayleigh, Sec. R. S.

Received October 4th, 1895. Read November 14th, 1895.

In former papers* I have considered the problem of the motion in two dimensions of inviscid incompressible fluid between two parallel walls. In the case where the steady motion is such that in each half of the layer included between the walls the vorticity is constant, it appeared that the motion is stable, small displacements of the surface separating the two vorticities being propagated as waves of constant amplitude. More particularly, if the velocity of the steady motion increase uniformly from zero at the walls to the value U in the middle stratum, a disturbance proportional to $e^{i(nt+kx)}$ requires that

$$n + kU = U/b \,.\, \tanh kb \qquad (1),$$

where $2b$ is the distance between the walls. The wave-length is $2\pi/k$, and the fact that n is real indicates that the disturbance is stable.

Discussions upon the difficult question of the nature of the instability manifested by fluids in their flow through pipes of moderate bore seemed to make it desirable to push the investigation of the disturbance from some simple case of steady motion so far at least as to include the squares of the small quantities.

In the present paper the problem chosen for the purpose is that above referred to, simplified by excluding the fixed walls, or, what comes to the same thing, by supposing them removed to a distance very great in comparison with the wave-length of the disturbance. We suppose, then, that in the steady motion the surface of separation coincides with $y = 0$, that when y is positive the vorticity is $+\omega$, and that when y is negative the vorticity is $-\omega$. In the disturbed motion the surface separating the two vorticities is displaced, so that its equation becomes $y = h\cos x$, k being put equal to unity for the sake of brevity.

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* "On the Stability or Instability of certain Fluid Motions," *Proc. Lond. Math. Soc.*, Vol. xi., p. 57, 1880; Vol. xix., p. 67, 1887.

In virtue of the incompressibility, the component velocities, denoted as usual by u and v, are connected with a stream-function ψ by the relations

$$u = d\psi/dy, \quad v = -d\psi/dx \qquad\qquad (2).$$

The vorticity is represented by $\frac{1}{2}\nabla^2\psi$, which is accordingly equal to $\pm\omega$. During the steady motion of the upper fluid, we have

$$\psi = \alpha + \beta y + \omega y^2 \qquad\qquad (3).$$

In consequence of the disturbance ψ deviates from the value given by (3); but, since, by a known theorem, the vorticity remains throughout equal to ω, the addition to ψ must satisfy $\nabla^2\psi = 0$. The additional terms must also satisfy the condition of being periodic in period 2π; and thus we obtain altogether as the expression for ψ during the disturbed motion

$$\psi = \alpha + \beta y + \omega y^2 + e^{-y}(A_1 \cos x + B_1 \sin x)$$
$$+ e^{-2y}(A_2 \cos 2x + B_2 \sin 2x) + \ldots \qquad\qquad (4),$$

positive exponents being excluded by the condition to be satisfied when $y = +\infty$. Similarly in the lower fluid

$$\psi' = \alpha' + \beta' y - \omega y^2 + e^{y}(A_1' \cos x + B_1' \sin x)$$
$$+ e^{2y}(A_2' \cos 2x + B_2' \sin 2y) + \ldots \qquad\qquad (5).$$

From these values of ψ, ψ' the velocities u, v at any point are deducible by (2).

We have still to satisfy the conditions at the surface of separation

$$y = h \cos x \qquad\qquad (6).$$

It is necessary that u and v, as given by ψ and ψ', should there be continuous, any sliding of the one body of fluid upon the other being equivalent to a vortex-sheet, and therefore excluded by the conditions of the problem. Thus at the surface we must have

$$d(\psi - \psi')/dx = 0, \quad d(\psi - \psi')/dy = 0 \qquad\qquad (7).$$

For the purposes of the first approximation, where only the first power of h is retained, y may be put equal to zero in the exponential terms so soon as the differentiations have been performed. Equations (7) give accordingly

$$-\sin x\,(A_1 - A_1') + \cos x\,(B_1 - B_1')$$
$$-2 \sin 2x\,(A_2 - A_2') + 2 \cos 2x\,(B_2 - B_2') - \ldots\ldots = 0$$

$$\beta-\beta'+4\omega h\cos x-\cos x\,(A_1+A_1')-\sin x\,(B_1+B_1')$$
$$-2\cos 2x\,(A_2+A_2')-2\sin 2x\,(B_2+B_2')-\ldots\ldots\ldots=0;$$

from which it appears that to this approximation all the coefficients with suffixes higher than unity must vanish. Also

$$B_1=0,\quad B_1'=0;\quad \beta'=\beta,\quad A_1'=A_1=2\omega h.$$

Thus
$$\psi=\alpha+\beta y+\omega y^2+2\omega h\,e^{-y}\cos x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$
$$\psi'=\alpha'+\beta y-\omega y^2+2\omega h\,e^{y}\cos x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

are the values of ψ determined in accordance with (6) and the other prescribed conditions. From (8) or (9), we find as the values of u and v at the surface

$$u=\beta,\quad v=2\omega h\sin x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

applicable when the form of the surface is that given by (6), at the moment, we may suppose, when $t=0$.

By means of (10) it is possible to determine the form and position of the surface of separation at time dt, and thus to trace out its transformation. In the present case it will be simplest merely to verify that the propagation of the form (6) with a certain velocity (V) satisfies all the conditions. If

$$F(x,y,t)=y-h\cos(x-Vt)=0\ \ldots\ldots\ldots\ldots\ldots(11)$$

be the equation of the surface, the condition to be satisfied* is

$$DF/Dt=0,$$

or
$$\frac{dF}{dt}+u\frac{dF}{dx}+v\frac{dF}{dy}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Here, when $t=0$,

$$\frac{dF}{dt}=-Vh\sin x,\quad \frac{dF}{dx}=h\sin x,\quad \frac{dF}{dy}=1;$$

so that (12) becomes, with use of (10),

$$(-V+\beta+2\omega)\,h\sin x=0,$$

showing that (11) continues to represent the surface of separation at time dt, provided that

$$V=\beta+2\omega\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

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* Lamb's *Hydrodynamics*, § 10.

Accordingly, if (13) be satisfied, equation (11) suffices to represent the changes in the surface of separation for any length of time, or, in other words, the disturbance is propagated as a simple wave.

From (8) it appears that β represents the velocity in the steady motion when $y = 0$, and the result is in accordance with (1), where $\tanh kb = 1$. The disturbance may be supposed to be got rid of by the introduction of a flexible lamina at the surface of separation. If, by forces applied to it, the lamina be straightened out so as to coincide with $y = 0$, and be held there at rest, the steady motion is recovered.

In proceeding to further approximations, in which higher powers of h are retained, it appears either from the equations, or immediately from the symmetries involved, that all the B's vanish, so that cosines only occur in (4) and (5), that

$$A_1' = A_1, \quad A_3' = A_3, \quad A_5' = A_5, \ \&c.\,;$$
$$A_2' = -A_2, \quad A_4' = -A_4, \ \&c.\,;$$

and further that $\beta' = \beta$. Equations (4) and (5) may thus be written

$$\psi = \alpha + \beta y + \omega y^2 + A_1 e^{-y}\cos x + A_2 e^{-2y}\cos 2x + A_3 e^{-3y}\cos 3x + \dots \qquad (14),$$

$$\psi' = \alpha' + \beta y - \omega y^2 + A_1 e^{y}\cos x - A_2 e^{2y}\cos 2x + A_3 e^{3y}\cos 3x - \dots \qquad (15).$$

A_1 is of order h, A_2 of order h^2, A_3 of order h^3, and so on. If we are content to neglect h^6, we may stop at A_5; and we find as the equations necessary in order to secure the continuity of u and v at the surface (6)

$$A_1\left(2 + \frac{3h^2}{4} + \frac{h^4}{12}\right) = 4\omega h + 2A_2\left(2h + \frac{4h^3}{3}\right) - 3A_3\frac{9h^2}{4},$$

$$2A_2(2 + 2h^2) = A_1\left(h + \frac{h^3}{12}\right) + 3A_3 \,.\, 3h,$$

$$3A_3\left(2 + \frac{9h^2}{2}\right) = -A_1\left(\frac{h^3}{4} + \frac{5h^4}{192}\right) + 2A_2(2h + h^3) + 4A_4 \,.\, 4h,$$

$$4A_4 \,.\, 2 = A_1\frac{h^3}{24} - 2A_2 \,.\, h^2 + 3A_3 \,.\, 3h,$$

$$5A_5 \,.\, 2 = -A_1\frac{h^4}{192} + 2A_2\frac{h^3}{3} - 3A_3\frac{9h^2}{4} + 4A_4 \,.\, 4h.$$

[redacted]

From these equations the values of the constants may be determined by successive approximations. Thus, if we retain terms of the order h^2, A_3, A_4, &c., vanish, and

$$A_1 = 2\omega h, \quad 2A_2 = \omega h^2.$$

This is the second approximation. The fifth approximation gives

$$A_1 = 2\omega h\left(1+\frac{h^2}{8}-\frac{h^4}{96}\right) \quad\text{.....................(16)},$$

$$2A_2 = \omega h^2\left(1+\frac{h^2}{3}\right) \quad\text{...........................(17)},$$

$$3A_3 = \frac{3\omega h^3}{4}\left(1+\frac{9h^2}{16}\right)\quad\text{............................(18)},$$

$$4A_4 = \frac{2\omega h^4}{3} \quad\text{.......................................(19)},$$

$$5A_5 = \frac{125\omega h^5}{192} \quad\text{....................................(20)},$$

which values are to be substituted in (14), (15).

The next step is the investigation of the values of u, v at the surface (6). They are most conveniently expressed as

$$\tfrac{1}{2}d(\psi+\psi')/dy, \quad -\tfrac{1}{2}d(\psi+\psi')/dx.$$

We get, correct as far as h^5,

$$u = \beta+\omega h^2-\tfrac{1}{4}\omega h^4+\tfrac{1}{12}\omega h^4\cos 2x \quad\text{..........................(21)},$$

$$v = 2\omega h\left(1-\frac{h^2}{4}-\frac{5h^4}{192}\right)\sin x+\frac{271\omega h^5}{96}\sin 3x \quad\text{.........(22)},$$

the terms containing $\cos 4x$ in (21), and $\sin 5x$ in (22), vanishing to this order. If we substitute these values in (12), we obtain

$$h\sin x\left\{-V+\beta+2\omega+\tfrac{1}{2}\omega h^2-\tfrac{11}{32}\omega h^4\right\}+\tfrac{275}{96}\omega h^5\sin 3x = 0 \quad\text{......(23)}.$$

So far, then, as terms in h^4, the surface of separation (6) is propagated as a simple wave with velocity given by

$$V = \beta+2\omega+\tfrac{1}{2}\omega h^2 \quad\text{..........................(24)};$$

but, if terms in h^5 are retained, a change of form manifests itself, corresponding to the term in $\omega h^5\sin 3x$ outstanding in (23).

Hitherto the wave-length has been supposed to be 2π, but, if we now take it to be $2\pi/k$, (24) becomes

$$V = \beta + 2\omega/k \, . \, (1 + \tfrac{1}{4}k^2h^2) \qquad \text{(25)},$$

as is evident by "dimensions." The velocity of propagation is that of the flow of the fluid in the steady motion at the place where

$$ky = 1 + \tfrac{1}{4}k^2h^2 \qquad \text{(26)}.$$

So far as the present investigation can reach, there is no sign of the amplitude of a wave tending spontaneously to increase.

Some Algebraical Theorems connected with the Theory of Partitions. By A. R. Forsyth.

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1. In a letter written to me last month (September), Major MacMahon suggested the consideration of the following problem:—

"Let the fraction, obtained by taking the reciprocal of

$$\begin{aligned} X = (1-ax)&\left(1-\frac{x}{a}\right) \\ (1-abx^2)&\left(1-\frac{x^2}{ab}\right) \\ (1-abcx^3)&\left(1-\frac{x^3}{abc}\right) \\ &\vdots \end{aligned},$$

where X contains n product-pairs, be expanded in ascending powers of x. In the expansion thus obtained, suppress every term containing a negative index for any one of the symbols $a, b, c, \ldots$; and in the surviving terms let each of these symbols be made unity. The sum of the resulting series is required."

The result of the selective and summative operations, which are to be effected on X^{-1} for the present purpose, may be denoted by $Sl\,X^{-1}$, and the like for more general fractions.

Special simple forms can easily be evaluated. Thus, when $n=1$, it is necessary to consider the fraction

$$\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)}.$$

Using a well-known theorem in trigonometry, we write

$$\frac{1-x^2}{(1-ax)\left(1-\frac{x}{a}\right)} = 1+x\left(a+\frac{1}{a}\right)+x^2\left(a^2+\frac{1}{a^2}\right)+\dots;$$

whence
$$Sl\,\frac{1-x^2}{(1-ax)\left(1-\frac{x}{a}\right)} = 1+x+x^2+\dots$$
$$= \frac{1}{1-x},$$

and therefore
$$Sl\,\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)} = \frac{1}{(1-x)(1-x^2)}.$$

Similarly, by expanding the two parts of the fraction

$$\frac{1}{\left\{(1-ax)\left(1-\frac{x}{a}\right)\right\}\left\{(1-aby)\left(1-\frac{y}{ab}\right)\right\}},$$

taking the product, and effecting the same operations, it can be proved that

$$Sl\,\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)(1-aby)\left(1-\frac{y}{ab}\right)}$$
$$= \frac{1-x^2y}{(1-x)(1-x^2)(1-xy)(1-y)(1-y^2)}$$

so that, by taking $y=x^2$, we have

$$Sl\,\frac{1}{(1-ax)\left(1-\frac{x}{a}\right)(1-abx^2)\left(1-\frac{x^2}{ab}\right)} = \frac{1}{(1-x)(1-x^2)^2(1-x^3)}.$$

This method becomes almost insuperably laborious as the number of product-pairs is increased, and seems to be ineffective for the general case; but it suggests the form of the result, viz.,

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\ldots(1-x^n)^2(1-x^{n+1})}$$

a form which Major MacMahon had inferred also from other considerations.

2. The following method leads to the general result.

The expansion of the fraction

$$\left[\left\{(1-a_1x_1)\left(1-\frac{x_1}{a_1}\right)\right\}\left\{(1-a_1x_1a_2x_2)\left(1-\frac{x_1x_2}{a_1a_2}\right)\right\}\ldots\right]^{-1},$$

where there are s product-pairs, is the product of the series

$$1+a_1x_1+\ldots+(a_1x_1)^{m_1}+\ldots,$$

$$1+\frac{x_1}{a_1}+\ldots+\left(\frac{x_1}{a_1}\right)^{n_1}+\ldots,$$

$$1+a_1x_1a_2x_2+\ldots+(a_1x_1a_2x_2)^{m_2}+\ldots$$

$$1+\frac{x_1x_2}{a_1a_2}+\ldots+\left(\frac{x_1x_2}{a_1a_2}\right)^{n_2}+\ldots,$$

$$\vdots \qquad ,$$

where the integers m, n have all values from 0 to ∞. Writing

$$x_1 = x_2 = \ldots = x,$$

the general term in the expansion of

$$\left[\left\{(1-a_1x)\left(1-\frac{x}{a_1}\right)\right\}\left\{(1-a_1a_2x^2)\left(1-\frac{x^2}{a_1a_2}\right)\right\}\ldots\right]^{-1}$$

has $$a_1^{m_1-n_1+m_2-n_2+m_3-n_3+\ldots}\,a_2^{m_2-n_2+m_3-n_3+\ldots}\,a^{m_3-n_3+\ldots}\ldots,$$

for the coefficient of the power of x represented by

$$x^{m_1+n_1+2(m_2+n_2)+3(m_3+n_3)+\ldots}.$$

This general term will survive if no one of the indices of the quantities a_2, a_3, ... be negative; hence, keeping the indices of a_2, a_3, ... not negative, the term survives for values of n_1 from 0 to

$m_1+(m_2-n_2)+(m_3-n_3)+\dots$. In each such term we may now make a_1 unity; and then, taking their sum, we have

$$SlX^{-1} = Sla_2^{m_2-n_2+m_3-n_3+\dots}\, a_3^{m_3-n_3+\dots} \dots$$

$$\dots \frac{1}{1-x}\left\{x^{m_1+(2m_2+2n_2)+(3m_3+3n_3)+\dots} - x^{1+2m_1+(3m_2+n_2)+(4m_3+2n_3)+\dots}\right\}.$$

The integer m_1 now occurs only in the indices of the two powers of x; and thus these two typical terms survive for all values of m_1 from 0 to ∞. Summing for these, we have

$$SlX^{-1} = Sl\,a_2^{m_2-n_2+m_3-n_3+\dots}\, a_3^{m_3-n_3+\dots} \dots$$

$$\dots\left\{\frac{1}{(1-x)^2}\, x^{(2m_2+2n_2)+(3m_3+3n_3)+\dots} - \frac{x}{(1-x)(1-x^2)}\, x^{(3m_2+n_2)+(4m_3+2n_3)+\dots}\right\}.$$

This concludes the first stage in the summation; its effect is to make a_1, m_1, n_1 disappear from the typical terms. And it is found that the summation can be effected by similar stages in succession, each consisting of two processes: the first is a summation of a limited number of terms, say for values of n_i; the second is a summation of an unlimited number of terms, for values of m_i; and the effect is to make a_i, m_i, n_i disappear from the typical terms.

The limits of n_2 are 0 to $m_2+(m_3-n_3)+\dots$ for terms that survive. Proceeding to take their sum when a_2 is made unity, we have

$$SlX^{-1} = Sla_3^{m_3-n_3+\dots} \dots \left[\frac{1}{(1-x)^2}\,\frac{1}{1-x^2}\left\{x^{2m_2+(3m_3+3n_3)+\dots} - x^{2+4m_2+(5m_3+n_3)+\dots}\right\}\right.$$

$$\left. - \frac{x}{(1-x)(1-x^2)}\,\frac{1}{1-x}\left\{x^{3m_2+(4m_3+2n_3)+\dots} - x^{1+4m_2+5m_3+n_3)+\dots}\right\}\right]$$

$$= Sla_3^{m_3-n_3+\dots} \dots \frac{1}{(1-x)^2(1-x^2)}\left\{x^{2m_2+(3m_3+3n_3)+\dots} - x^{1+3m_2+(4m_3+2n_3)+\dots}\right\}.$$

Now summing for m_2 for values 0 to ∞, we have

$$SlX^{-1} = Sla_3^{m_3-n_3+\dots} \dots \left\{\frac{x^{(3m_3+3n_3)+\dots}}{(1-x)^2(1-x^2)^2} - x\,\frac{x^{(4m_3+2n_3)+\dots}}{(1-x)^2(1-x^2)(1-x^3)}\right\},$$

being the result at the end of the second stage.

Proceeding similarly through the third stage, by summing first for n_3 from 0 to $m_3+(m_4-n_4)+\dots$, after making a_3 unity, and then for m_3 from 0 to ∞, we find the result at the end of the stage in the form

$$SlX^{-1} = Sla_4^{m_4-n_4+\dots}\dots\left\{\frac{x^{(4m_4+4n_4)+\dots}}{(1-x)^2(1-x^2)^2(1-x^3)^2} - x\frac{x^{(5m_4+3n_4)+\dots}}{(1-x)^2(1-x^2)^2(1-x^3)(1-x^4)}\right\}.$$

It is now clear that the result at the end of $i-1$ stages is

$$SlX^{-1} = Sla_i^{m_i-n_i+\dots}\dots\left[\frac{x^{(im_i+in_i)+\dots}}{(1-x)^2(1-x^2)^2\dots(1-x^{i-1})^2} - x\frac{x^{[(i+1)m_i+(i-1)n_i]+\dots}}{(1-x)^2(1-x^2)^2\dots(1-x^{i-2})^2(1-x^{i-1})(1-x^i)}\right],$$

the index of the power of x in the first term in the bracket being

$$\{im_i+in_i\}+\{(i+1)\,m_{i+1}+(i+1)\,n_{i+1}\} + \{(i+2)\,m_{i+2}+(i+2)\,n_{i+2}\}+\dots,$$

and the index in the second term being

$$\{(i+1)\,m_i+(i-1)\,n_i\}+\{(i+2)\,m_{i+1}+in_{i+1}\} + \{(i+3)\,m_{i+2}+(i+1)\,n_{i+2}\}+\dots.$$

The inductive establishment of the result is so simple after the preceding explanations that it need not be reproduced here.

At the end of s stages, there being s product-pairs, all the necessary summations have been effected, the symbolical coefficient involving the symbols a is now unity, and all the indices m_{s+1}, n_{s+1}, ... are zero. The end of the s^{th} stage gives the final result, so that we have

$$\begin{aligned} SlX^{-1} &= \frac{1}{(1-x)^2(1-x^2)^2\dots(1-x^s)^2} - \frac{x}{(1-x)^2(1-x^2)^2\dots(1-x^{s-1})^2(1-x^s)(1-x^{s+1})} \\ &= \frac{1}{(1-x)(1-x^2)^2(1-x^3)^2\dots(1-x^s)^2(1-x^{s+1})}, \end{aligned}$$

which is the result required.

3. It may be added that, of course, the summation need not be taken in the order implied. Thus, dealing with a simple case for illustration—that of two product-pairs—we have

$$SlX^{-1} = Sl\,a_1^{m_1-n_1+m_2-n_2}\,a_2^{m_2-n_2}\,x^{m_1+n_1+2m_2+2n_2},$$

and we may sum, first with regard to m_1, and then with regard to n_1. Taking $m_2-n_2=\theta$, the typical term will survive when m_1 ranges from 0 to ∞ if $n_1 \leqslant \theta$, and, when m_1 ranges from $n_1-\theta$ to ∞, if $n_1 \geqslant \theta+1$. Hence, summing for m_1, we have

$$SlX^{-1} = Sl\,a_2^{\theta}\,\frac{x^{n_1+4n_2+2\theta}}{1-x} \text{ for } n_1 = 0 \text{ to } \theta$$

$$+Sl\,a_2^{\theta}\,\frac{x^{2n_1+4n_2+\theta}}{1-x} \text{ for } n_1 = \theta+1 \text{ to } \infty$$

$$= Sl\,\frac{a_2^{\theta}}{1-x}\left\{\frac{x^{4n_2+2\theta}-x^{1+4n_2+3\theta}}{1-x}+\frac{x^{2+4n_2+3\theta}}{1-x^2}\right\}$$

$$= Sl\,\frac{a_2}{1-x}\left\{\frac{x^{4n_2+2\theta}}{1-x}-x\,\frac{x^{4n_2+3\theta}}{1-x^2}\right\},$$

agreeing with the preceding form at the end of the first stage for the case $s=2$.

In the present case, we may now sum for θ from 0 to ∞, and for n_2 from 0 to ∞, obtaining the result

$$SlX^{-1} = \frac{1}{(1-x)(1-x^2)^2(1-x^3)}.$$

But had there been more than two product-pairs, so that at the end of the first stage there would have been more than one symbol a_3, the summation would have had to be separated as above; viz., taking

$$m_3-n_3+\ldots = \phi,$$

then m_3 ranges from 0 to ∞ if $n_3 \leqslant \phi$, and m_3 ranges from $n_3-\phi$ to ∞ if $n_3 \geqslant \phi+1$. And so on for other cases. A slight consideration will show that the previous sequence in summation is the more convenient, save at the last stage; an example of the latter is furnished in § 10 (*post*).

4. In another letter written a few days after the first, Major MacMahon propounded the similar problem for a more general case when there are s sets of products as above, but each product contains

$(1+r)$ factors (instead of two, as above). External considerations had led him to infer the form of the result in the limit when both s and r are infinite.

The preceding method can be applied: and some cases are solved in what follows. The general result is inferred, but it is not completely established, partly owing to the very laborious character of the algebra; and the limiting form, when both r and s are infinite, agrees with that which was inferred by Major MacMahon.

Let X denote the product

$$\begin{array}{l}
(1-a_r x_1)\left(1-\frac{a_{r-1}}{a_r}x_1\right)\left(1-\frac{a_{r-2}}{a_{r-1}}x_1\right)\ \ldots\ \left(1-\frac{a_1}{a_2}x_1\right)\left(1-\frac{x_1}{a_1}\right) \\
(1-a_r x_1 b_r x_2)\left(1-\frac{a_{r-1}}{a_r}x_1\frac{b_{r-1}}{b_r}x_2\right)\ \ldots\ldots\ldots\ \left(1-\frac{a_1}{a_2}x_1\frac{b_1}{b_2}x_2\right)\left(1-\frac{x_1x_2}{a_1b_1}\right) \\
(1-a_r x_1 b_r x_2 c_r x_3)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \left(1-\frac{x_1x_2x_3}{a_1b_1c_1}\right) \\
\qquad\vdots
\end{array},$$

where there are s sets of symbols $a, b, c, \ldots$, so that there are s lines; and each line contains $r+1$ factors. Then X^{-1} is to be expanded in positive powers of $x_1, x_2, \ldots$; when all the variables $x_1, x_2, \ldots$ are made equal to x, the general term has

$$\begin{array}{llll}
a_r^{\,r-l_{r-1}+m_r-m_{r-1}+n_r-n_{r-1}+\ldots} & a_{r-1}^{\,l_{r-1}-l_{r-2}+m_{r-1}-m_{r-2}+n_{r-1}-n_{r-2}+\ldots} & \ldots & a_1^{\,l_1-l_0+m_1-m_0+n_1-n_0+\ldots} \\
b_r^{\,m_r-m_{r-1}+n_r-n_{r-1}+\ldots} & b_{r-1}^{\,m_{r-1}-m_{r-2}+n_{r-1}-n_{r-2}+\ldots} & \ldots\ldots & b_1^{\,m_1-m_0+n_1-n_0+\ldots} \\
c_r^{\,n_r-n_{r-1}+\ldots} & c_{r-1}^{\,n_{r-1}-n_{r-2}+\ldots} & \ldots\ldots\ldots & c^{\,n_1-n_0+\ldots} \\
\ddots & \ddots & \ddots & \ddots
\end{array}$$

as the coefficient of the power of x represented by

$$x^{(l_0+l_1+\ldots+l_r)+2(m_0+m_1+\ldots+m_r)+3(n_0+n_1+\ldots+n_r)+\ldots}.$$

In this expansion, all terms containing a negative exponent for any one (or for more than one) of the symbols $a, b, c, \ldots$ are to be suppressed; in those which survive, each of the symbols $a, b, c, \ldots$ is to be made unity; and the sum of the latter is required. The result, as before, is denoted by $Sl\,X^{-1}$.

The summation is carried out in as many stages as there are sets of symbols a, b, c, ..., viz., in s stages. In each stage, there are $r+1$ summations; in the first stage, they are taken with regard to $l_0, l_1, \dots, l_r$ in respective succession; in the second stage, with regard to $m_0, m_1, \dots, m_r$ in respective succession; and so with the others.

Owing to the length of the expressions for the indices, and the coefficients in the general case, we shall first take a particular case, say $r=3$; then state the result for the case $r=4$, which can easily be established; it will then appear that the form of the result for the general case can be inferred, and can be inductively established, all these remarks applying to the first stage. Moreover, for the sake of brevity, we shall denote the quantity

$$\begin{aligned} &\lambda m_0+\mu m_1+\nu m_2+\rho m_3+\dots \\ &+(\lambda+1)\,n_0+(\mu+1)\,n_1+(\nu+1)\,n_2+(\rho+1)\,n_3+\dots \\ &+(\lambda+2)\,p_0+(\mu+2)\,p_1+(\nu+2)\,p_2+(\rho+2)\,p_3+\dots \\ &+\dots \end{aligned}$$

by the symbol $(\lambda, \mu, \nu, \rho, \dots)$; it will appear that the part of the index of x depending upon the integers $m, n, p, \dots$ always takes this form with different values of $\lambda, \mu, \nu, \rho, \dots$ from term to term, subject to the condition that the integer

$$\lambda+\mu+\nu+\rho+\dots$$

$(=2r+2$ in general$)=8$ in the first special form considered. Further, also for the sake of brevity, it will be convenient to write

$$1-x^a=(a):$$

so that, for instance, $(1)^2\,(2)$ means $(1-x)^2\,(1-x^2)$, and so on.

5. Let Θ denote the part of the coefficient depending on the symbols $b, c, \dots$; it is the same for all the possible values of $l_0, l_1, \dots$, and is therefore unaffected by summation with regard to them.

Summing first for l_0, the limits are 0—because of the origin of the term—to

$$l_1+(m_1-m_0)+(n_1-n_0)+\dots;$$

higher values of l_0 give rise to terms that are to be suppressed. In the terms thus selected, we make a_1 equal to unity, and the result of summation gives

$$Sl\,X^{-1}=Sl\,\Theta\,a_3^{l_3-l_2+\quad{}_3-m_2+\dots}\,a_2^{l_2-l_1+m_2-m_1+\dots}$$
$$\frac{1}{1-x}\left\{x^{l_1+l_2+l_3+(2,2,2,2)}-x^{1+2l_1+l_2+l_3+(1,\ 3,\ ,2)}\right\}.$$

The summation with regard to l_1 extends over terms given by values of l_1 from 0 to

$$l_2+(m_2-m_1)+\dots;$$

in these terms we make a_2 equal to unity, and find that, after summing, we have

$$SlX^{-1} = Sl\Theta\, a^{l_3-l_2+m_2-m_3+\dots}$$

$$\frac{1}{1-x}\left[\frac{x^{l_2+l_3+(2,2,2,2)}-x^{1+2l_2+l_3+(2,1,3,2)}}{1-x}\right.$$

$$\left.-\frac{x}{1-x^2}\left\{x^{l_2+l_3+(1,3,2,2)}-x^{2+3l_2+l_3+(1,1,4,2)}\right\}\right].$$

The limits for l_2 that leave surviving terms are 0 to

$$l_3+(m_3-m_2)+\dots.$$

Taking these terms, making $a_3=1$, and summing, we have

$$SlX^{-1} = Sl\Theta\left[\frac{1}{(1)^2}\left\{\frac{x^{l_3+(2,2,2,2)}-x^{1+2l_3+(2,2,1,3)}}{(1)}\right\}\right.$$

$$-\frac{x}{(1)^2}\left\{\frac{x^{l_3+(2,1,3,2)}-x^{2+3l_3+(2,1,1.4)}}{(2)}\right\}$$

$$-\frac{x}{(1)(2)}\left\{\frac{x^{l_3+(1,3,2,2)}-x^{1+2l_3+(1,3,1,3)}}{(1)}\right\}$$

$$\left.+\frac{x^3}{(1)(2)}\left\{\frac{x^{l_3+(1,1,4,2)}-x^{3+4l_3+(1,1,1,5)}}{(3)}\right\}\right].$$

In the particular case under consideration, there remains only a single operation in the first stage—summation with regard to l_3 from 0 to ∞. Effecting this, we have as the result of the first stage, in the case of $r=3$,

$$SlX^{-1} = Sl\Theta\left[\frac{x^{(2,2,2,2)}}{(1)^4}\right.$$

$$-\frac{x}{(1)^3(2)}\left\{x^{(2,2,1,3)}+x^{(2,1,3,2)}+x^{(1,3,2,2)}\right\}$$

$$+\frac{x^2}{(1)^2(2)^2}x^{(1,3,1,3)}$$

$$+\frac{x^3}{(1)^2(2)(3)}\left\{x^{(2,1,1,4)}+x^{(1,1,4,2)}\right\}$$

$$\left.-\frac{x^6}{1)(2)(3)(4)}\left(x^{1,1,1,5)}\right)\right].$$

6. To see what this becomes when there is only a single set of symbols a, it is sufficient to take all the indices $m, n, \ldots$ zero. Each index of the form $(\lambda, \mu, \nu, \rho)$ is then zero; and we have

$$Sl\,\frac{1}{(1-a_3x)\left(1-\frac{a_2}{a_3}x\right)\left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)}$$

$$=\frac{1}{(1)^4}-\frac{3x}{(1)^2(2)}+\frac{x^2}{(1)^2(2)^2}+\frac{x^3}{(1)^2(2)(3)}-\frac{x^6}{(1)(2)(3)(4)}$$

$$=\frac{1}{(1)(2)(3)(4)}$$

after reductions,

$$=\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

7. In the case of $r=4$, the summation with regard to l_3 is from 0 to $l_4+(m_4-m_3)+\ldots$; and the final operation is a summation with regard to l_4 from 0 to ∞. The result is

$$SlX^{-1}=Sl\Theta\left[\frac{x^{(2,2,2,2,2)}}{(1)^5}\right.$$

$$-\frac{x}{(1)^4(2)}\left\{x^{(3,2,2,1,3)}+x^{(2,2,1,3,2)}+x^{(2,1,3,2,2)}+x^{(1,3,3,2,2)}\right\}$$

$$+\frac{x^2}{(1)^3(2)^2}\left\{x^{(2,1,3,1,3)}+x^{(1,3,2,1,3)}+x^{(1,3,1,3,2)}\right\}$$

$$+\frac{x^3}{(1)^3(2)(3)}\left\{x^{(2,2,1,1,4)}+x^{(2,1,1,4,2)}+x^{(1,1,4,2,2)}\right\}$$

$$-\frac{x^4}{(1)^2(2)^2(3)}\left\{x^{(1,3,1,1,4)}+x^{(1,1,4,1,3)}\right\}$$

$$-\frac{x^6}{(1)^2(2)(3)(4)}\left\{x^{(2,1,1,1,5)}+x^{(1,1,1,5,2)}\right\}$$

$$\left.+\frac{x^{10}}{(1)(2)(3)(4)(5)}\,x^{(1,1,1,1,6)}\right].$$

8. The result of the first stage of operations in the general case of an unrestricted value of r can now be inferred. We remark:

(i.) The number of terms associated with $Sl\Theta_2$, each in the form of a fraction, is 2^r.

(ii.) Each term is composed of two parts—a coefficient, such as $\frac{x}{(1^4)(2)}$ above, and a power, such as $x^{(2,2,2,1,3)}$ above.

(iii.) The first fraction is of the form

$$\frac{1}{(1)^{r+1}}x^{(2,2,2,2,\ldots)},$$

where 2 occurs $r+1$ times in the index of x.

(iv.) As regards the (fractional) coefficient of any term, its denominator is of the form

$$(1)^{\rho_1}(2)^{\rho_2}(3)^{\rho_3}(4)^{\rho_4}\ldots,$$

where
$$\rho_1+\rho_2+\rho_3+\rho_4+\ldots=r+1;$$
and its numerator is
$$(-1)^{\theta}x^{\rho_3+2\rho_3+3\rho_4+\ldots},$$

where θ is the sum of the integers $\rho_2, \rho_3, \rho_4, \ldots$, and thus θ is $r+1-\rho_1$. Hence the (fractional) coefficient of a term is of the form

$$\frac{(-1)^{r+1-\rho_1}x^{\rho_3+2\rho_3+3\rho_4+\ldots}}{(1)^{\rho_1}(2)^{\rho_2}(3)^{\rho_3}(4)^{\rho_4}\ldots},$$

subject to the condition

$$\rho_1+\rho_2+\rho_3+\rho_4+\ldots=r+1.$$

(v.) As regards the index of the power of x, wherever 3 occurs it follows 1; wherever 4 occurs it follows 1, 1; wherever 5 occurs it follows 1, 1, 1, and so on.

When 13 occurs in the index there is a factor (2) in the denominator of the coefficient; when 114 occurs in the index there is a factor (3) in that denominator; when 1115, then a factor (4), and so on.

Two terms have the same coefficient, if the indices of the powers of x in them differ only by permutation of the symbols 2, 13, 114, 1115, ... Hence the total number of terms having the same coefficient is the same as the number of permutations of the symbols 2, 13, 114, 1115, ..., which occur in the index-symbol of any one;

and any one index-symbol can be written down from an inspection of the denominator. For example, an index-symbol to be associated with the coefficient in (iv.) is

$$2\,.\,2\,\ldots\quad 13\,.\,13\,\ldots\quad 114\,.\,114\,\ldots\quad 1115\,.\,1115\,\ldots\quad \ldots,$$

where the numbers of the symbols 2, 13, 114, 1115, ... are $\rho_1, \rho_2, \rho_3, \rho_4$, ... respectively.

Every symbol is a partition of $2r+2$ into $r+1$ of the integers 1, 2, 3, ..., $r+2$, each repeated any proper number of times. Each such partition, in which the symbol 1 is properly associated with the symbols 3, 4, 5, ...—it is easy to see that the appropriate number of symbols 1 will be found in the partition—gives rise to an index-symbol; and the possible permutations are then to be formed as in the preceding paragraph.

The expression at the end of the last summation is now completely deducible in the general case.

Similar explanations apply as regards the expression before the end of the last summation; for the complete establishment, an inductive proof would be necessary in statement. It is not difficult—only lengthy—after what precedes.

9. There is interest in a special case, viz., $s=1$, when there is only a single set of symbols a; the first stage, as above, is the only one necessary, and the result—with all the symbols $(\lambda, \mu, \nu, \rho, \ldots)$ made zero—is the value of

$$U = Sl\left[(1-a_r x)\left(1-\frac{a_{r-1}}{a_r}x\right)\ldots\left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)\right]^{-1}.$$

It can be expressed in a simpler form, as follows :—

Let α denote $\dfrac{1}{(1)}$,

β " $-\dfrac{x}{(1)(2)}$,

γ " $\dfrac{x^3}{(1)(2)(3)}$,

δ " $-\dfrac{x^6}{(1)(2)(3)(4)}$,

$\vdots$,

and so on. Form the product of the expressions

$$1+zu\alpha+\frac{z^2}{2!}u^2\alpha^2+\frac{z^3}{3!}u^3\alpha^3+\dots,\quad \text{ad inf.},$$

$$1+zu^2\beta+\frac{z^2}{2!}u^4\beta^2+\frac{z^3}{3!}u^6\beta^3+\dots,$$

$$1+zu^3\gamma+\frac{z^2}{2!}u^6\gamma^2+\frac{z^3}{3!}u^9\gamma^3+\dots,$$

$$\vdots$$

After multiplying up, change z^m into $m!$; it is easy to see that the numerical coefficient of any particular term in the product is the number of permutations of the index-symbol in a corresponding term of the preceding investigation, and thus is the proper numerical coefficient to be associated with

$$\frac{(-1)^{r+1-p_1}x^{p_2+2p_3+3p_4+\dots}}{(1)^{p_1}(2)^{p_2}(3)^{p_3}(4)^{p_4}}$$

in the present case when the index-symbols are zero. It therefore follows* from all the preceding considerations that U is the coefficient of u^{r+1} in this product. But the product is

$$e^{zu\alpha+zu^2\beta+zu^3\gamma+\dots},$$

or, taking v to denote $u\alpha+u^2\beta+u^3\gamma+\dots$, the product is

$$e^{zv}$$

$$=1+zv+\frac{z^2}{2!}v^2+\frac{z^3}{3!}v^3+\dots,$$

whatever z may be. Now realizing, the product is

$$1+v+v^2+v^3+\dots$$

$$=\frac{1}{1-v}$$

$$=\frac{1}{1-u\alpha-u^2\beta-u^3\gamma-\dots},$$

so that U is the coefficient of u^{r+1} in

$$\frac{1}{1-\frac{u}{(1)}+\frac{u^2x}{(1)(2)}-\frac{u^3x^3}{(1)(2)(3)}+\frac{u^4x^6}{(1)(2)(3)(4)}-\dots \text{ ad inf.}},$$

---

* This was suggested by Jeffery's method of finding the number of permutations of a number of things when they are the same in sets; see Todhunter's *Algebra*, p. 537.

that is, in

$$\frac{1}{(1-u)(1-xu)(1-x^2u)(1-x^3u)\ \dots\ \text{ad inf.}},$$

that is, in

$$1+\frac{u}{(1)}+\frac{u^2}{(1)(2)}+\frac{u^3}{(1)(2)(3)}+\frac{u^4}{(1)(2)(3)(4)}+\dots,$$

that is, it is

$$\frac{1}{(1)(2)\dots(r+1)}.$$

Hence

$$Sl\left[(1-a_r x)\left(1-\frac{a_{r-1}}{a_r}x\right)\dots\left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)\right]^{-}$$

$$=\frac{1}{(1-x)(1-x^2)(1-x^3)\dots(1-x^{r+1})}.$$

10. This particular result can also be established as follows:—In the expansion of the fraction, the general term is of the form

$$a_r^{p_r-p_{r-1}}\,a_{r-1}^{p_{r-1}-p_{r-2}}\,a_{r-2}^{p_{r-2}-p_{r-3}}\dots a_2^{p_2-p_1}\,a_1^{p_1-p_0}\,x^{p_r+p_{r-1}+p_{r-2}+\dots+p_1+p_0}.$$

Summing first with regard to p_r, the limits are p_{r-1} to ∞ for terms that survive; the result is

$$\frac{1}{1-x}\,a_{r-1}^{p_{r-1}-p_{r-2}}\,a_{r-2}^{p_{r-2}-p^{r-3}}\dots a_2^{p_2-p_1}\,a_1^{p_1-p_0}\,x^{2p_{r-1}+p_{r-2}+p_{r-3}+\dots+p_1+p_0}.$$

Summing next for p_{r-1}, the limits are p_{r-2} to ∞ for terms that survive; the result is

$$\frac{1}{(1-x)(1-x^2)}\,a_{r-2}^{p_{r-2}\ \ p_{r-3}}\dots a_2^{p_2-p_1}\,a_1^{p_1-p_0}\,x^{3p_{r-2}+p_{r-3}+\dots+p_1+p_0}.$$

Proceeding in this way, the result of r summations—the last being from p_0 to ∞—is

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^r)}\,x^{(r+1)p_0}.$$

And p_0 can now have any value from 0 to ∞; hence a last summation leads to the result as stated above.

11. Proceeding to the second stage, the summation can be carried out in a similar manner, either for the aggregate of terms together,

or for each term separately. The latter was the mode adopted with the following result.

Denoting by Φ the part of the symbolical coefficient Θ which is independent of $c, d, \ldots$, and by $|\lambda, \mu, \nu, \rho|$ the expression

$$\begin{aligned}&\lambda n_0+\mu n_1+\nu n_2+\rho n_3\\ &+(\lambda+1)p_0+(\mu+1)p_1+(\nu+1)p_2+(\rho+1)p_3\\ &+\ldots,\end{aligned}$$

it can be proved that

$$\begin{aligned}SlX^{-1} = Sl\Phi\Bigg[&\frac{x^{|3,3,3,3|}}{(1)^4(2)^4}\\ &-\frac{x}{(1)^4(2)^3(3)}\left\{x^{|3,3,2,4|}+x^{|3,2,4,3|}+x^{|2,4,3,3|}\right\}\\ &+\frac{x^2}{(1)^4(2)^2(3)^2}x^{|2,4,2,4|}\\ &+\frac{x^3}{(1)^3(2)^3(3)^2}\left\{x^{|3,1,4,4|}+x^{|1,4,4,3|}\right\}\\ &+\frac{x^3}{(1)^4(2)^2(3)(4)}\left\{x^{|3,2,2,5|}+x^{|2,2,5,3|}\right\}\\ &-\frac{x^4}{(1)^3(2)^3(3)(4)}\left\{x^{|3,1,3,5|}+x^{|1,3,5,3|}\right\}\\ &-\frac{x^5}{(1)^3(2)^2(3)^2(4)}\left\{x^{|1,4,2,5|}+x^{|2,1,5,4|}\right\}\\ &-\frac{x^6}{(1)^4(2)(3)(4)(5)}x^{|2,2,2,6|}\\ &+\frac{x^7}{(1)^3(2)^2(3)(4)(5)}\left\{x^{|2,1,3,6|}+x^{|1,3,2,6|}\right\}\\ &+\frac{x^8}{(1)^2(2)^2(3)^2(4)^2}x^{|1,1,5,5|}\\ &-\frac{x^9}{(1)(2)^2(3)^2(4)(5)}x^{|1,1,4,6|}\Bigg],\end{aligned}$$

which is the result at the end of the second stage.

12. In particular, if only two sets of symbols occur in the original expression for X^{-1}, we have $\Phi=1$; and all the symbolic indices $|\lambda, \mu, \nu, \rho|$ are zero. Then

$$SlX^{-1} = \frac{1}{(1)^4(2)^4} - \frac{3x}{(1)^4(2)^3(3)} + \frac{x^2}{(1)^4(2)^2(3)^2} + \frac{2x^3}{(1)^3(2)^3(3)^2}$$

$$+ \frac{2x^3}{(1)^4(2)^2(3)(4)} - \frac{2x^4}{(1)^3(2)^3(3)(4)} - \frac{2x^5}{(1)^3(2)^2(3)^2(4)}$$

$$- \frac{x^6}{(1)^4(2)(3)(4)(5)} + \frac{2x^7}{(1)^3(2)^2(3)(4)(5)}$$

$$+ \frac{x^8}{(1)^2(2)^2(3)^2(4)^2} - \frac{x^9}{(1)^2(2)^2(3)^2(4)(5)}$$

$$= \frac{1}{(1)(2)^2(3)^2(4)^2(5)}, \text{ multiplied by}$$

x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
1	3	8	14	22	27	30	27	22	14	8	3	1
	−3	−9	−21	−36	−51	−60	−60	−51	−36	−21	−9	−3
		1	3	6	10	13	14	13	10	6	3	1
			2	4	8	12	14	14	12	8	4	2
			2	6	12	18	22	22	18	12	6	2
				−2	−4	−8	−10	−12	−10	−8	−4	−2
					−2	−4	−6	−8	−8	−6	−4	−2
						−1	−3	−5	−6	−5	−3	−1
							2	4	6	6	4	2
								1	1	1	1	1
									−1	−1	−1	−1
1	±3	±9	±21	±38	±57	±73	±79	±76	±61	±41	±21	±9

$$= \frac{1}{(1)(2)^2(3)^2(4)^2(5)};$$

and therefore

$$Sl\left\{\frac{1}{(1-a_3x)\left(1-\dfrac{a_2}{a_3}x\right)\left(1-\dfrac{a_1}{a_2}x\right)\left(1-\dfrac{x}{a_1}\right)}\right.$$

$$\left.\times\frac{1}{(1-a_3b_3x^2)\left(1-\dfrac{a_2b_2}{a_3b_3}x^2\right)\left(1-\dfrac{a_1b_1}{a_2b_2}x^2\right)\left(1-\dfrac{x^2}{a_1b_1}\right)}\right\}$$

$$=\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)}.$$

13. From the results obtained, the general expression at the end of the second stage in the case of s sets of symbols and $r+1$ factors involving any one set does not present itself as an obvious generalizable result from the particular case. But the last expression, and corresponding expressions which I have obtained in simpler cases, suggest (but, of course, do not prove) that in the general case the required sum is the reciprocal of the product of the expressions

$$\begin{array}{l}(1)(2)(3)(4)\ \dots\ (r+1),\\ \quad(2)(3)(4)\ \dots\ \quad (r+2),\\ \qquad(3)(4)\ \dots\ \quad (r+2)(r+3),\\ \qquad\quad\dots\ \ \dots\ \ \dots\ \ \dots\\ \qquad\qquad\qquad s(1+s)(2+s)\ \dots\ (r+s).\end{array}$$

14. Further, we have at once

$$Sl\frac{1}{(1-a_1x)(1-a_1a_2x^2)\dots(1-a_1a_2\dots a_nx^n)}=\frac{1}{(1-x)(1-x^2)\dots(1-x^n)},$$

and we have proved that

$$Sl\frac{1}{(1-a_{n-1}x)\left(1-\dfrac{a_{n-2}}{a_{n-1}}x\right)\dots\left(1-\dfrac{a_1}{a_2}x\right)\left(1-\dfrac{x}{a_1}\right)}$$

$$=\frac{1}{(1-x)(1-x^2)\dots(1-x^n)}.$$

We have also proved that

$$Sl\frac{1}{(1-a_1x)\left(1-\frac{x}{a_1}\right)(1-a_1a_2x^2)\left(1-\frac{x^2}{a_1a_2}\right)\dots n \text{ product-pairs}}$$

$$=\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2\dots(1-x^n)^2(1-x^{n+1})},$$

and we have inferred that the latter expression is also the value of

$$Sl\left\{\frac{1}{(1-a_{n-1}x)\left(1-\frac{a_{n-2}}{a_{n-1}}x\right)\dots\left(1-\frac{a_1}{a_2}x\right)\left(1-\frac{x}{a_1}\right)}\right.$$

$$\left.\times\frac{1}{(1-a_{n-1}b_{n-1}x^2)\left(1-\frac{a_{n-2}b_{n-2}}{a_{n-1}b_{n-1}}x^2\right)\dots\left(1-\frac{a_1b_1}{a_2b_2}x^2\right)\left(1-\frac{x^2}{a_1b_1}\right)}\right\}.$$

Comparing these results with the diagram of the general expression given in § 13, it can be inferred that there is a symmetry of value between two expressions, one having m rows and n factors in each row, the other having n rows and m factors in each row. The establishment of this symmetry can be effected with the aid of the theory of partitions, by which subject, indeed, all these fractions were primarily suggested.

2nd October, 1895.

Note on Matrices. By J. Brill, M.A. Received October 14th, 1895. Communicated November 14th, 1895.

My object in the following short communication is to obtain the most general form of the differential of a matrix which admits of its being commutative with the matrix itself.

If m be the matrix, and $\lambda_1, \lambda_2, \dots, \lambda_n$ its latent roots, which we shall suppose to be all different, then m satisfies the equation

$$(m-\lambda_1)(m-\lambda_2)\dots(m-\lambda_n)=0 \quad \dots\dots\dots\dots\dots\dots(1).$$

[redacted]

Differentiating this, we have

$$\begin{aligned}&(dm-d\lambda_1)(m-\lambda_2) \;\ldots\ldots\; (m-\lambda_n)\\ +&(m-\lambda_1)(dm-d\lambda_2) \;\ldots\ldots\; (m-\lambda_n)\\ +&\text{\&c.} = 0.\end{aligned}$$

If we now introduce the condition that dm shall be commutative with m, we may write this equation in the form

$$\begin{aligned}&(dm-d\lambda_1)(m-\lambda_2) \;\ldots\ldots\ldots\ldots\ldots\; (m-\lambda_n)\\ +&(dm-d\lambda_2)(m-\lambda_1)(m-\lambda_3) \;\ldots\ldots\; (m-\lambda_n)\\ +&\text{\&c.}\\ +&(dm-d\lambda_n)(m-\lambda_1) \;\ldots\ldots\ldots\ldots\ldots\; (m-\lambda_{n-1}) = 0.\end{aligned}$$

Multiplying this equation by $(m-\lambda_1)$, and taking account of equation (1), we have

$$\begin{aligned}&(dm-d\lambda_2)(m-\lambda_1)^2(m-\lambda_3) \;\ldots\ldots\; (m-\lambda_n)\\ +&\text{\&c.}\\ +&(dm-d\lambda_n)(m-\lambda_1)^2(m-\lambda_2) \;\ldots\ldots\; (m-\lambda_{n-1}) = 0 \ldots\ldots\ldots (2).\end{aligned}$$

Now

$$\begin{aligned}m-\lambda_1 &= m-\lambda_2+\lambda_2-\lambda_1\\ &= m-\lambda_3+\lambda_3-\lambda_1\\ &= \text{\&c.}\end{aligned}$$

Taking account of these relations, and also of equation (1), we easily reduce equation (2) to the form

$$\begin{aligned}&(dm-d\lambda_2)(m-\lambda_1)(m-\lambda_3) \;\ldots\ldots\ldots\ldots\ldots\; (m-\lambda_n)(\lambda_2-\lambda_1)\\ +&(dm-d\lambda_3)(m-\lambda_1)(m-\lambda_2)(m-\lambda_4) \;\ldots\ldots\; (m-\lambda_n)(\lambda_3-\lambda_1)\\ +&\text{\&c.}\\ +&(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2) \;\ldots\ldots\ldots\ldots\ldots\; (m-\lambda_{n-1})(\lambda_n-\lambda_1) = 0.\end{aligned}$$

Multiplying this by $(m-\lambda_2)$, it becomes

$$(dm-d\lambda_3)(m-\lambda_1)(m-\lambda_2)^2(m-\lambda_4) \;\ldots\ldots\; (m-\lambda_n)(\lambda_3-\lambda_1)+\text{\&c.} = 0;$$

and, treating this equation in a manner similar to that in which we treated equation (2), we find that it reduces to

$$\begin{aligned}&(dm-d\lambda_3)(m-\lambda_1)(m-\lambda_2)(m-\lambda_4) \;\ldots\ldots\; (m-\lambda_n)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\\ +&\text{\&c.}\\ +&(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2) \;\ldots\ldots\ldots\ldots\ldots\; (m-\lambda_{n-1})(\lambda_n-\lambda_1)(\lambda_n-\lambda_2)\\ &= 0.\end{aligned}$$

This process may evidently be continued until we arrive at the result

$$(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_{n-1})(\lambda_n-\lambda_1)\ldots(\lambda_n-\lambda_{n-1})=0.$$

Now the product $(\lambda_n-\lambda_1)\ldots\ldots(\lambda_n-\lambda_{n-1})$

is scalar, and does not vanish. We may therefore divide out by it, and we obtain

$$(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2)\ldots\ldots(m-\lambda_{n-1})=0.$$

By proceeding in a similar manner, we may also obtain the following :—

$$(dm-d\lambda_1)(m-\lambda_2)(m-\lambda_3)\ldots\ldots(m-\lambda_n)=0,$$

$$(dm-d\lambda_2)(m-\lambda_1)(m-\lambda_3)\ldots\ldots(m-\lambda_n)=0,$$

&c., &c.

Thus we readily obtain

$$dm\left\{\frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}+\frac{(m-\lambda_1)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\ldots(\lambda_2-\lambda_n)}+\&c.\right\}$$

$$=d\lambda_1\frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}+\&c.$$

Now, by means of Sylvester's Interpolation Theorem, we have that, if $f(m)$ be a function framed on the model of a scalar function, then

$$f(m)=\frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}f(\lambda_1)+\&c.;$$

and therefore

$$1=m^0=\Sigma\frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}.$$

Thus we obtain, for the most general form of dm that shall be commutative with m, the expression

$$d\lambda_1\frac{(m-\lambda_2)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}+d\lambda_2\frac{(m-\lambda_1)(m-\lambda_3)\ldots(m-\lambda_n)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\ldots(\lambda_2-\lambda_n)}$$

$$+\&c.+d\lambda_n\frac{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_{n-1})}{(\lambda_n-\lambda_1)(\lambda_n-\lambda_2)\ldots(\lambda_n-\lambda_{n-1})}.$$

If there be equalities existing among the latent roots of the matrix, this formula will require modifying in a similar manner to Sylvester's Interpolation Theorem.

[As some points connected with the above paper have been misunderstood, I have thought it advisable to add a few words by way of further explanation. The question arose from an attempt of mine to apply the theory of matrices to obtain solutions of linear differential equations with constant coefficients.* I came across the theorem as applicable to the binary matrix. Struck with the resemblance to Sylvester's Interpolation Theorem, I first of all verified the theorem for the case of the ternary matrix, and then sought a method of establishing it for the general case, with the result given above.

If a matrix be capable of continuous variation, this is only possible through the continuous variation of certain scalar elements involved in its expression. The matrix m satisfies an identical equation, viz. (1). The matrix $m+dm$ will also satisfy an identical equation of similar form, and the main assumption introduced in the above paper is that the latent roots involved in this equation differ infinitesimally from those involved in equation (1). The work will therefore hold only where this is the case. I cannot at present conceive of any case in which the infinitesimal variation of any scalar quantity involved in the expression of a matrix would give rise to a finite change in any of its latent roots. If any such cases should arise, they would need special investigation.

The above work thus furnishes us with a test as to whether, when a matrix is varied by means of the infinitesimal variation of certain scalar elements involved in its expression, the differential of the matrix be commutative with the matrix itself.—*December* 14*th*, 1895.]

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* "On the Application of the Theory of Matrices to the Discussion of Linear Differential Equations with Constant Coefficients," *Proc. Camb. Phil. Soc.*, VIII., 201–210. See also "On Quaternion Functions, with especial reference to the Discussion of Laplace's Equation," *Proc. Camb. Phil. Soc.*, VII., 151–156.

Determination of the Volumes of certain Species of Tetrahedra without employment of the Method of Limits.

By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received October 22nd, 1895. Read November 14th, 1895.

The object of this communication is to prove the existence of certain species of tetrahedra whose volumes are determinable without employment of the method of limits.

Abstract.

Art. 1. It is shown that symmetrical tetrahedra have equal volumes.

Art. 2. It is shown that tetrahedra which are images of one another with regard to a common face have equal volumes.

Art. 3. It is shown that a tetrahedron in which the straight line bisecting a pair of opposite edges is perpendicular to those edges can be bisected into two superposable tetrahedra, by drawing a plane through either of these edges and the middle point of the other.

Art. 4. If now (see the figure of Art. 4) $ABCD$ be a tetrahedron, and DE, CF be drawn equal and parallel to BA, and if EA, AF, FE, EC be joined, then it follows by Art. 2, if BE be perpendicular to the plane ACD, that the tetrahedra $ABCD$, $ACDE$ are equal.

In like manner, if DF be perpendicular to the plane ACE, the tetrahedra $ADCE$, $AECF$ are equal.

Hence the tetrahedron $ABCD$ is a third of the prism $BDCFEA$.

The two conditions (each of which amounts to two restrictions) (1) that BE is perpendicular to the plane ACD and (2) that DF is perpendicular to the plane ACE result in the expression of the lengths of the six edges of the tetrahedron in terms of two positive quantities a, r as follows:—

$$AC = a\sqrt{9-3r^2},$$

$$AD = BC = 2a,$$

$$AB = BD = DC = a\sqrt{1+r^2}.$$

Tetrahedra of this kind will be called tetrahedra of the first type.

Art. 5. It is shown that the tetrahedron $ABCD$ of the first type can be bisected into two superposable tetrahedra by a plane drawn through BD and the middle point J of AC; or by a plane through AC and the middle point K of BD.

Arts. 6, 7. The last article leads to the second and third types of tetrahedra.

The tetrahedron $BDJA$ of the second type has its edges

$$AD = 2a,$$
$$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$
$$BA = BD = a\sqrt{1+r^2}.$$
$$JB = JD = \tfrac{1}{2}a\sqrt{1+5r^2}.$$

The tetrahedron $ACKB$ of the third type has its edges

$$AC = a\sqrt{9-3r^2},$$
$$BA = 2BK = a\sqrt{1+r^2},$$
$$BC = 2a,$$
$$KA = KC = \tfrac{1}{2}a\sqrt{9+r^2}.$$

Art. 8. It is shown that the three types of tetrahedra are distinct.

Art. 9. In the special case $r^2 = 2$, it is possible by Art. 2 to bisect the tetrahedron $ABCD$ of the first type into two superposable tetrahedra by a plane through AD and the middle point O of BC.

The tetrahedron $ABOD$ is therefore one whose volume is known.

The edges are

$$AO = OD = a\sqrt{2},$$
$$AB = BD = a\sqrt{3},$$
$$AD = 2\,(OB) = 2a.$$

It is not included in any of the three preceding types, but by drawing a plane through BO and the middle point of AD it can be divided into two equal tetrahedra of the first type for which $r^2 = 1$.

Again, all the faces of the tetrahedron of the first type for which $r^2 = 2$ are equal, and all the tetrahedra which have a common vertex at the centre M of the sphere circumscribing that tetrahedron and which stand on the faces of the tetrahedron are equal. Hence the tetrahedron $MABD$ is one whose volume is known.

In this case $MA = MB = MD = \frac{1}{2}a\sqrt{5}$,

$$AB = BD = a\sqrt{3},$$

$$AD = 2a.$$

This does not belong to any of the previous types. But a plane through MB and the middle point of AD divides it into two equal tetrahedra of the third type for which $r^2 = 1$.

Art. 1. *To prove that symmetrical tetrahedra have equal volumes.*

Let $ABCD$ be a tetrahedron, and let the edges BA, CA, DA meeting at the vertex A be produced through A respectively to E, F, G, so that

$$BA = AE, \quad CA = AF, \quad DA = AG;$$

and let EF, FG, GE be joined; then the tetrahedra $ABCD$, $AEFG$ will be shown to be equal.

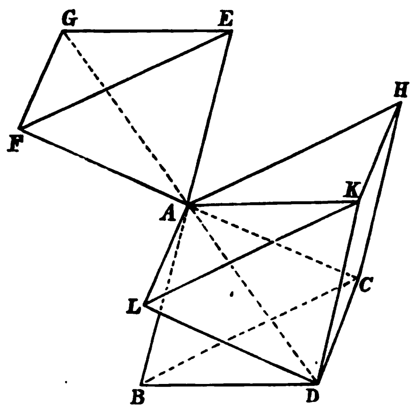

Fig. 1.

Draw CH, DK equal and parallel to BA; and DL equal and parallel to CA.

Join AH, HK, KA, AL, LK.

Then $BCDHKA$ is a prism on base BCD, and its altitude is the perpendicular from A on BCD.

Also $DLKHCA$ is a prism on base ACH, and its altitude is the perpendicular from D on ACH. It is therefore equal to a prism on base ABC, with altitude equal to the perpendicular from D on ABC.

Now the base BCD : the base BCA

$=$ the perpendicular from D on BC : the perpendicular from A on BC

$=$ the perpendicular from D on ABC : the perpendicular from A on BCD.

Hence the prism on base BCD whose altitude is the perpendicular from A on BCD is equal to the prism on base ABC whose altitude is the perpendicular from D on ABC.

Hence the prisms $BCDHKA$, $DLKHCA$ are equal.

The prisms have common the pyramid whose vertex is A, and whose base is the parallelogram $DCKH$.

Taking this away from each prism, it follows that the tetrahedra $ABCD$, $ADKL$ are equal.

Now DA, DL, DK are equal and parallel to AG, AF, AE, respectively, and are drawn in the same directions.

Hence the tetrahedron $ADKL$ can be superposed on the tetrahedron $AEFG$.

Therefore the tetrahedra $ABCD$, $AEFG$ are equal in volume. They are not superposable, but are said to be symmetrically equal.

Art. 2. *To prove that tetrahedra which are images of one another with regard to a common face have equal volumes.*

Let $ABCD$ be a tetrahedron, and let the edge DA be perpendicular to the edges BA, CA, and let DA be produced to E, so that

$$DA = AE,$$

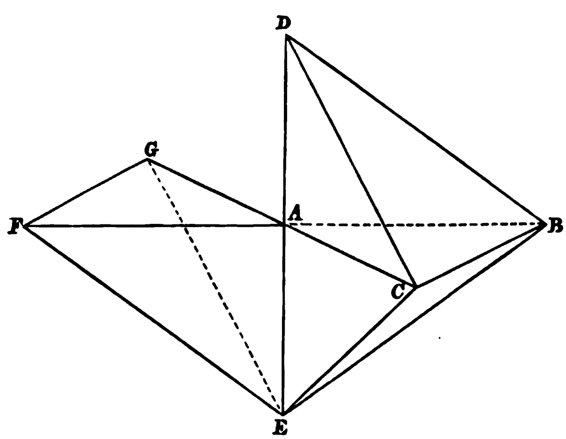

Fig. 2.

and let EB, EC be joined; then the tetrahedra $ABCD$, $ABCE$ will be shown to be equal.

Produce BA, CA to F and G, respectively, so that

$$BA = AF, \quad CA = AG.$$

Then, by the last article, the tetrahedra $ABCD$, $AEFG$ are equal.

But the tetrahedron $AEFG$ can be superposed on the tetrahedron $ABCE$.

Hence the tetrahedra $ABCD$, $ABCE$ are equal.

If, now, ABC be any plane triangle, and P, Q two points on opposite sides of its plane such that PQ is bisected by the plane of ABC at R, then, by the above, the tetrahedra $PRBC$, $PRCA$, $PRAB$ are respectively equal to $QRBC$, $QRCA$, $QRAB$.

Hence the tetrahedra $PABC$, $QABC$ are equal, and these two tetrahedra are images of one another with regard to the common face ABC.

Art. 3. *To prove that a tetrahedron in which the straight line bisecting a pair of opposite edges is perpendicular to those edges can be bisected into two superposable tetrahedra by drawing a plane through either of these edges and the middle point of the other.*

Let AB bisect at right angles CD, EF, a pair of opposite edges of the tetrahedron $CDEF$; then the tetrahedra $CDBF$, $CDBE$ are superposable; as also are the tetrahedra $EFAC$, $EFAD$.

Let the figure revolve through two right angles about the straight line AB, which is supposed fixed.

Then the final positions of the points A, B, C, D, E, F are A, B, D, C, F, E, respectively.

Hence the tetrahedron $CDBF$ can be superposed on the tetrahedron $DCBE$.

Hence each of them is equal to half the tetrahedron $CDEF$.

Also the tetrahedron $EFAC$ can be superposed on the tetrahedron $FEAD$.

Hence each of them is equal to half the tetrahedron $CDEF$.

Art. 4. *To construct a tetrahedron whose volume can be determined without using the proposition that tetrahedra on equal bases and having equal altitudes are equal, except in the special form demonstrated in Art.* 2.

(The demonstration of the general case of the proposition just mentioned has not been effected hitherto without employing the method of limits.)

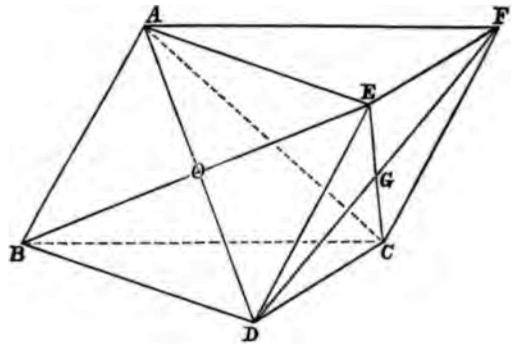

Fig. 3.

Let $ABCD$ be a tetrahedron.

Draw CF, DE equal and parallel to BA.

Join AE, AF, FE, EC.

Let BE and AD meet at O, DF and CE at G.

Then BE, AD bisect each other at O; DF and CE bisect each other at G.

The tetrahedra $ACDB$, $ACDE$ are on the base ACD and have equal altitudes.

The tetrahedra $AECD$, $AECF$ are on the base AEC and have equal altitudes.

Now, if BE be perpendicular to the plane ACD, then the tetrahedra $ADCE$, $ADCB$ are images of one another with regard to their common face ADC. Hence they are equal.

Again, if DF be perpendicular to the plane AEC, it can be shown in the same way that the tetrahedra $ADCE$, $AECF$ are equal.

Hence, if BE be perpendicular to the plane ACD, and if also DF be perpendicular to the plane ACE, then the tetrahedra $ABCD$, $AECD$, $AECF$ are all equal.

Hence the tetrahedron $ABCD$ is equal to one-third of the prism $BCDEFA$.

It remains to find the possible forms of the tetrahedron $ABCD$ which satisfy the conditions that BE, DF are perpendicular to the planes ACD, ACE, respectively.

The length of CE will first be found,

$$CE^2+CB^2 = 2CO^2+2OB^2;$$

therefore $$CE^2+CB^2+AD^2 = 2\,(CO^2+OD^2)+2\,(OB^2+OD^2)$$
$$= AC^2+CD^2+AB^2+BD^2;$$

therefore $CE^2 = (AB^2 + CD^2) + (AC^2 + BD^2) - (BC^2 + AD^2)$ ......(I.).

Next, if BE be perpendicular to the plane ACD, BE is perpendicular to AD and OC.

If BE, *i.e.*, BO, be perpendicular to AD, then

$$AB = BD \qquad \text{.......................(II.)}.$$

If BE be perpendicular to OC, then

$$BC = CE \qquad \text{.......................(III.)}.$$

Hence, by (I.),

$$AD^2 + 2BC^2 = (AB^2 + CD^2) + (AC^2 + BD^2) \qquad \text{.........(IV.)}.$$

In like manner DF will be perpendicular to the plane AEC, if

$$DC = DE \qquad \text{.......................(V.)},$$

and $$CE^2 + 2AD^2 = AE^2 + AC^2 + DE^2 + DC^2 \qquad \text{............(VI.)}.$$

But $$DE = AB,$$

$$CE = CB,$$

$$AE = DB.$$

Hence (V.) and (VI.) become

$$DC = AB \qquad \text{.......................(VII.)},$$

$$BC^2 + 2AD^2 = (DB^2 + AC^2) + (AB^2 + DC^2) \qquad \text{.........(VIII.)}.$$

From (II.) and (VII.),

$$AB = BD = DC \qquad \text{.......................(IX.)}.$$

From (IV.) and (VIII.),

$$BC = AD \qquad \text{.......................(X.)}.$$

From (VIII.), (IX.) and (X.),

$$3BC^2 = 3AB^2 + AC^2 \qquad \text{.......................(XI.)}.$$

Now put $$BC = AD = 2a,$$

and since $$BD + DC > BC,$$

therefore $$2DC > 2a;$$

therefore $$DC > a.$$

Hence it is possible to put

$$DC = a\sqrt{1+r^2};$$

therefore $$AB = BD = DC = a\sqrt{1+r^2}.$$

Then, by (XI.), $AC = a\sqrt{9-3r^2}$.

Hence the edges of the tetrahedron $ABCD$ are given by

$$AB = BD = DC = a\sqrt{1+r^2},$$

$$BC = AD = 2a,$$

$$AC = a\sqrt{9-3r^2}.$$

It is obvious that $0 < r^2 < 3$, and that $BO = ar$.

Any tetrahedron of this kind may be called a tetrahedron of the first type.

It may be shown that the dihedral angles whose edges are BC, AD are right angles.*

The dihedral angle whose edge is CA is 60°.

The cosines of the dihedral angles whose edges are AB, CD are each $\frac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is BD is $\frac{1}{2}(r^2-1)$.

The volume of the tetrahedron will now be found.

It is one-third of the base multiplied by the height, and any side may be taken as base.

Take ABD as base. This face is perpendicular to the face ACD by construction.

Hence volume of $ABCD$

$$= \tfrac{1}{3}\,(\text{area of } ABD) \times (\text{perpendicular from } C \text{ on } AD).$$

Now $$\cos CAD = \frac{9-3r^2+4-(1+r^2)}{4\sqrt{9-3r^2}} = \frac{\sqrt{3-r^2}}{\sqrt{3}}$$

therefore $$\sin CAD = \frac{r}{\sqrt{3}};$$

therefore perpendicular from C on $AD = ar\sqrt{3-r}$.

Since ABD is an isosceles triangle,

the perpendicular from B on $AD = ar$;

therefore area of $ABD = a^2 r$.

---

* In any tetrahedron $ABCD$ the cosine of the dihedral angle between the planes BCA and BCD is

$$\frac{(AC^2-AB^2)(DB^2-DC^2)+(BC)^2\left[AC^2+AB^2+BD^2+DC^2-BC^2-2AD^2\right]}{16\,(\text{area of } ABC)(\text{area of } DBC)}.$$

Hence the volume of *ABCD*

$$= \tfrac{1}{3}a^3r^2\sqrt{3-r^2}.$$

It may be noticed that the three tetrahedra *ABCD*, *ACDE*, *ACEF* have their edges all equal in this particular case. It is possible to superpose *ABCD* on *ACEF* by rotation round *AC* through 120°. But *ACDE* is symmetrically equal to either of the other two tetrahedra.

It will now be shown that the tetrahedron *ABCD* of the first type can be dissected into eight equal tetrahedra of which six are superposable; the other two are also superposable, but they are symmetrically equal with regard to any one of the six.

As in the third proposition of Euclid's Twelfth Book (see Fig. 4),

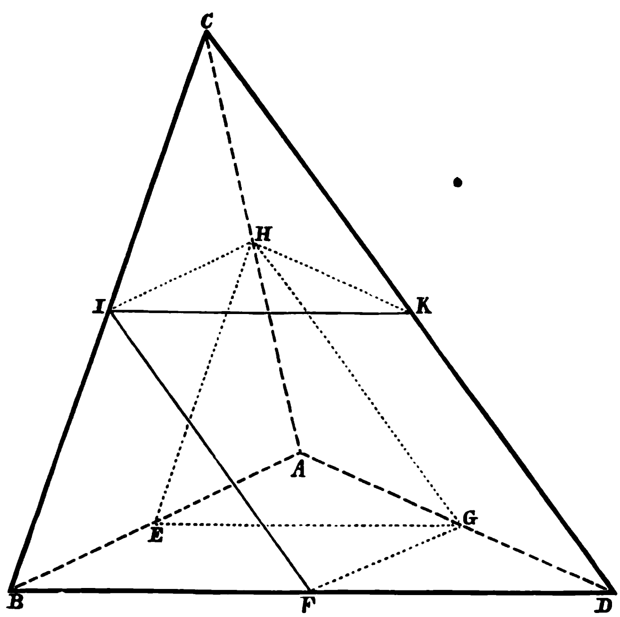

FIG. 4.

take *E*, *F*, *G*, *H*, *I*, *K* the middle points of *AB*, *BD*, *DA*, *AC*, *BC*, *DC*, respectively. Then the tetrahedron *ABCD* breaks up into the two tetrahedra *CIHK*, *AHEG*, and the two prisms *FDGHIK*, *BFIHGE*.

The tetrahedra *CIHK*, *AHEG* are superposable to one another and similar to *ABCD*.

On examining Fig. 3 it will be seen that the prism *BDCFEA* of that figure is drawn on the isosceles base *BDC* with its edges parallel to *BA*, the side of the tetrahedron equal to either of the equal sides of the isosceles base.

Now, taking in Fig. 4

$$AB = BD = DC = a\sqrt{1+r^2},$$

$$AD = BC = 2a,$$

$$AC = a\sqrt{9-3r^2},$$

it is at once seen that the prism *FDGHIK* stands on the isosceles triangle *FGD* as base, and has its edges parallel to *DK*, which is equal to the equal sides of the isosceles base *FGD*. Hence this prism breaks up into the tetrahedron *FDGK*, one superposable to it, and one symmetrically equal to it.

In like manner, the tetrahedron *BFIHGE* stands on the isosceles triangle *BFI* as base, and has its edges parallel to *BE*, which is equal to the equal sides of the isosceles base *BFI*. Hence this prism is decomposable into the tetrahedron *BIEF*, one superposable to it, and one symmetrically equal to it.

The tetrahedra *FDGK*, *BIEF*, *CIHK*, *AHEG* are superposable to one another, and similar to the whole tetrahedron.

Hence the whole tetrahedron *ABCD* is decomposable into six tetrahedra equal and superposable to *AHEG*, and two tetrahedra symmetrically equal to *AHEG*. All these tetrahedra are equal. Their linear dimensions are half of those of *ABCD*.

Art. 5. *To derive other types of tetrahedra from the first type.*

The tetrahedron *ABCD* is symmetrical with regard to the line joining the middle points of *AC*, *BD*.

Hence this joining line is perpendicular to *AC*, *BD*.

It will be useful to prove this in another way.

Let *J*, *K* be the middle points of *AC*, *BD*, respectively.

Then
$$2JD^2+2AJ^2 = AD^2+DC^2,$$

$$2JB^2+2AJ^2 = AB^2+BC^2.$$

Now
$$AD = BC = 2a,$$

$$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$

$$AB = DC = a\sqrt{1+r^2};$$

therefore
$$JB = JD = \frac{a}{2}\sqrt{1+5r^2}.$$

Also $$KB = KD,$$ $$JK = JK;$$

therefore JK is perpendicular to BD.

Again $$2KA^2+2KB^2 = AB^2+AD^2,$$ $$2KC^2+2KB^2 = BC^2+CD^2,$$ $$AD = BC = 2a,$$ $$KB = \tfrac{1}{2}a\sqrt{1+r^2},$$ $$AB = CD = a\sqrt{1+r^2};$$

therefore $$KA = KC = \frac{a}{2}\sqrt{9+r^2}.$$

Also $$JA = JC,$$ $$KJ = KJ;$$

therefore KJ is perpendicular to AC.

Hence KJ is perpendicular to the opposite edges AC and BD and bisects them both.

Hence, applying the proposition of Art. 3, it follows that the tetrahedra $ACKB$, $ACKD$, $BDJA$, $BDJC$ are each half the tetrahedron $ABCD$.

Hence the tetrahedra $BDJA$, $ACKB$ (which are respectively superposable on the other two) will give the two new types.

Art. 6. *The second type of tetrahedron.*

Consider first the tetrahedron $BDJA$, which will be referred to as the tetrahedron of the second type.

The sides are $$AD = 2a,$$ $$AJ = \tfrac{1}{2}a\sqrt{9-3r^2},$$ $$BA = BD = a\sqrt{1+r^2},$$ $$JB = JD = \tfrac{1}{2}a\sqrt{1+5r^2}.$$

The dihedral angle whose edge is AD is a right angle.

The dihedral angle whose edge is AJ is 60°.

The cosine of the dihedral angle whose edge is AB is $\frac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is DB is $\frac{1}{2}\sqrt{1+r^2}$.

The cosine of the dihedral angle whose edge is JD is $\dfrac{\sqrt{3-r^2}}{2\sqrt{1+r^2}}$.

The cosine of the dihedral angle whose edge is JB is $-\dfrac{\sqrt{3-r^2}}{2\sqrt{1+r^2}}$.

[The tetrahedron of the second type may be derived from that of the first type in other ways, for example:—

If O be the middle point of AD, and CO be produced to H, so that $CO = OH$, then $BCDH$ is a tetrahedron of the second type.

Its sides are

$$HC = 2a\sqrt{4-r^2},$$
$$HD = a\sqrt{9-3r^2},$$
$$BC = BH = 2a,$$
$$BD = DC = a\sqrt{1+r^2}.$$

Putting

$$a = \tfrac{1}{2}A\sqrt{1+R^2},$$
$$r^2 = \frac{4R^2}{1+R^2},$$
$$HC = 2A,$$
$$HD = \tfrac{1}{2}A\sqrt{9-3R^2}$$
$$BC = BH = A\sqrt{1+R^2},$$
$$BD = DC = \tfrac{1}{2}A\sqrt{1+5R^2},$$

showing that the tetrahedron belongs to the second type.]

Art. 7. *The third type of tetrahedron.*

Consider next the tetrahedron $ACKB$, which will be referred to as the tetrahedron of the third type.

Its sides are

$$AC = a\sqrt{9-3r^2},$$
$$BK = \tfrac{1}{2}a\sqrt{1+r^2},$$
$$AB = a\sqrt{1+r^2},$$
$$BC = 2a,$$
$$KA = KC = \tfrac{1}{2}a\sqrt{9+r^2}.$$

The dihedral angle whose edge is BC is a right angle.

The dihedral angle whose edge is AC is 30°.

The cosine of the dihedral angle whose edge is AB is $\tfrac{1}{2}\sqrt{3-r^2}$.

The cosine of the dihedral angle whose edge is BK is $\tfrac{1}{2}(r^2-1)$.

The cosine of the dihedral angle whose edge is KC is $\dfrac{\sqrt{3-r^2}}{2\sqrt{3}}$.

The cosine of the dihedral angle whose edge is KA is $-\dfrac{\sqrt{3-r^2}}{2\sqrt{3}}$.

Art. 8. *Comparison of the three types of tetrahedra.*

The tetrahedron of the first type has three equal sides, a character not possessed *in general* by either of the tetrahedra of the second and third types.

Hence the tetrahedron of the first type is distinct from the tetrahedra of the second and third types.

The tetrahedron of the second type has two isosceles faces, whilst that of the third type has only one isosceles face.

Hence the three types are all distinct.

Art. 9. *Special tetrahedra not included in any of the previously mentioned types.*

(A) A very simple case of the tetrahedron of the first type is when $r^2 = 2$.

Then $$AB = BD = DC = CA = a\sqrt{3},$$
$$AD = BC = 2a.$$

Let O be the middle point of BC.

Then BC is perpendicular to the plane DOA, and O is the middle point of BC.

Hence the tetrahedra $ABDO$, $ACDO$ are equal by Art. 2.

Hence the tetrahedron $ABDO$ is one whose volume can be found without using the method of limits.

Its sides are $$AO = OD = a\sqrt{2},$$
$$AB = BD = a\sqrt{3},$$
$$AD = 2a,$$
$$OB = a.$$

This tetrahedron does not belong to any of the preceding types.

But, if it be bisected by drawing a plane through BO and the middle point E of AD, then the tetrahedron $ABEO$ has the sides
$$BO = OE = EA = a,$$
$$AO = BE = a\sqrt{2},$$
$$AB = a\sqrt{3},$$
and this belongs to the first type, r^2 being equal to 1.

(B) All the faces of the tetrahedron of the first type when $r^2 = 2$ are equal. Hence from the symmetry of the figure it follows that the four tetrahedra which have a common vertex at the centre M of the circumscribing sphere are all equal, *i.e.*, superposable or symmetrically equal.

The radius of the circle circumscribing ABD is

$$\frac{AB \,.\, BD \,.\, DA}{4 \text{ area of } ABD} = \tfrac{3}{4} a \sqrt{2}.$$

Hence the distance of this centre from the middle point P of AD is $\frac{1}{4} a \sqrt{2}$.

Hence the distance of the centre of the circumscribing sphere from P is $\frac{1}{2} a$.

Hence the radius of the circumscribing sphere is $\frac{1}{2} a \sqrt{5}$.

The tetrahedra whose bases are ABD, ACD, and which have a common vertex at the centre of the circumscribing sphere, though not superposable, are symmetrically equal.

Hence the tetrahedron $MABD$ is one whose volume is known, where

$$MA = MB = MD = \tfrac{1}{2} a \sqrt{5},$$

$$AB = BD = a \sqrt{3},$$

$$AD = 2a.$$

But, if a plane be drawn through MB and P the middle point of AD, then $MABD$ is divided into two equal tetrahedra. Consider $MPDB$.

$$MP = \tfrac{1}{2} a,$$

$$PD = a,$$

$$MB = MD = \tfrac{1}{2} a \sqrt{5},$$

$$PB = a \sqrt{2},$$

$$BD = a \sqrt{3}.$$

Putting

$$A \sqrt{9 - 3R^2} = a \sqrt{3},$$

$$A \sqrt{1 + R^2} = a,$$

$$2A = a \sqrt{2},$$

$$A \sqrt{9 + R^2} = a \sqrt{5},$$

all which are satisfied by putting

$$A = a/\sqrt{2}$$

and $$R^2 = 1,$$

showing that it is a tetrahedron of the third type.

Supplementary Remark.—If $ABCD$ be a tetrahedron of the first type, and if P be the middle point of BC and O the middle point of AD, then the tetrahedron $APOC$ belongs to the first type.

In this case $$PD = OB = ar,$$

$$OC = PA = a\sqrt{4-r^2},$$

whence $$PO = a.$$

The sides of $APOC$ are therefore

$$AO = OP = PC = a,$$

$$AP = OC = a\sqrt{4-r^2},$$

$$AC = a\sqrt{9-3r^2},$$

or, putting $$a = A\sqrt{1+R^2},$$

$$r^2 = \frac{4R^2}{1+R^2},$$

$$a\sqrt{4-r^2} \text{ becomes } 2A,$$

and $$a\sqrt{9-3r^2} \text{ becomes } A\sqrt{9-3R^2},$$

showing that it belongs to the first type.

Note by Lieut.-Col. Allan Cunningham, R.E.

Lt.-Col. Allan Cunningham, R.E., announced the discovery of a new criterion to determine when 2 is a 16-ic residue of a prime (p), viz.:

p must be of the two quadratic forms

$$p = (8\alpha \pm 1)^2 + (16\beta')^2 = (8\gamma \pm 1)^2 + 2(4\delta)^2,$$

in order that p may be of the required form $p = 16m+1$, and also have 2 for an 8-ic residue.

The new criterion is

$$2^{\frac{1}{16}(p-1)} \equiv (-1)^{\beta'+\delta} \pmod{p},$$

so that

$$2^{\frac{1}{16}(p-1)} = -1 \quad \text{or} \quad +1,$$

according as $(\beta'+\delta)$ is *odd* or *even*. Proof was promised on a future occasion.

2. He also drew attention to *two* statements in the works of the late M. Ed. Lucas relative to two of Mersenne's numbers $(2^{67}-1)$ and $(2^{127}-1)$. In § 14 of his *Recherches sur plusieurs ouvrages de Léonard de Pise*, Rome, 1877, he says, "J'ai ainsi vérifié, mais une seule fois, je l'avoue, que le nombre $A = 2^{127}-1$ est un nombre premier." Now Lucas's process for the verification of a prime has the rare advantage of being a *direct* process, the prime character being proved by the *success* of the procedure; such a process is far less likely to be vitiated by arithmetical mistakes than an *indirect* process (wherein the prime character is proved by the *failure* of a long procedure). It is therefore highly probable that $(2^{127}-1)$ is really a prime.

Again, on p. 11 of his pamphlet *Sur la Théorie des Nombres premiers*, Turin, 1876, he states "... suivant une assertion du *P.* Mersenne, les nombres $2^{67}-1$, $2^{127}-1$, $2^{257}-1$ seraient premiers. Je ne pense pas qu'il en soit ainsi du premier de ces nombres, que j'ai déjà essayé par ma méthode."

This must throw great doubt upon Mersenne's assertion that $2^{67}-1$ is a prime. As regards proof of a number being a composite, Lucas's process is *indirect* (the composite character being proved by the *failure* of the procedure), and is more liable to be vitiated by arithmetical error than a *direct* process would be.

This note may be looked on as an addendum to Mr. W. W. Rouse Ball's article on "Mersenne's Numbers" in the *Messenger of Mathematics*, Vol. XXI. (1891–92), p. 35, wherein these two numbers are marked as still *unverified.*

On the Representation of a Number as a Sum of Squares. By G. B. Mathews.

Received November 4th, 1895. Read November 14th, 1895.

Let us adopt the convention that two integral solutions $(\xi_1, \xi_2, \dots \xi_k)$, $(\xi'_1, \xi'_2, \dots \xi'_k)$ of the diophantine equation

$$x_1^2+x_2^2+\dots+x_k^2=n$$

are to be considered distinct, unless

$$\xi_i=\xi'_i \qquad [i=1, 2, \dots k].$$

Then the number of solutions is equal to the coefficient of q^n in the expansion of

$$\theta_1^k=(\sum_{-\infty}^{+\infty} q^{s^2})^k$$

$$=(1+2q+2q^4+2q^9+\dots)^k$$

in ascending powers of q.

It is known that

$$\theta_1=\frac{1+q}{1-q}\,\frac{1-q^2}{1+q^2}\,\frac{1+q^3}{1-q^3}\dots$$

(Jacobi, *Werke* I., p. 238); therefore, taking logarithms and differentiating,

$$\frac{1}{\theta_1}\frac{d\theta_1}{dq}=\frac{2}{1-q^2}-\frac{4q}{1-q^4}+\frac{6q^2}{1-q^6}-\dots$$

$$=2(1+q^2+q^4+\dots)$$

$$-4q(1+q^4+q^8+\dots)$$

$$+6q(1+q^6+q^{12}+\dots)$$

$$+\dots$$

$$=2\Sigma(-)^{s-1}sq^{(2r+1)s-1} \qquad \begin{bmatrix} r=0, 1, 2, \dots \\ s=1, 2, 3, \dots \end{bmatrix}.$$

If we put $(2r+1)s=n,$

n assumes all the values 1, 2, 3, ... ; and

$$\mu=2r+1$$

will denote any *odd* divisor of n.

Therefore, arranging the expansion according to powers of q, we have

$$\frac{1}{\theta_1}\,\frac{d\theta_1}{dq} = -\sum_1^{\infty} \psi(n)\, q^{n-1},$$

where
$$\psi(n) = 2\sum_{\mu} (-)^{n/\mu}\,\frac{n}{\mu},$$

the summation with regard to μ relating to all odd divisors of n.

The arithmetical function $\psi(n)$ may be expressed in a more convenient form. Thus, if n is odd, we may put

$$n = \mu\mu',$$

and then
$$\psi(n) = -2\Sigma\mu' = -2\zeta(n),$$

where $\zeta(n)$ denotes the sum of the divisors of n.

If n is even, let
$$n = 2^{\alpha} m,$$

where m is odd; then

$$\psi(n) = 2^{\alpha+1}\sum \frac{m}{\mu} = 2^{\alpha+1}\,\zeta(m).$$

Both results are included in the formula

$$\psi(n) = -(-)^{n/m}\,\{\zeta(n)+\zeta(m)\},$$

where m is the largest odd divisor of n.

It is easy to calculate the values of $\psi(n)$ from a table of divisor-sums; thus the first twenty values are as follows:—

n	$\psi(n)$	n	$\psi(n)$
1	-2	11	-24
2	$+4$	12	$+32$
3	-8	13	-28
4	$+8$	14	$+32$
5	-12	15	-48
6	$+16$	16	$+32$
7	-16	17	-36
8	$+16$	18	$+52$
9	-26	19	-40
10	$+24$	20	$+48$

Now, if we assume

$$\theta_1^k = 1 + c_1 q + c_2 q^2 + \ldots + c_n q^n + \ldots,$$

it follows by logarithmic differentiation that

$$\frac{\sum_1^\infty n c_n q^{n-1}}{1 + \sum_1^\infty c_n q_n} = \frac{k}{\theta_1}\,\frac{d\theta_1}{dq} = -k \sum_1^\infty \psi(n)\, q^{n-1};$$

hence, multiplying up and equating coefficients,

$$c_1 + k\psi(1) = 0,$$

$$2c_2 + kc_1\psi(1) + k\psi(2) = 0,$$

$$3c_3 + kc_2\psi(1) + kc_1\psi(2) + k\psi(3) = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$nc_n + kc_{n-1}\psi(1) + kc_{n-2}\psi(2) + \ldots + k\psi(n) = 0.$$

These equations enable us to find c_1, c_2, &c., successively, or again to write down c_n in the form of a determinant.

Thus $c_1 = -k\psi(1)$,

$$c_2 = -\tfrac{1}{2}\begin{vmatrix} k\psi(2), & k\psi(1) \\ k\psi(1), & 1 \end{vmatrix},$$

$$c_3 = \frac{1}{3!}\begin{vmatrix} k\psi(3), & k\psi(2), & k\psi(1) \\ k\psi(2), & k\psi(1), & 2 \\ k\psi(1), & 1, & 0 \end{vmatrix},$$

$$c_4 = \frac{1}{4!}\begin{vmatrix} k\psi(4), & k\psi(3), & k\psi(2), & k\psi(1) \\ k\psi(3), & k\psi(2), & k\psi(1), & 3 \\ k\psi(2), & k\psi(1), & 2, & 0 \\ k\psi(1), & 1, & 0, & 0 \end{vmatrix},$$

and so on; the general formula being

$$c_n = \frac{(-)^{\frac{1}{2}n(n-1)}}{n!} \begin{vmatrix} k\psi(n), & k\psi(n-1), & \dots & k\psi(2), & k\psi(1) \\ k\psi(n-1), & k\psi(n-2), & \dots & k\psi(1), & n-1 \\ k\psi(n-2), & k\psi(n-3), & \dots & n-2, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ k\psi(2), & k\psi(1), & \dots & 0, & 0 \\ k\psi(1), & 1, & \dots & 0, & 0 \end{vmatrix}.$$

This expression is therefore, in a sense, an explicit arithmetical formula for the number of representations of a given positive integer n as the sum of k integral squares. It is not suggested, of course, that this is the best practical way of calculating the number of representations; but it is noticeable that such a simple analysis should lead to a definite arithmetical formula which applies uniformly to all possible cases.

Very curious theorems may be obtained by comparing this formula with those which have already been discovered for small values of k. The concordance of the results in particular cases may be tested by means of the table of $\psi(n)$ given above.

Thus let $k = 2$, and, for greater clearness, write $F(n, 2)$ for the number of integral solutions of $x^2+y^2 = n$.

Then, if n is not a multiple of 4, one way of expressing $F(n, 2)$ is

$$F(n, 2) = 4(d_1-d_3),$$

where d_1, d_3 denote the number of divisors of n which are of the form $4m+1$, $4m+3$ respectively, and this must agree with the value of c_n when $k = 2$.

Thus, to take the simplest case,

$$c_2 = -\tfrac{1}{2}\begin{vmatrix} 2\psi(2), & 2\psi(1) \\ 2\psi(1), & 1 \end{vmatrix} = -\begin{vmatrix} 4, & -4 \\ -2, & 1 \end{vmatrix}$$

$$= 4,$$

which is right, the solutions being obtained from $(\pm 1, \pm 1)$ by variations of sign.

Similarly, $$F(3,2) = -\frac{1}{3!}\begin{vmatrix} 2\psi(3), & 2\psi(2), & 2\psi(1) \\ 2\psi(2), & 2\psi(1), & 2 \\ 2\psi(1), & 1, & 0 \end{vmatrix}$$

$$= -\tfrac{1}{3}\begin{vmatrix} -8, & 8, & -4 \\ 4, & -4, & 2 \\ -2, & 1, & 0 \end{vmatrix} = 0,$$

and in general $F(n, 2) = 0$ if n is divisible by an odd power of a prime $p = 4m+3$, the quotient being prime to p.

Other results of the same kind may be obtained by comparison with Jacobi's formula for $k = 4$, and those of Eisenstein, Smith, and Hermite for $k = 5$. The first of these is particularly interesting because $F(n, 4)$ can be very simply expressed as a divisor-sum, and therefore as a ψ-function. The other case gives a very curious instance of the identity of two arithmetical expressions constructed by entirely different algorithms; indeed, it seems very difficult to prove their identity by the direct transformation of one into the other.

One property of $\psi(n)$ seems to deserve attention. Since

$$\frac{d\theta_1}{dq} = 2\sum_1^\infty n^2 q^{n^2-1},$$

and, on the other hand,

$$\frac{d\theta_1}{dq} = -\theta_1 \sum_1^\infty \psi(n)\, q^{n-1}$$

$$= -\sum_{-\infty}^{+\infty} q^{k^2} \times \sum_1^\infty \psi(n)\, q^{n-1}$$

$$= -\sum_1^\infty \sum_{\pm k} \{\psi(n-k^2)\}\, q^{n-1} \qquad [k^2 < n^2],$$

it follows that $$\sum_{\pm k} \psi(n-k^2) = 0 \quad \text{or} \quad -2n,$$

according as n is not or is a square.

Thus $$\psi(8)+2\psi(7)+2\psi(4)=0,$$

while $$\psi(9)+2\psi(8)+2\psi(5)=-18.$$

It is easily seen that this property defines $\psi(n)$ in much the same way as the class-number for a given negative determinant is defined by Kronecker's well-known formulæ.

Evaluation of a certain Dialytic Determinant. *By* W. W. Taylor.

Received and Read November 14th, 1895.

1. In a paper "On the Existence of a Root of a Rational Integral Equation," read before this Society on March 8th, 1894,* by Professor E. B. Elliott, taking

$$F(x, y) \equiv \sum_{r=0}^{r=n} a_r x^{n-r} y^r,$$

he remarks with regard to the dialytic determinant Δ of the two expressions $F(\rho x, y)$, $F(x, \rho y)$, *i.e.*,

$$\Delta \equiv \left| \begin{array}{cccccccc} a_0\rho^n, & a_1\rho^{n-1}, & \dots & a_{n-1}\rho, & a_n, & & \dots & \\ & a_0\rho^n, & \dots & a_{n-2}\rho^2, & a_{n-1}\rho, & a_n, & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & a_0\rho^n, & a_1\rho^{n-1}, & a_2\rho^{n-2}, & \dots & a_n \\ a_n, & a_{n-1}\rho, & \dots & a_1\rho^{n-1}, & a_0\rho^n, & & \dots & \\ & a_n, & \dots & a_2\rho^{n-2}, & a_1\rho^{n-1}, & a_0\rho^n, & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & \dots & a_n, & a_{n-1}\rho, & a_{n-2}\rho^2, & \dots & a_0\rho_n \end{array} \right|,$$

* *Proc. Lond. Math. Soc.*, Vol. xxv., p. 176.

"It is unfortunate for the simplicity of the argument of this paper that the property of such a determinant as Δ that, after division by its obvious factors $F(\rho, 1)$ and $F(-\rho, 1)$, it leaves a perfect square as quotient, is one which direct algebraical methods have, as far as I know, not yet supplied" (p. 184).

It is the object of the present paper to supply such a proof.

For the sake of brevity I shall write $a, b, c, \ldots$ for $a_0\rho^n$, $a_1\rho^{n-1}$, $a_2\rho^{n-2}, \ldots$; and for the sake of clearness, while describing my method in general terms, I shall illustrate it by the case where $n = 6$.

Then $$F(\rho, 1) = a+b+c+d+e+f+g,$$

$$F(\rho, -1) = a-b+c-d+e-f+g,$$

$$\Delta = \begin{vmatrix} a & b & c & d & e & f & g & . & . & . & . & . \\ . & a & b & c & d & e & f & g & . & . & . & . \\ . & . & a & b & c & d & e & f & g & . & . & . \\ . & . & . & a & b & c & d & e & f & g & . & . \\ . & . & . & . & a & b & c & d & e & f & g & . \\ . & . & . & . & . & a & b & c & d & e & f & g \\ g & f & e & d & c & b & a & . & . & . & . & . \\ . & g & f & e & d & c & b & a & . & . & . & . \\ . & . & g & f & e & d & c & b & a & . & . & . \\ . & . & . & g & f & e & d & c & b & a & . & . \\ . & . & . & . & g & f & e & d & c & b & a & . \\ . & . & . & . & . & g & f & e & d & c & b & a \end{vmatrix}.$$

2. In the determinant Δ add all columns together for a new n^{th} column, and to the $(n+1)^{\text{th}}$ column add if it is an odd column all the other odd columns, and if it is an even column all the other even columns; double it as so increased and subtract from it the new n^{th} column.

The n^{th} column thus becomes $F(\rho, 1)$ throughout, and the $(n+1)^{\text{th}}$ column consists of alternate terms $+F(\rho, -1)$ and $-F(\rho, -1)$, the first being $(-1)^n F(\rho, -1)$, and by these operations we have doubled the determinant. Divide by $F(\rho, 1)$, $F(\rho, -1)$, and we obtain the result

$$2\Delta/\{F(\rho, 1)\,F(\rho, -1)\}$$

$$= \left|\begin{array}{cccccccccccc}
a & b & c & d & e & 1 & 1 & . & . & . & . & . \\
. & a & b & c & d & 1 & -1 & g & . & . & . & . \\
. & . & a & b & c & 1 & 1 & f & g & . & . & . \\
. & . & . & a & b & 1 & -1 & e & f & g & . & . \\
. & . & . & . & a & 1 & 1 & d & e & f & g & . \\
. & . & . & . & . & 1 & -1 & c & d & e & f & g \\
g & f & e & d & c & 1 & 1 & . & . & . & . & . \\
. & g & f & e & d & 1 & -1 & a & . & . & . & . \\
. & . & g & f & e & 1 & 1 & b & a & . & . & . \\
. & . & . & g & f & 1 & -1 & c & b & a & . & . \\
. & . & . & . & g & 1 & 1 & d & c & b & a & . \\
. & . & . & . & . & 1 & -1 & e & d & c & b & a
\end{array}\right|.$$

3. Add the n^{th} column to the $(n+1)^{\text{th}}$ for a new $(n+1)^{\text{th}}$ column, and divide it by 2, subtract the new $(n+1)^{\text{th}}$ column from the n^{th}, and we see that

$$\Delta/F(\rho, 1)\,F(\rho, -1)$$

$$= \left|\begin{array}{cccccccccccc}
a & b & c & d & e & 0 & 1 & . & . & . & . & . \\
. & a & b & c & d & 1 & 0 & g & . & . & . & . \\
. & . & a & b & c & 0 & 1 & f & g & . & . & . \\
. & . & . & a & b & 1 & 0 & e & f & g & . & . \\
. & . & . & . & a & 0 & 1 & d & e & f & g & . \\
. & . & . & . & . & 1 & 0 & c & d & e & f & g \\
g & f & e & d & c & 0 & 1 & . & . & . & . & . \\
. & g & f & e & d & 1 & 0 & a & . & . & . & . \\
. & . & g & f & e & 0 & 1 & b & a & . & . & . \\
. & . & . & g & f & 1 & 0 & c & b & a & . & . \\
. & . & . & . & g & 0 & 1 & d & c & b & a & . \\
. & . & . & . & . & 1 & 0 & e & d & c & b & a
\end{array}\right|$$

4.* From the first row subtract the last but one for a new first row, and from the last row subtract the second for a new last row.

From the second row subtract the last but two for a new second row, and from the last but one subtract the third for a new last but one.

... ...

From the r^{th} row subtract the $(2n-r)^{\text{th}}$ for a new r^{th} row, and from the $(2n-r+1)^{\text{th}}$ row subtract the $(r+1)^{\text{th}}$ for a new $(2n-r+1)^{\text{th}}$ row.

Leave the n^{th} and $(n+1)^{\text{th}}$ rows unchanged.

Thus $\Delta / F(\rho, 1)\, F(\rho, -1)$

$$
= \begin{vmatrix}
a & b & c & d & e-g & 0 & 0 & -d & -c & -b & -a & . \\
. & a & b & c-g & d-f & 0 & 0 & g-c & -b & -a & . & . \\
. & . & a-g & b-f & c-e & 0 & 0 & f-b & g-a & . & . & . \\
. & -g & -f & a-e & b-d & 0 & 0 & e-a & f & g & . & . \\
-g & -f & -e & -d & a-c & 0 & 0 & d & e & f & g & . \\
. & . & . & . & . & 1 & 0 & c & d & e & f & g \\
g & f & e & d & c & 0 & 1 & & & & & \\
. & g & f & e & d & 0 & 0 & a-c & -d & -e & -f & -g \\
. & . & g & f & e-a & 0 & 0 & b-d & a-e & -f & -g & . \\
. & . & . & g-a & f-b & 0 & 0 & c-e & b-f & a-g & . & . \\
. & . & -a & -b & g-c & 0 & 0 & d-f & c-g & b & a & . \\
. & -a & -b & -c & -d & 0 & 0 & e-g & d & c & b & a
\end{vmatrix}
$$

In each of the first $(n-1)$ rows the sum of the constituents in the r^{th} and $(2n-r)^{\text{th}}$ columns is zero for all values of r from 1 to $(n-2)$; and in each of the last $n-1$ rows the sum of the constituents in the $(r+1)^{\text{th}}$ and $(2n-r+1)^{\text{th}}$ columns is zero for all values of r from 1 to $n-2$.

5. As there are single terms in the n^{th} and $(n+1)^{\text{th}}$ columns, and each of these unity, the determinant can now be reduced to one with $2n-2$ columns.

---

* The same final result is obtained by performing upon Δ in its original form the processes of §§ 4 and 6, with some slight modifications due to the fact that the n^{th} and $(n+1)^{\text{th}}$ columns do not in that case degenerate (as in § 5), but give finally in the n^{th} and $(n+1)^{\text{th}}$ rows the terms

$$
\begin{array}{ll}
a+c+e\,..., & b+d+f\,..., \\
b+d+f\,...; & a+c+e\,...,
\end{array}
$$

which on the expansion of Δ give the factors $F(\rho, 1)$, $F(\rho, -1)$.

$$\Delta / F(\rho, 1)\, F(\rho, -1)$$

$$= \begin{vmatrix} a & b & c & d & e-g & -d & -c & -b & -a & . \\ . & a & b & c-g & d-f & g-c & -b & -a & . & . \\ . & . & a-g & b-f & c-e & f-b & g-a & . & . & . \\ . & -g & -f & a-e & b-d & e-a & f & g & . & . \\ -g & -f & -e & -d & a-c & d & e & f & g & . \\ . & g & f & e & d & a-c & -d & -e & -f & -g \\ . & . & g & f & e-a & b-d & a-e & -f & -g & . \\ . & . & . & g-a & f-b & c-e & b-f & a-g & . & . \\ . & . & -a & -b & g-c & d-f & c-g & b & a & . \\ . & -a & -b & -c & -d & e-g & d & c & b & a \end{vmatrix}$$

6. Now add the first column to the last but one for a new last but one; and add the last column to the second for a new second.

Add the new second thus formed to the last but two for a new last but two; and add the new last but one to the third for a new third.

... ...

Add the new r^{th} column to the $(2n-2-r)^{\text{th}}$ for a new $(2n-2-r)^{\text{th}}$; and add the new $(2n-1-r)^{\text{th}}$ to the $(r+1)^{\text{th}}$ for a new $(r+1)^{\text{th}}$.

Continue this process for all values of r up to $n-2$.

This process has not altered the constituents of the determinant lying in the first $n-1$ columns and the first $n-1$ rows, or those lying in the last $n-1$ columns and the last $n-1$ rows, but it has cancelled all the rest.

Moreover, the constituent that lies in the r^{th} column and the s^{th} row is the same as the constituent that lies in the $(2n-1-r)^{\text{th}}$ column and the $(2n-s-1)^{\text{th}}$ row.

Therefore the determinant is the square of the determinant whose constituents are the terms lying in the first $(n-1)$ columns and first $(n-1)$ rows; *i.e.*,

$$\Delta / F(\rho, 1)\, F(\rho, -1) = \begin{vmatrix} a & b & c & d & e-g \\ . & a & b & c-g & d-f \\ . & . & a-g & b-f & c-e \\ . & -g & -f & a-e & b-d \\ -g & -f & -e & -d & a-c \end{vmatrix}^2 = \delta^2.$$

7. In the general case this determinant can be written thus:

$$\delta = \left|\begin{array}{lllllll} a_0\rho^n, & a_1\rho^{n-1}, & a_2\rho^{n-2}, & \dots & a_{n-4}\rho^4, & a_{n-3}\rho^3, & a_{n-2}\rho^2 - a_n \\ . & a_0\rho^n, & a_1\rho^{n-1}, & \dots & a_{n-5}\rho^5, & a_{n-4}\rho^4 - a_n, & a_{n-3}\rho^3 - a_{n-1}\rho \\ . & . & a_0\rho^n, & \dots & a_{n-6}\rho^6 - a_n, & a_{n-5}\rho^5 - a_{n-1}\rho, & a_{n-4}\rho^4 - a_{n-2}\rho^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & -a_n, & \dots & a_0\rho^n - a_6\rho^{n-6}, & a_1\rho^{n-1} - a_5\rho^{n-5}, & a_2\rho^{n-2} - a_4\rho^n \\ . & -a_n, & -a_{n-1}\rho, & \dots & -a_5\rho^{n-5}, & a_0\rho^n - a_4\rho^{n-4}, & a_1\rho^{n-1} - a_3\rho^{n-3} \\ -a_n, & -a_{n-1}\rho, & -a_{n-2}\rho^2, & \dots & -a_4\rho^{n-4}, & -a_3\rho^{n-3}, & a_0\rho^n - a_2\rho^{n-2} \end{array}\right|,$$

where the constituent that lies in the r^{th} column and the s^{th} row is

$$a_{r-s}\rho^{n-r+s} - a_{2n-r-s}\rho^{r+s-n},$$

provided that 0 be substituted for a_{r-s} if $r-s$ is negative, and also for a_{2n-r-s} if n is greater than $r+s$.

8. The function δ is of the degree $n(n-1)$ in ρ, and moreover δ is a function of ρ^2, for the effect of changing ρ into $-\rho$ in it is to alter the signs of alternate columns and then the signs of alternate rows, *i.e.*, is to multiply it by $(-1)^{2n}$, *i.e.*, not to alter it.

This also comes at once from the general form given for a constituent of the determinant, for this can be written

$$\rho^{n-r+s}(a_{r-s} - a_{2n-r-s}\,\rho^{2(r-n)}),$$

and the order of any term in the expansion of δ differs from

$$n(n-1) - \sum_{r=1}^{r=n-1} r + \sum_{s=1}^{s=n-1} s,$$

i.e., from $n(n-1)$ by an even number; therefore δ is an equation of the order $\frac{1}{2}n(n-1)$ in ρ^2.

9. The equations whose roots are the products of pairs of the roots of

$$ax^3 + bx^2 + cx + d = 0 \quad \text{and} \quad ax^4 + bx^3 + cx^2 + dx + e = 0$$

are found by evaluating δ when $n = 3$ and when $n = 4$.

They are
$$a^2\rho^6 - ac\rho^4 + bd\rho^2 - d^2 = 0,$$

and $(a\rho^4 + e)^3 - c(a\rho^4 + e)^2 + (bd - 4ae)(a\rho^4 + e) + 4ace - ad^2 - b^2e = 0.$

10. The paper of Professor Elliott referred to above proves that every equation of the n^{th} degree has n roots of the form $p + q\sqrt{(-1)}$.

If the coefficients in $F(\rho, 1)$ be all real, to each root $p + q\sqrt{(-1)}$ there corresponds a root $p - q\sqrt{(-1)}$, unless q vanishes.

Therefore there are always at least $\frac{n}{2}$ or $\frac{n-1}{2}$ real products of the roots taken in pairs, according as n is even or odd.

Therefore there are at least $\frac{n}{2}$ or $\frac{n-1}{2}$ real values of ρ^2 which satisfy the equation $\delta = 0$.

11. It is perhaps well to add that δ_n may be readily written down by writing the positive and negative sets of terms in the constituents of δ separately, and then adding corresponding terms together. Thus in the case of δ_7 we write down

$$\begin{vmatrix} a & b & c & d & e & f \\ . & a & b & c & d & e \\ . & . & a & b & c & d \\ . & . & . & a & b & c \\ . & . & . & . & a & b \\ . & . & . & . & . & a \end{vmatrix} \text{ and } \begin{vmatrix} . & . & . & . & . & -h \\ . & . & . & . & -h & -g \\ . & . & . & -h & -g & -f \\ . & . & -h & -g & -f & -e \\ . & -h & -g & -f & -e & -d \\ -h & -g & -f & -e & -d & -c \end{vmatrix},$$

and obtain the result

$$\delta_7 = \begin{vmatrix} a & b & c & d & e & f-h \\ . & a & b & c & d-h & e-g \\ . & . & a & b-h & c-g & d-f \\ . & . & -h & a-g & b-f & c-e \\ . & -h & -g & -f & a-e & b-d \\ -h & -g & -f & -e & -d & a-c \end{vmatrix}.$$

Thursday, December 12th, 1895.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

The following gentlemen were elected members of the Society:—Cuthbert Edmund Cullis, M.A., Fellow of Caius College, Cambridge; Andrew Munro, B.A., Fellow and Junior Bursar of Queens' College, Cambridge; Laurence Crawford, M.A., Fellow of King's College, Cambridge, Lecturer in Mathematics, Mason College, Birmingham; and Sidney Samuel Hough, B.A., Fellow of St. John's College, Cambridge.

The Auditor (the Rev. T. R. Terry) presented his report, and at the same time complimented the Treasurer for the admirable manner in which he had performed the duties of his office. Mr. Kempe proposed, and Mr. Bickmore seconded, a motion for the acceptance of the Treasurer's report, coupling with the motion a vote of thanks to the Auditor for the careful way in which he had discharged his office. Both motions were carried unanimously.

Prof. Hill gave a sketch of a "Note on the Convergency of Series," by Dr. R. Bryant.

Lt.-Col. Cunningham, R.E., dealt at some length upon the "Criteria of 2 as a 16-ic Residue." Messrs. Bickmore, Kempe, and the President joined in a discussion of the paper.

Dr. Hobson read a short note "On the Distribution of Electricity induced on an Infinite Disc with a Circular Hole in it," by Mr. H. M. Macdonald.

A paper by Dr. R. Lachlan, entitled "On the Double Foci of a Bicircular Quartic and the Nodal Focal Curves of a Cyclide," was taken as read.

The following presents were received:—

"Imperial University of Japan Calendar 1894-5."

"Atti del R. Istituto Veneto di Scienze, Lettere ed Arti," Tomo LII., Serie Settima, Tomo Quinto, Disp. 4-9, 1893-4; Tomo LIII., Serie Settima, Tomo Sesto, Disp. 1, 2, 3, 1894-5.

"Proceedings of the Royal Society," Vol. LVIII., No. 352.

"Proceedings of the Cambridge Philosophical Society," Vol. VIII., Pt. 5; 1895.

"Reale Istituto Lombardo—Rendiconti," Serie 2, Vol. XXIV.; Milano, 1891.

"Bulletin of the American Mathematical Society," Series 2, Vol. II., No. 2; November, 1895.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 3; Coimbra, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 3, Vol. I., Fasc. 8-10; 1895.

"Rendiconti del Circolo Matematico di Palermo," Tomo IX., Fasc. 5, 6; 1895.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki VII., Dai 2.

"Educational Times," December, 1895.

"Journal für die reine und angewandte Mathematik," Bd. CXV., Heft 4; Berlin, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Serie 5, Vol. IV., Fasc. 9, Sem. 2; Roma, 1895.

Fischer, O.—"Beitrage zu einer Muskel-dynamik," Abth. 1, "Uber die Wirkungsweise eingelenkiger Muskeln," Roy. 8vo; Leipzig, 1895.

"Memorie del reale Istituto Lombardo di Scienze e Lettere (Classe di Scienze Matematiche e Naturali)," Vol. XVI., 1891, VIII. della Serie 3, Fasc. 3; Vol. XVII., 1892, VIII. della Serie 3, Fasc. 1; Milano.

"Indian Engineering," Vol. XVIII., Nos. 17–20.

The Electrical Distribution induced on an Infinite Plane Disc with a Circular Hole in it. *By* H. M. MACDONALD. Received and read December 12th, 1895.

The function

$$W_n = \mu^{\frac{1}{2}} \int_0^\infty e^{-\kappa z} \kappa^{\frac{1}{2}} J_n(\kappa r)\, d\kappa \int_a^\infty J_{n+\frac{1}{2}}(\kappa r')\, J_{n+\frac{1}{2}}(\mu r')\, r' dr'$$

is such that for points in the plane $z = 0$, $r > a$,

$$W_n = J_n(\mu r),$$

and $r < a$,

$$\frac{\partial W_n}{\partial z} = 0.$$

The inducing system may be taken to be on the positive side of the plane; then in the neighbourhood of the plane the potential V_1 due to it may be represented by

$$V_1 = \Sigma\Sigma A_{n\mu} e^{\mu z} J_n(\mu r) \cos(n\phi + \alpha_\mu);$$

then V, the potential due to the induced charge, is given by

$$V = -\Sigma\Sigma A_{n\mu} W_n \cos(n\phi + \alpha_\mu),$$

whence the distribution may be obtained as in Vol. XXVI., p. 260.

Note on the Convergency of Series. By Robert Bryant.

Received November 26th, 1895. Read December 12th, 1895.

In what follows the n^{th} term of a series is used to denote the series itself.

By Cauchy's condensation test the convergency of the series $f(n)$ is the same as that of the series

$$a^n f(a^n),\quad a^n a^{a^n} f(a^{a^n})\ \dots,\qquad a > 1.$$

Continue this series backwards; then the series preceding $f(n)$ is

$$\frac{f(\log_a n)}{n}.$$

Hence we have the following rule :—

Change n into $\log_a n$ and divide by n.

Putting $\log_a = l,\quad \log_a \log_a = l^2,$ &c.,

then $f(n)$ and $\dfrac{f(ln)}{n}$ are both convergent or both divergent.

The following series are therefore convergent or divergent according as $r <$ or $\nless 1$,

$$r^n,\quad \frac{r^{ln}}{n},\quad \frac{r^{l^2n}}{nln},\quad \frac{r^{l^3n}}{nlnl^2n},\quad \dots\dots\quad \frac{r^{l^kn}}{nlnl^2n\dots l^{k-1}n}.$$

Comparing $f(n)$ with $\dfrac{r^{l^kn}}{nlnl^2n\dots l^{k-1}n}$, we see that it is convergent

if $$f(n) < \frac{r^{l^kn}}{nlnl^2n\dots l^{k-1}n}\quad \text{and}\quad r < 1.$$

Assuming n so large that $l^kn > 0$, this gives

$$nlnl^2n\dots l^{k-1}nf(n) < r^{l^kn};$$

therefore $$l\left[nlnl^2n\dots l^{k-1}nf(n)\right] < l^k n \log_a r;$$

therefore $$\frac{l\left[nlnl^2n\dots l^{k-1}nf(n)\right]}{l^kn} < \log_a r;$$

therefore $$\frac{l\left[nlnl^2n\dots l^{k-1}nf(n)\right]}{l^kn} < 0.$$

It is divergent if this expression > 0.

In the case where it is $=0$, we cannot assert anything.

This is Chrystal, *Algebra*, Vol. II., p. 112.

It is well known that the series $\frac{1}{n^{1+a}}$ is convergent if a be positive, and divergent if a be negative. Hence, by the above rule, the following series are convergent or divergent according as a is positive or negative,

$$\frac{1}{n^{1+a}}\; \frac{1}{n\,(ln)^{1+a}},\; \frac{1}{n\,ln\,(l^2n)^{1+a}},\; \ldots\ldots\; \frac{1}{n\,ln\,l^2n \ldots l^{k-1}n\,(l^kn)^{1+a}}.$$

This is Chrystal, p. 113.

Compare $f(n)$ with $\frac{r^{l^kn}}{n\,ln\,l^2n \ldots l^{k-1}n}$; therefore compare

$$\frac{f(n+1)}{f(n)} \text{ with } \frac{n\,ln\,l^2n \ldots l^{k-1}n}{(n+1)\,l\,(n+1)\,l^2\,(n+1) \ldots l^{k-1}\,(n+1)}\;\frac{r^{l^k(n+1)}}{r^{l^k(n)}};$$

or, denoting $n\,ln\,l^3n \ldots l^kn$ by $P_k(n)$, therefore compare

$$\frac{f(n+1)\,P_{k-1}(n+1)}{f(n)\,P_{k-1}(n)} \text{ with } r^{l^k(n+1)-l^k(n)},$$

therefore $f(n)$ is convergent, if

$$\frac{f(n+1)\,P_{k-1}(n+1)}{f(n)\,P_{k-1}(n)} < 1,$$

i.e., according as $\frac{f(n+1)}{f(n)}\,P_{k-1}(n+1) - P_{k-1}(n)$ is negative or positive.

This is Chrystal, p. 115.

It appears to be not generally known that in Cauchy's condensation test a may be any quantity greater than unity. This was discovered by Kohn. Consequently it must be noted that the base of the logarithms above may be any quantity greater than unity. Also, from and after a certain value of n, $f(n)$ must be positive and continually diminish as n increases.

[redacted]

On the Double Foci of a Bicircular Quartic, and the Nodal Focal Curves of a Cyclide. By R. LACHLAN. Received December 11th, 1895. Communicated December 12th, 1895.

In this paper I propose to obtain equations to determine the double foci of a bicircular quartic by the method of power-coordinates, explained in my memoir on "Systems of Circles and Spheres" (*Phil. Trans.*, Vol. CLXXVII., 1886). It will be shown that the locus of the double foci of a system of confocal bicircular quartics consists of the two circular cubics of the system. The double foci of a bicircular quartic are identical with the foci of the focal conics; consequently the problem of determining their locus is the same as that of finding the locus of the foci of a conic which passes through four given concyclic points. The more general problem of finding the locus when the four points are unrestricted is given in Salmon (*Conics*, § 228, Ex. 10), where it is shown that the locus is in general a sextic curve which breaks up into two cubics when the points are concyclic.

In the present paper the analogous problem for cyclides is also discussed, and it is shown that the nodal focal curves of a system of confocal cyclides are plane sections of the three cubic cyclides of the system.

In the original memoir, the coordinates of a point were taken as the powers of the point referred to four orthogonal circles divided by the radii of the respective circles. In some respects it may be more convenient to take such coordinates, but considerable advantage is lost owing to the fact that the connexion with ordinary "three-line" and "three-point" coordinates is less obvious. In this paper the coordinates are always taken to be proportional to the powers of the element (point, line, circle) with respect to the circles of reference.

PART I.—DOUBLE FOCI OF A BICIRCULAR QUARTIC.

Preliminary Formulæ, §§ 1–5.

1. Let r_1, r_2, r_3, r_4 be the radii of four circles which form an orthogonal system; and let π_1, π_2, π_3, π_4 denote the powers of any other circle with respect to them.

The power (π) of any two circles is given by [*T.* § 25 (23)]*

$$-2\pi = \frac{\pi_1\pi_1'}{r_1^2} + \frac{\pi_2\pi_2'}{r_2^2} + \frac{\pi_3\pi_3'}{r_3^2} + \frac{\pi_4\pi_4'}{r_4^2}.$$

In this formula either of the circles may be replaced by a point, a line, or the line at infinity.

2. Let x, y, z, w denote the powers of any point with respect to the given circles; then we deduce from the above relation

$$\frac{x}{r_1^2} + \frac{y}{r_2^2} + \frac{z}{r_3^2} + \frac{w}{r_4^2} = -2;$$

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} + \frac{w^2}{r_4^2} = 0.$$

3. Let ξ, η, ζ, ω denote the perpendiculars from the centres of the given circles on any straight line; then we deduce from the relation in § 1

$$\frac{\xi}{r_1^2} + \frac{\eta}{r_2^2} + \frac{\zeta}{r_3^2} + \frac{\omega}{r_4^2} = 0,$$

$$\frac{\xi^2}{r_1^2} + \frac{\eta^2}{r_2^2} + \frac{\zeta^2}{r_3^2} + \frac{\omega^2}{r_4^2} = 1.$$

4. If X, Y, Z, W denote the powers of any circle, we have

$$\frac{X}{r_1^2} + \frac{Y}{r_2^2} + \frac{Z}{r_3^2} + \frac{W}{r_4^2} = 0,$$

$$\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{W^2}{r_4^2} = 4\rho^2,$$

where ρ is the radius of the circle.

5. We may also deduce as a particular case of the relation in § 1

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} = 0.$$

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* References to my memoir on "Circles and Spheres" (*Phil. Trans.*, Vol. CLXXVII.) will be denoted by *T.*

Point-Equations, §§ 6, 7.

6. Any equation of the first degree in point-coordinates (x, y, z, w) clearly represents a circle, or a straight line.

The equation $$lx+my+nz+pw = 0$$

represents the circle whose powers are given by

$$\frac{X}{lr_1^2} = \frac{Y}{mr_2^2} = \frac{Z}{nr_3^2} = \frac{W}{pr_4^2} = \frac{-2}{l+m+n+p}.$$

If $l+m+n+p = 0$, the equation represents the straight line

$$\frac{\xi}{lr_1^2} = \frac{\eta}{mr_2^2} = \frac{\zeta}{nr_3^2} = \frac{\omega}{pr_4^2}.$$

7. An equation of the second degree in point-coordinates in general represents a bicircular quartic [*T.* § 69].

In particular an equation of the form

$$ax^2+by^2+cz^2+dw^2 = 0$$

represents a bicircular quartic having the coordinate circles for principal circles.

If an equation of the second degree be satisfied by $x = y = z = w$, the curve is a circular cubic.

Line-Equations, §§ 8–14.

8. An equation of the first degree in line-coordinates $(\xi\eta\zeta\omega)$ must represent a point.

Thus the equation $$l\xi+m\eta+n\zeta+p\omega = 0$$

represents a point, but the coordinates of the point are not necessarily proportional to lr_1^2, mr_2^2, nr_3^2, pr_4^2. These quantities are, in fact, the coordinates of some circle concentric with the point.

But the equation may be written in the form

$$l\xi+m\eta+n\zeta+p\omega+\lambda\left(\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}+\frac{\omega}{r_4^2}\right) = 0;$$

and then, if we choose λ so that

$$\Sigma\frac{(\lambda+lr_1^2)^2}{r_1^2} = 0,$$

i.e., $$2\lambda\Sigma l+\Sigma l^2r_1^2 = 0,$$

the coordinates of the point will be given by

$$\frac{x}{lr_1^2+\lambda} = \frac{y}{mr_2^2+\lambda} = \frac{z}{nr_3^2+\lambda} = \frac{w}{pr_4^2+\lambda}.$$

9. It should be noticed that the equation

$$l\xi+m\eta+n\zeta+p\omega = 0$$

may be written in the form

$$\left(\frac{l}{r_4^2}-\frac{p}{r_1^2}\right)\xi+\left(\frac{m}{r_4^2}-\frac{p}{r_2^2}\right)\eta+\left(\frac{n}{r_4^2}-\frac{p}{r_3^2}\right)\zeta = 0.$$

Hence, if α, β, γ be the areal coordinates of the point referred to the triangle formed by the centres of circles whose radii are r_1, r_2, r_3,

$$\frac{\alpha}{\dfrac{l}{r_4^2}-\dfrac{p}{r_1^2}} = \frac{\beta}{\dfrac{m}{r_4^2}-\dfrac{p}{r_2^2}} = \frac{\gamma}{\dfrac{n}{r_4^2}-\dfrac{p}{r_3^2}}.$$

Hence, if x, y, z, w be the power-coordinates of a point referred to four circles (1, 2, 3, 4), and α, β, γ the areal coordinates of the same point referred to the triangle formed by the centres of the circles (1, 2, 3),

$$\frac{\alpha}{\dfrac{x-w}{r_1^2}} = \frac{\beta}{\dfrac{y-w}{r_2^2}} = \frac{\gamma}{\dfrac{z-w}{r_3^2}}.$$

10. An equation of the second degree in line-coordinates will in general represent a conic.

The condition that an equation of the second degree should represent a point-pair may be found by forming the discriminant of the equation when one variable (ω, say) has been eliminated by means of the relation

$$\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}+\frac{\omega}{r_4^2} = 0.$$

11. Thus the equation

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2 = 0$$

represents a point-pair if the equation

$$\frac{a}{r_4^4}\xi^2+\frac{b}{r_4^4}\eta^2+\frac{c}{r_4^4}\zeta^2+d\left(\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}\right)^2 = 0$$

is the product of two linear factors.

It is easy to show that the discriminant of this is

$$\frac{1}{ar_1^4}+\frac{1}{br_2^4}+\frac{1}{cr_3^4}+\frac{1}{dr_4^4}=0.$$

12. When the equation

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2=0$$

represents a point-pair, the equations in point-coordinates which determine them may be found as follows.

Let x, y, z, w be the power-coordinates of either of the points; then it is evident that the equation

$$\frac{x-w}{r_1^2}\xi+\frac{y-w}{r_2^2}\eta+\frac{z-w}{r_3^2}\zeta=0$$

represents one of the points in which the polar of the point

$$\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}=0$$

cuts the conic $$a\xi^2+b\eta^2+c\zeta^2=0.$$

Hence $$\frac{x-w}{ar_1^4}+\frac{y-w}{br_2^4}+\frac{z-w}{cr_3^4}=0,$$

and $$\frac{(x-w)^2}{ar_1^4}+\frac{(y-w)^2}{br_2^4}+\frac{(z-w)^2}{cr_3^4}=0.$$

In virtue of the relation

$$\frac{1}{ar_1^4}+\frac{1}{br_2^4}+\frac{1}{cr_3^4}+\frac{1}{dr_4^4}=0,$$

these equations may be written

$$\frac{x}{ar_1^4}+\frac{y}{br_2^4}+\frac{z}{cr_3^4}+\frac{w}{dr_4^4}=0,$$

$$\frac{x^2}{ar_1^4}+\frac{y^2}{br_2^4}+\frac{z^2}{cr_3^4}+\frac{w^2}{dr_4^4}=0;$$

the former representing a line, and the latter a circular cubic having its asymptote parallel to the line [*T.*, § 111].

13. The equation $\frac{\xi^2}{r_1^2}+\frac{\eta^2}{r_2^2}+\frac{\zeta^2}{r_3^2}+\frac{\omega^2}{r_4^2}=0$

represents a point-pair, namely, the circular points at infinity.

14. The equation

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2+\lambda\left(\frac{\xi^2}{r_1^2}+\frac{\eta^2}{r_2^2}+\frac{\zeta^2}{r_3^2}+\frac{\omega^2}{r_4^2}\right)=0$$

represents a system of confocal conics.

Hence the foci of the conic

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2=0$$

will be given by the equation

$$\left(a+\frac{\lambda}{r_1^2}\right)\xi^2+\left(b+\frac{\lambda}{r_2^2}\right)\eta^2+\left(c+\frac{\lambda}{r_3^2}\right)\zeta^2+\left(d+\frac{\lambda}{r_4^2}\right)\omega^2=0,$$

where λ is given by the quadratic equation

$$\frac{1}{r_1^2(\lambda+ar_1^2)}+\frac{1}{r_2^2(\lambda+br_2^2)}+\frac{1}{r_3^2(\lambda+cr_3^2)}+\frac{1}{r_4^2(\lambda+dr_4^2)}=0.$$

The equations in point-coordinates determining the foci will therefore be (§ 12)

$$\frac{x}{r_1^2(\lambda+ar_1^2)}+\frac{y}{r_2^2(\lambda+br_2^2)}+\frac{z}{r_3^2(\lambda+cr_3^2)}+\frac{w}{r_4^2(\lambda+dr_4^2)}=0,$$

$$\frac{x^2}{r_1^2(\lambda+ar_1^2)}+\frac{y^2}{r_2^2(\lambda+br_2^2)}+\frac{z^2}{r_3^2(\lambda+cr_3^2)}+\frac{w^2}{r_4^2(\lambda+dr_4^2)}=0.$$

Double Foci of a Bicircular Quartic, §§ 15–19.

15. The equation of a bicircular quartic referred to its principal circles takes the form

$$ax^2+by^2+cz^2+dw^2=0.$$

Since we also have $\frac{x^2}{r_1^2}+\frac{y^2}{r_2^2}+\frac{z^2}{r_3^2}+\frac{w^2}{r_4^2}=0,$

the above equation may be written

$$\left(\frac{a}{r_4^2}-\frac{d}{r_1^2}\right)x^2+\left(\frac{b}{r_4^2}-\frac{d}{r_2^2}\right)y^2+\left(\frac{c}{r_4^2}-\frac{d}{r_3^2}\right)z^2=0.$$

The equation of the focal conic which is the locus of the centres of the bitangent circles which cut the principal circle $w = 0$ orthogonally is [*T.*, §§ 116, 117]

$$\frac{\alpha^2}{\frac{a}{r_4^2} - \frac{d}{r_1^2}} + \frac{\beta^2}{\frac{b}{r_4^2} - \frac{d}{r_2^2}} + \frac{\gamma^2}{\frac{c}{r_4^2} - \frac{d}{r_3^2}} = 0,$$

where α, β, γ denote areal coordinates of a point referred to the centres of the principal circles (1, 2, 3).

Hence the tangential equation of this focal conic is

$$\left(\frac{a}{r_4^2} - \frac{d}{r_1^2}\right)\xi^2 + \left(\frac{b}{r_4^2} - \frac{d}{r_2^2}\right)\eta^2 + \left(\frac{c}{r_4^2} - \frac{d}{r_3^2}\right)\zeta^2 = 0,$$

i.e.,
$$\frac{1}{r_4^2}\left(a\xi^2 + b\eta^2 + c\zeta^2 + d\omega^2\right) - d\left(\frac{\xi^2}{r_1^2} + \frac{\eta^2}{r_2^2} + \frac{\zeta^2}{r_3^2} + \frac{\omega^2}{r_4^2}\right) = 0.$$

16. Hence the four focal conics of the bicircular quartic

$$ax^2 + by^2 + cz^2 + dw^2 = 0$$

are given by

$$\Sigma - ar_1^2\Sigma_0 = 0,$$
$$\Sigma - br_2^2\Sigma_0 = 0,$$
$$\Sigma - cr_3^2\Sigma_0 = 0,$$
$$\Sigma - dr_4^2\Sigma_0 = 0;$$

where
$$\Sigma \equiv a\xi^2 + b\eta^2 + c\zeta^2 + d\omega^2,$$

$$\Sigma_0 \equiv \frac{\xi^2}{r_1^2} + \frac{\eta^2}{r_2^2} + \frac{\zeta^2}{r_3^2} + \frac{\omega^2}{r_4^2}.$$

The form of these equations shows that the four focal conics are confocal with the conic
$$\Sigma = 0.$$

17. Now the double foci of the bicircular quartic are the foci of its focal conics.

Hence the equations determining the double foci of the bicircular quartic
$$ax^2 + by^2 + cz^2 + dw^2 = 0$$

are (§ 14)

$$\frac{x}{r_1^2(\lambda+ar_1^2)}+\frac{y}{r_2^2(\lambda+br_2^2)}+\frac{z}{r_3^2(\lambda+cr_3^2)}+\frac{w}{r_4^2(\lambda+dr_4^2)}=0,$$

$$\frac{x^2}{r_1^2(\lambda+ar_1^2)}+\frac{y^2}{r_2^2(\lambda+br_2^2)}+\frac{z^2}{r_3^2(\lambda+cr_3^2)}+\frac{w^2}{r_4^2(\lambda+dr_4^2)}=0,$$

where λ is either root of the equation

$$\frac{1}{r_1^2(\lambda+ar_1^2)}+\frac{1}{r_2^2(\lambda+br_2^2)}+\frac{1}{r_3^2(\lambda+cr_3^2)}+\frac{1}{r_4^2(\lambda+dr_4^2)}=0.$$

18. Thus the double foci of the bicircular quartic lie on two circular cubics.

The equation of either cubic may be written in the form

$$\Sigma\frac{\lambda x^2}{r_1^2(\lambda+ar_1^2)}-\Sigma\frac{x^2}{r_1^2}=0,$$

i.e., in the form
$$\Sigma\frac{ax^2}{\lambda+ar_1^2}=0,$$

which may be written

$$\frac{x^2}{\frac{1}{a}+\frac{r_1^2}{\lambda}}+\frac{y^2}{\frac{1}{b}+\frac{r_2^2}{\lambda}}+\frac{z^2}{\frac{1}{c}+\frac{r_3^2}{\lambda}}+\frac{w^2}{\frac{1}{d}+\frac{r_4^2}{\lambda}}=0.$$

When the equation of the cubic is written thus it is evident that it is confocal with (*i.e.*, has the same single foci as) the quartic [see *T.*, § 91].

19. Hence the locus of the double foci of a system of confocal bicircular quartics consists of the two circular cubics of the system.*

Each quartic has two double foci on each of the circular cubics. It follows also that the double focus of either cubic lies on the other.

* A proof of this theorem by Burnside is given in Salmon (*Conics*, § 228, Ex. 10).

PART II.—NODAL FOCAL CURVES OF A CYCLIDE.

Preliminary Formulæ, §§ 20–24.

20. Let r_1, r_2, r_3, r_4, r_5 be the radii of five spheres which form an orthogonal system; and let π_1, π_2, π_3, π_4, π_5 denote the powers of any other sphere with respect to them.

The power (π) of any two spheres is given by [*T.*, § 212 (223)]

$$-2\pi = \frac{\pi_1\pi_1'}{r_1^2} + \frac{\pi_2\pi_2'}{r_2^2} + \frac{\pi_3\pi_3'}{r_3^2} + \frac{\pi_4\pi_4'}{r_4^2} + \frac{\pi_5\pi_5'}{r_5^2}.$$

In this formula either of the spheres may be replaced by a point, a plane, or the plane at infinity.

21. If x, y, z, w, v denote the powers of any point with respect to the system of reference, we have from the above relation

$$\frac{x}{r_1^2} + \frac{y}{r_2^2} + \frac{z}{r_3^2} + \frac{w}{r_4^2} + \frac{v}{r_5^2} = -2,$$

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} + \frac{w^2}{r_4^2} + \frac{v^2}{r_5^2} = 0.$$

22. If ξ, η, ζ, ω, ϖ denote the perpendiculars from the centres of the five spheres on any plane, we have

$$\frac{\xi}{r_1^2} + \frac{\eta}{r_2^2} + \frac{\zeta}{r_3^2} + \frac{\omega}{r_4^2} + \frac{\varpi}{r_5^2} = 0,$$

$$\frac{\xi^2}{r_1^2} + \frac{\eta^2}{r_2^2} + \frac{\zeta^2}{r_3^2} + \frac{\omega^2}{r_4^2} + \frac{\varpi^2}{r_5^2} = 1.$$

23. If X, Y, Z, W, V denote the powers of any sphere, we have

$$\frac{X}{r_1^2} + \frac{Y}{r_2^2} + \frac{Z}{r_3^2} + \frac{W}{r_4^2} + \frac{V}{r_5^2} = -2,$$

$$\frac{X^2}{r_1^2} + \frac{Y^2}{r_2^2} + \frac{Z^2}{r_3^2} + \frac{W^2}{r_4^2} + \frac{V^2}{r_5^2} = 4\rho^2,$$

where ρ is the radius of the sphere.

24. We may also deduce, as a particular case of the relation in § 20, the formula

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}=0.$$

Point-Equations, §§ 25, 26.

25. Any equation of the first degree in point-coordinates represents in general a sphere.

Thus the equation

$$lx+my+nz+pw+qv=0$$

represents the sphere whose powers are given by

$$\frac{X}{lr_1^2}=\frac{Y}{mr_2^2}=\frac{Z}{nr_3^2}=\frac{W}{pr_4^2}=\frac{V}{qr_5^2}=\frac{-2}{l+m+n+p+q}.$$

If $l+m+n+p+q=0$, the equation represents the plane

$$\frac{\xi}{lr_1^2}=\frac{\eta}{mr_2^2}=\frac{\zeta}{nr_3^2}=\frac{\omega}{pr_4^2}=\frac{\varpi}{qr_5^2}.$$

26. An equation of the second degree in point-coordinates in general represents a cyclide having the circle at infinity for a nodal curve [*T.*, § 241]. If, however, the equation be satisfied by

$$x=y=z=w=v,$$

it will represent a cubic cyclide.

Plane-Equations, §§ 27–32.

27. An equation of the first degree in plane-coordinates $(\xi, \eta, \zeta, \omega, \varpi)$ represents a point.

Thus the equation

$$l\xi+m\eta+n\zeta+p\omega+q\varpi=0$$

will represent the point given by

$$\frac{x}{lr_1^2+\lambda}=\frac{y}{mr_2^2+\lambda}=\frac{z}{nr_3^2+\lambda}=\frac{w}{pr_4^2+\lambda}=\frac{v}{qr_5^2+\lambda},$$

where λ is given by $$\Sigma\frac{(\lambda+lr_1^2)^2}{r_1^2}=0,$$

i.e. by $$2\lambda\Sigma l+\Sigma l^2r_1^2=0.$$

The equation $$l\xi+m\eta+n\zeta+p\omega+q\varpi=0$$

may be written in the form

$$\left(\frac{l}{r_5^2}-\frac{q}{r_1^2}\right)\xi+\left(\frac{m}{r_5^2}-\frac{q}{r_2^2}\right)\eta+\left(\frac{n}{r_5^2}-\frac{q}{r_3^2}\right)\zeta+\left(\frac{p}{r_5^2}-\frac{q}{r_4^2}\right)\omega=0.$$

Hence, if α, β, γ, δ be the tetrahedral coordinates of the point referred to the tetrahedron formed by the centres of the spheres (1, 2, 3, 4),

$$\frac{\alpha}{\dfrac{l}{r_5^2}-\dfrac{q}{r_1^2}}=\frac{\beta}{\dfrac{m}{r_5^2}-\dfrac{q}{r_2^2}}=\frac{\gamma}{\dfrac{n}{r_5^2}-\dfrac{q}{r_3^2}}=\frac{\delta}{\dfrac{p}{r_5^2}-\dfrac{q}{r_4^2}}.$$

28. An equation of the second degree in plane-coordinates represents in general a quadric surface.

The condition that an equation of the second degree should represent a plane curve may be found by forming the discriminant of the equation when one variable (ϖ, say) has been eliminated by means of the relation

$$\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}+\frac{\omega}{r_4^2}+\frac{\varpi}{r_5^2}=0.$$

29. Thus the condition that the equation

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2+e\varpi^2=0$$

should represent a plane curve is found, by forming the discriminant of

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2+r_5^4e\left(\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}+\frac{\omega}{r_4^2}\right)^2=0,$$

to be $$\frac{1}{ar_1^4}+\frac{1}{br_2^4}+\frac{1}{cr_3^4}+\frac{1}{dr_4^4}+\frac{1}{er_5^4}=0.$$

30. When the equation

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2+e\varpi^2=0$$

represents a plane curve, its equations in point-coordinates may be found thus:—

Let x, y, z, w, v be any point on the curve; then the equation

$$\frac{x-v}{r_1^2}\xi+\frac{y-v}{r_2^2}\eta+\frac{z-v}{r_3^2}\zeta+\frac{w-v}{r_4^2}\omega=0$$

represents a point lying in the curve of intersection of the quadric

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2=0$$

by the polar of the point

$$\frac{\xi}{r_1^2}+\frac{\eta}{r_2^2}+\frac{\zeta}{r_3^2}+\frac{\omega}{r_4^2}=0.$$

Hence
$$\frac{x-v}{ar_1^4}+\frac{y-v}{br_2^4}+\frac{z-v}{cr_3^4}+\frac{w-v}{dr_4^4}=0,$$

and
$$\frac{(x-v)^2}{ar_1^4}+\frac{(y-v)^2}{br_2^4}+\frac{(z-v)^2}{cr_3^4}+\frac{(w-v)^2}{dr_4^4}=0;$$

which equations, since

$$\frac{1}{ar_1^4}+\frac{1}{br_2^4}+\frac{1}{cr_3^4}+\frac{1}{dr_4^4}+\frac{1}{er_5^4}=0,$$

may be written
$$\frac{x}{ar_1^4}+\frac{y}{br_2^4}+\frac{z}{cr_3^4}+\frac{w}{dr_4^4}+\frac{v}{er_5^4}=0,$$

$$\frac{x^2}{ar_1^4}+\frac{y^2}{br_2^4}+\frac{z^2}{cr_3^4}+\frac{w^2}{dr_4^4}+\frac{v^2}{er_5^4}=0.$$

The former of these equations represents a plane, and the latter a cubic cyclide having its asymptotic plane parallel to the former.

31. The equation

$$\Sigma_0\equiv\frac{\xi^2}{r_1^2}+\frac{\eta^2}{r_2^2}+\frac{\zeta^2}{r_3^2}+\frac{\omega^2}{r_4^2}+\frac{v^2}{r_5^2}=0$$

represents a plane curve, namely, the circle in the plane at infinity.

32. The equation
$$\Sigma+\lambda\Sigma_0=0$$

represents a quadric confocal with $\Sigma=0$.

Hence, if
$$\Sigma\equiv a\xi^2+b\eta^2+c\zeta^2+d\omega^2+e\varpi^2,$$

the focal conics of $\Sigma=0$ will be given by the equation

$$\Sigma+\lambda\Sigma_0=0,$$

where λ is a root of the cubic equation

$$\Sigma \frac{1}{r_1^2(\lambda + ar_1^2)} = 0.$$

The point-equations of the focal conics of $\Sigma = 0$ will therefore be given by the equations

$$\Sigma \frac{x}{r_1^2(\lambda + ar_1^2)} = 0,$$

$$\Sigma \frac{x^2}{r_1^2(\lambda + ar_1^2)} = 0.$$

The first of these equations evidently represents a plane, and the latter a cubic cyclide. The plane intersects the cyclide in a conic and a straight line, namely, the line in which the cyclide cuts the line at infinity [*T.*, § 264].

Nodal Focal Curves of a Cyclide, §§ 33–37.

33. The equation of a cyclide referred to its principal spheres takes the form

$$ax^2 + by^2 + cz^2 + dw^2 + ev^2 = 0,$$

where

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} + \frac{z^2}{r_3^2} + \frac{w^2}{r_4^2} + \frac{v^2}{r_5^2} = 0.$$

The equation may therefore be written in the form

$$\left(\frac{a}{r_5^2} - \frac{e}{r_1^2}\right)x^2 + \left(\frac{b}{r_5^2} - \frac{e}{r_2^2}\right)y^2 + \left(\frac{c}{r_5^2} - \frac{e}{r_3^2}\right)z^2 + \left(\frac{d}{r_5^2} - \frac{e}{r_4^2}\right)w^2 = 0.$$

The equation of the focal quadric which is the locus of the centres of the bitangent spheres which cut the principal sphere $v = 0$ is [*T.*, § 281]

$$\frac{\alpha^2}{\dfrac{a}{r_5^2} - \dfrac{e}{r_1^2}} + \frac{\beta^2}{\dfrac{b}{r_5^2} - \dfrac{e}{r_2^2}} + \frac{\gamma^2}{\dfrac{c}{r_5^2} - \dfrac{e}{r_3^2}} + \frac{\delta^2}{\dfrac{d}{r_5^2} - \dfrac{e}{r_4^2}} = 0,$$

where $\alpha, \beta, \gamma, \delta$ denote tetrahedral coordinates of a point referred to the centres of the principal spheres (1, 2, 3, 4).

Hence the tangential equation of this focal quadric is

$$\left(\frac{a}{r_5^2}-\frac{e}{r_1^2}\right)\xi^2+\left(\frac{b}{r_5^2}-\frac{e}{r_2^2}\right)\eta^2+\left(\frac{c}{r_5^2}-\frac{e}{r_3^2}\right)\zeta^2+\left(\frac{d}{r_5^2}-\frac{e}{r_4^2}\right)\omega^2=0,$$

i.e., $$\Sigma-er_5^2\Sigma_0=0,$$

where $$\Sigma\equiv a\xi^2+b\eta^2+c\zeta^2+d\omega^2+e\varpi^2,$$

$$\Sigma_0\equiv\frac{\xi^2}{r_1^2}+\frac{\eta^2}{r_2^2}+\frac{\zeta^2}{r_3^2}+\frac{\omega^2}{r_4^2}+\frac{\varpi^2}{r_5^2}.$$

34. It follows that the equations of the five focal quadrics are given by

$$\Sigma-ar_1^2\Sigma_0=0,$$

$$\Sigma-br_2^2\Sigma_0=0,$$

$$\Sigma-cr_3^2\Sigma_0=0,$$

$$\Sigma-dr_4^2\Sigma_0=0,$$

$$\Sigma-er_5^2\Sigma_0=0,$$

the form of which equations shows that the five quadrics are confocal with the quadric

$$\Sigma=0.$$

35. Now the nodal focal curves of the cyclide are the focal conics of the focal quadrics.

Hence the equations which determine the nodal focal curves of the cyclide

$$ax^2+by^2+cz^2+dw^2+ev^2=0$$

are (§ 32)

$$\Sigma\frac{x}{r_1^2(\lambda+ar_1^2)}=0,$$

$$\Sigma\frac{x^2}{r_1^2(\lambda+ar_1^2)}=0,$$

where λ is any one of the three roots of the equation

$$\Sigma\frac{1}{r_1^2(\lambda+ar_1^2)}=0.$$

36. Thus the nodal focal curves of the cyclide lie on three cubic cyclides.

The equation of any one of these cubics may be written in the form

$$\Sigma \frac{\lambda x^2}{r_1^2(\lambda + a r_1^2)} - \Sigma \frac{x^2}{r_1^2} = 0,$$

i.e.,

$$\Sigma \frac{a x^2}{\lambda + a r_1^2} = 0,$$

which may be written

$$\Sigma \frac{x^2}{\frac{1}{a} + \frac{r_1^2}{\lambda}} = 0;$$

in which form it is evident that the cubic is confocal with the given cyclide [see *T.*, § 259].

37. Hence the locus of the nodal focal curves of a system of confocal cyclides consists of the three cubic cyclides of the system.

Each cyclide has one focal curve on each of the cubics.

It follows also that the nodal focal curves of either of the cubic cyclides are plane sections of the two confocal cubics.

On 2 as a 16-ic Residue. By Lt.-Col. ALLAN CUNNINGHAM, R.E., Fellow of King's College, London. Read December 12th, 1895. Revised and returned February 13th, 1896.

Preface.

In the following paper all the symbols used are to be taken as *integers*, and generally as *positive integers* (unless otherwise stated). N stands for *any* number (composite or prime), p denotes a *prime*. Also ω, Ω are always to be read as *odd* integers; whilst ϵ, e, E are used sometimes for *even* integers, and sometimes as (integer) *exponents* or as *residue-indices* (not necessarily even).

[The author wishes to acknowledge here the great help received in preparing this paper, and in revising the proofs, from Mr. Charles E. Bickmore.]

1. *Introduction.*

When p is a *prime*, Fermat's theorem gives

$$a^{p-1} \equiv 1, \quad (\text{mod } p), \quad \text{when } a \text{ is prime to } p \quad \ldots\ldots\ldots\ldots (1),$$

and it may happen that

$$a^{(p-1)/e} \equiv 1, \text{ (mod } p\text{), when (and only when) } p = ef+1 \ldots (2).$$

In this case a is said to be a *residue* of the prime p of order e, and e is called the *residue-index*. This result is conveniently expressed in Lejeune-Dirichlet's notation as

$$\left(\frac{a}{p}\right)_e = +1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It is useful to note that (2) or (3) always involves the ambiguous

$$a^{(p-1)/2e} \equiv \pm 1, \text{ (mod } p\text{), or } \left(\frac{a}{p}\right)_{2e} = \pm 1, \ \left[\text{if } \frac{p-1}{e} \text{ be } \textit{even}\right] \ldots (4),$$

whereas *either* of the latter results (4) involves the *definite* result (2, 3).

2. *Objects.*

The object of this paper is to bring forward a new criterion for the division of Fermat's exponent $(p-1)$ by 16 when the base $a = 2$, *i.e.*, for determining when 2 is a *sextodecimic* residue of a prime p, or when

$$2^{\frac{1}{16}(p-1)} \equiv 1, \text{ (mod } p\text{), or } \left(\frac{2}{p}\right)_{16} = +1 \ \ldots\ldots\ldots\ldots (5).$$

As the new criterion depends at present *wholly on induction* from numerous known instances, the rule will be first stated (Arts. 3, 4); some applications will then be given (Arts. 5–23); and, lastly, the evidence of the rule will be given in full (Arts. 24–26). Five tables of primes of which 2 is an 8-ic (but non-16-ic) or a 16-ic residue are given at the end (Art. 27).

3. *Preliminary Sextodecimic Conditions.*

In order that (5) may be true, the following conditions must be satisfied in the first place:—

$$\text{i. } \left(\frac{2}{p}\right)_2 = +1, \ \text{ ii. } \left(\frac{2}{p}\right)_4 = +1, \ \text{ iii. } \left(\frac{2}{p}\right)_8 = +1 \ \ldots\ldots (6);$$

i.e., 2 must be a *quadratic*, *quartic*, and *octavic* residue of p, and, further,

$$\tfrac{1}{16}(p-1) = m \text{ (an integer), or } p = 16m+1 \ \ldots\ldots\ldots (7).$$

i. *Quadratic Condition.*—Euler's condition* that 2 should be a 2-ic residue is simply $p = 8i \pm 1$; this is fully satisfied by (7).

ii. *Quartic Condition.*—Since $p = 8i+1$, it is expressible in *each* of the four quadratic forms

$$p = \mathrm{a}^2+\mathrm{b}^2 = \mathrm{c}^2+2\mathrm{d}^2 = \mathrm{e}^2-2\mathrm{f}^2 = 2\mathrm{g}^2-\mathrm{h}^2 \quad \text{............(8)};$$

of these, the (a, b) and (c, d) expressions are *unique*† (p being prime), whilst the (e, f) and (g, h) expressions may be formed in an infinity of ways.

Now, since $p = 16m+1$, these take the forms

$$p = (8\alpha_0 \pm 1)^2 + (4\beta_0)^2 = \mathrm{a}^2+\mathrm{b}^2 \quad \text{..............................(9)},$$

$$= (4\gamma_0 \pm 1)^2 + 2\,(2\delta_0)^2 = \mathrm{c}^2+2\mathrm{d}^2, \ [\text{with } (\pm\gamma_0+\delta_0) \textit{ even}] \text{...(10)},$$

$$= (4\epsilon_0 \pm 1)^2 - 2\,(2\zeta_0)^2 = \mathrm{e}^2-2\mathrm{f}^2, \ [\text{with } (\pm\epsilon_0-\zeta_0) \textit{ even}] \text{...(11)},$$

$$= 2(4\eta_0 \pm 1)^2 - (8\theta_0 \pm 1)^2 = 2\mathrm{g}^2-\mathrm{h}^2 \text{..............................(12)}.$$

The conditions that 2 should be a *quartic* residue of a prime p of the (a, b), (c, d) forms were established by Gauss,‡ and those of the (e, f), (g, h) forms have been found by Mr. C. E. Bickmore. These are

$$\beta_0,\ \gamma_0 \text{ to be both } \textit{even}; \text{ say } \beta_0 = 2\beta,\ \gamma_0 = 2\gamma \text{...........(13)},$$

$$\left.\begin{array}{ll} & \mathrm{e} = (8\epsilon+1),\ \mathrm{f} = 4\zeta \\ \text{or} & \mathrm{e} = (8\epsilon+3),\ \mathrm{f} = (4\zeta+2) \end{array}\right\};\quad \mathrm{g} = (4\eta+1) \text{ ....(14)}.$$

These conditions mutually involve one another, so as to amount virtually to *only one* condition; and the final forms of p are (dropping the subscript ciphers throughout)

$$p = (8\alpha \pm 1)^2 + (8\beta)^2 \text{ ....(15)},$$

$$= (8\gamma \pm 1)^2 + 2\,(4\delta)^2 \text{..(16)},$$

$$= (8\epsilon+1)^2 - 2\,(4\zeta)^2, \text{ or } (8\epsilon+3)^2 - 2\,(4\zeta+2)^2 \text{ ......(17)},$$

$$= 2\,(4\eta+1)^2 - (8\zeta \pm 1)^2 \text{ ....(18)}.$$

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* Proved by Lagrange, *Nouveaux Mém. de l'Acad. des Sciences de Berlin*, 1775, pp. 349–351, and more simply by Cauchy, *Théorie des Nombres*, p. 451.

† See a celebrated letter from Fermat to Sir K. Digby, written in 1648 (printed in Wallis's *Commercium Epistolicum*).

‡ Gauss's *Werke*, Band II., p. 89.

iii. *Octavic Condition.*—Reuschle's condition* that 2 should be an *octavic* residue of p a prime of form (15) is simply that

$$\beta = \text{an } even \text{ number, say} = 2\beta' \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

so that p takes the form

$$p = (8\alpha \pm 1)^2 + (16\beta')^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and for every prime of this form (20) p is an *octavic* residue, so that

$$\left(\frac{2}{p}\right)_8 = +1, \quad \text{and} \quad \left(\frac{2}{p}\right)_{16} = \pm 1 \quad \ldots\ldots\ldots\ldots\ldots (21).$$

4. *Sextodecimic Criterion.*

A criterion is now required to distinguish between the signs (±) in result (21). The new criterion now proposed is

$$\left(\frac{2}{p}\right)_{16} = (-1)^{\mu}, \quad \text{where } \mu = \beta' + \delta = \tfrac{1}{16}\text{b} + \tfrac{1}{4}\text{d} \quad \ldots\ldots\ldots (22);$$

whence $\left(\frac{2}{p}\right)_{16} = -1,$ when $(\beta' + \delta)$ or $(\frac{1}{16}\text{b} + \frac{1}{4}\text{d})$ is *odd*...(23),

$= +1,$ when $(\beta' + \delta)$ or $(\frac{1}{16}\text{b} + \frac{1}{4}\text{d})$ is *even*...(24).

[This criterion depends only on the (a, b) and (c, d) forms of p, and is independent of the (e, f), (g, h) forms. In fact, the latter two forms are not really required for the purposes of this paper; but, in view of future developments to residues of higher orders, it has been thought worth while to give the *four* forms.]

5. *Application to given Primes.*

To determine then whether a *given* prime (p), which is of the form $p = 16m + 1$, has 2 for a *sextodecimic* residue, it should first be

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* "Neue Zahlentheoretische Tabellen," by Prof. Reuschle, in the *Program zum Schlusse des Schuljahrs* 1855–56 *am Königlichen Gymnasium zu Stuttgart*, 1856, p. 14. Reuschle's condition is not expressed in a convenient form. A more convenient form has been given to it in Mr. C. E. Bickmore's paper "On the Numerical Factors of $(a^n - 1)$," in the *Messenger of Mathematics*," Vol. xxv., 1895, Art. 13 (4), on p. 18; viz., that if

$$p = (4\alpha \pm 1)^2 + (8\beta)^2 = 8m + 1,$$

then

$$\left(\frac{2}{p}\right)_8 = (-1)^{\alpha+\beta},$$

from which the condition (19) above is adapted. Reuschle's condition is virtually included in a similar criterion for the base 10 in a letter from Jacobi, dated 1846, printed at p. 9 of same *Program*. Reuschle communicated his criterion to Jacobi in 1851 (see p. 11 of same).

expressed in the form $p = a^2 + b^2$. If (and only if) these turn out to be of forms

$$a = 8\alpha \pm 1, \quad b = 16\beta' \quad \text{.......................(25)},$$

then 2 is certainly an *octavic* residue, so that

$$\left(\frac{2}{p}\right)_{16} = \pm 1.$$

To distinguish between the signs, p must next be expressed in the form $p = c^2 + 2d^2$, whereupon c, d will be found to be of forms [see (16)]

$$c = 8\gamma \pm 1, \quad d = 4\delta; \quad \text{and } {}^*cd = 3\kappa \ (\text{when } p = prime) \ldots (26).$$

Then, by (22), $\left(\frac{2}{p}\right)_{16} = -1$ or $+1$, according as $\beta' + \delta$ is *odd* or even.

When p is a high prime the partitions $p = a^2 + b^2 = c^2 + 2d^2$ are very laborious:† the arithmetical work is much reduced by noting that a, b, c, d must necessarily contain the coefficients 8, 16, 8, 4 respectively, and *one** of c, d must contain the factor 3; the partition $p = a^2 + b^2$ is usually the easier of the two, as the high coefficient 16 in b limits the number of trials; it is useful also to note that,

$$\text{when } p = 32m + 1, \text{ then } a = 16\alpha' \pm 1, \ c = 16\gamma' \pm 1 \quad \text{...........(27)}.$$

[As in some cases the values of (e, f), (g, h) are much more easily found than those of (a, b), (c, d), it is convenient to have a means of utilizing the former in computing the latter: it is hoped to publish this in a future paper.]

6. *Computation of Primes of which 2 is a 16-ic Residue.*

A series of numbers N may readily be computed so as to be of *one* of the required quadratic forms $p = a^2 + b^2$ or $= c^2 + 2d^2$. When a prime of either form has thus been found, one pair (a, b) or (c, d) is known; the prime must then be resolved into the other form, so that the sextodecimic criterion (22) may be applied. Thus

i. With any *assumed* values of α, β', compute a series of numbers

$$N = a^2 + b^2 = (8\alpha \pm 1)^2 + (16\beta')^2.$$

---

* Euler (*Opera posthuma*, t. I.) proves that $cd = 3\kappa$ unless $c^2 + 2d^2 = 3\kappa$. Also $c = 3\kappa$, when $p = 3\varpi - 1$; and $d = 3\kappa$, when $p = 3\varpi + 1$.

† There is a table of a, b from $p = 5$ to 11981, and of c, d from $p = 17$ to 5953 [of form $(8m + 1)$ only], in Jacobi's *Opuscula Math.*, Vol. I., pp. 326 and 332; also published in *Crelle's Journal*, Vol. XXX. (1846), pp. 174 and 180. There is a larger table of a, b from $p = 5$ to 12,377 complete, and thence at intervals to $p = 24{,}917$, also of c, d (for primes of form $p = 8m + 1$ only) from $p = 17$ to 12,377 complete, and thence (very incompletely) to 24,889, in Reuschle's *Neue Zahlentheoretische Tabellen*, p. 32. With Reuschle's tables, it is now easy to form a complete list of all primes up to 12,400 of which 2 is a 16-ic residue. [A *great many* errata in Reuschle's tables, and some in Jacobi's, have been detected by Mr. C. E. Bickmore and by the author.]

If any of these be prime ($N = p$), then $p = 16m+1$, and 2 is necessarily an *octavic* residue. Proceed with the partition $p = c^2+2d^2$, which gives δ; then apply the 16-ic criterion.

ii. With any *assumed* values of γ, δ (such that one of c, d $= 3\kappa$), compute a series of numbers
$$N = c^2+2d^2 = (8\gamma \pm 1)^2 + 2(4\delta)^2.$$

If any of these be prime ($N = p$), then $p = 16m+1$, and 2 is necessarily a *quartic* residue. Proceed with the partition $p = a^2+b^2$. Note that a, b will certainly be of forms $a = 8\alpha \pm 1$, $b = 8\beta$, by (15). If $b = 16\beta'$, then 2 will be an octavic residue, and the sextodecimic criterion may now be applied.

[Thus in both these processes *only one* quadratic partition has to be effected; they both have the defect that many of the numbers N may turn out to be *composite*, whereas primes are required. Method i. is the more convenient of the two, as it provides for *octavic* residues directly, whereas Method ii. only provides for *quartic* residues; about half the primes resulting from Method ii. may be expected to be *octavic* non-residues.]

7. *Direct Computation of Primes of which* 2 *is a* 16-*ic Residue.*

A method will now be developed for computing a series of numbers N which shall be at once of both the required quadratic forms

$$N = a^2+b^2, \quad \text{where } a = (8\alpha \pm 1),\ b = 16\beta' \text{............(28)},$$

$$N = c^2+2d^2, \quad \text{where } c = (8\gamma \pm 1),\ d = 4\delta \text{ ....(29)}.$$

Let $$\beta' : \delta = \mu : \nu, \quad \text{or} \quad \beta' = \mu\theta,\ \delta = \nu\theta \text{..................(30)},$$

where μ, ν are *mutually prime* [so that $\mu : \nu$ is in its lowest terms] ....(31).

Now $$N = \left[a^2+b^2-2d^2\right]+2d^2$$
$$= \left[a^2+2(8\mu^2-\nu^2)(4\theta)^2\right]+2d^2 \text{...............(32)}.$$

Writing $$k = 2(8\mu^2-\nu^2) = k_1k_2, \text{ suppose ....(33)},$$

then $$a^2+k(4\theta)^2 = c^2 \text{..............................(34)}.$$

Now, supposing μ, ν to be *given*, this is a diophantine equation (in a, 4θ, c, as unknowns), the general solution of which depends on the composition of k; as to which

$k = k_1k_2$ is always *finite*, and *even*, and may be either ($\pm$)...(35*a*),

k_1, k_2 are both *finite*; one (say k_2) is always *even*, the other (k_1) is *odd* and may $= 1$; each may be $\pm$ ....(35*b*),

k is of *one* of the forms 2Ω, 8Ω, 16Ω (depending on ν), viz....(35*c*),

$$\left.\begin{array}{ll}(1) \text{ If } \nu \text{ is } odd, \; k = 2\,(8\mu^2-\nu^2) = 2\Omega, & \\ (2) \text{ If } \nu = 2\omega, \; k = 8\,(2\mu^2-\omega^2) = 8\Omega, & \text{and } \mu \text{ is } odd \\ (3) \text{ If } \nu = 4\nu'', \; k = 16\,(\mu^2-2\nu''^2) = 16\Omega, & \text{and } \mu \text{ is } odd\end{array}\right\} \text{...(35}d\text{)}.$$

The general solution of the diophantine (34) may now be expressed in terms of two arbitrary numbers m, n,

m, n any mutually prime integers; one (say m) *odd*, the other (n) *even* ....(36),

in following system :—

$$\pm \text{a} = k_1 m^2 \sim k_2 n^2, \quad \text{c} = k_1 m^2 + k_2 n^2, \quad \theta = \tfrac{1}{2} mn \text{.........(37)};$$

where the sign of a is given by $\text{a} \equiv +1$, (mod 8) ....(37*a*).

whence $\beta' = \tfrac{1}{2}\mu mn$, $\text{b} = 8\mu mn$; $\delta = \tfrac{1}{2}\nu mn$, $\text{d} = 2\nu mn$.........(38),

and $$(\beta'+\delta) = (\mu+\nu)\, m\frac{n}{2}, \; (m \text{ is } odd, \; n \text{ even}) \text{ ....(39)}.$$

Thus, for every way in which k can be resolved into two factors k_1, k_2, there exists a system of solutions of type (37, 38), and the final value of N is

$$N = k_1^2 m^4 + 4\,(8\mu^2+\nu^2)\, m^2 n^2 + k_2^2 n^4 \text{ ....(40)}.$$

Note that the coefficient of m^2n^2 is always $\not< 36$, and is independent of k_1, k_2, the factors of k.

Finally, if N be prime ($N = p$), then 2 is necessarily an octavic residue, because p is of the proper quadratic forms (28, 29), and, by (39),

$$\left(\frac{2}{p}\right)_{16} = (-1)^{\frac{1}{2}(\mu+\nu)n} \text{ ....(41)},$$

$$= -1 \text{ or } +1, \text{ according as } \tfrac{1}{2}(\mu+\nu)\, n \text{ is } odd \text{ or } even$$

............(42).

8. *Case of* $\pm k$ *a Perfect Square.*

This case deserves separate detail. It can only happen in Case (3) of (35*d*), viz.,

when $\nu = 4\nu''$, $k = 16\,(\mu^2 - 2\nu''^2)$, where μ is *odd*.........(43).

Here two cases arise according as

$$\mu^2 - 2\nu''^2 = \pm\kappa^2, \text{ and } k = \pm 16\kappa^2, \quad (\kappa \text{ is } odd) \text{.........(44)}.$$

9. *Case* i.

$$\mu^2 - 2\nu''^2 = -\kappa^2, \text{ or } \mu^2 + \kappa^2 = 2\nu''^2, \ k^2 = -16\kappa^2 \ldots\ldots\ldots(45).$$

This is possible when (and only when) ν''^2 is of form

$$\nu''^2 = s^2 + t^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46),$$

in which case $\mu = s \pm t, \quad \kappa = s \mp t$ ....(47).

With this condition (46), the *two* general solutions are

$$\text{a} = \kappa^2 m^2 + 16n^2, \quad \pm\text{c} = \kappa^2 m^2 \sim 16n^2, \quad \theta = \tfrac{1}{2}mn \ \ldots\ldots(48a),$$

or

$$\text{a} = m^2 + 16\kappa^2 n^2, \quad \pm\text{c} = m^2 \sim 16\kappa^2 n^2, \quad \theta = \tfrac{1}{2}mn \ \ldots\ldots(48b),$$

the sign of c being given (in both cases) by $\text{c} \equiv +1$, (mod 8) ...(48*c*);

whence $N = (\kappa m)^4 + 32(\kappa^2 + 2\mu^2)\, m^2 n^2 + (4n)^4$ ....(49*a*),

or $N = m^4 + 32(\kappa^2 + 2\mu^2)\, m^2 n^2 + (4\kappa n)^4$ ....(49*b*).

Hence, if $N = p$ (a prime), $\left(\dfrac{2}{p}\right)_{16} = (-1)^{\frac{1}{2}n}$, [for $(\mu+\nu)$ is *odd*]

$= -1$, if $\frac{1}{2}n$ be *odd*;

or $+1$, if $\frac{1}{2}n$ be *even* ....(50).

10. *Sub-Case of* i. (b = d.)

There is one sub-case of special interest in above, viz.,

when $s = 1, \ t = 0$; giving $\kappa = \mu = \nu'' = 1, \ \nu = 4$ ....(51).

Hence $\beta' = \theta, \quad \delta = 4\theta = 4\beta', \quad \beta' + \delta = 5\beta'$...............(52),

and $\text{b} = \text{d}$; whence $\text{a}^2 = \text{b}^2 + \text{c}^2$.....................(53),

the solutions of which take the simple *unique* form

$\text{a} = m^2 + (4n)^2, \ \pm\text{c} = m^2 \sim (4n)^2, \ \text{b} = \text{d} = 8mn$, ($m$ *odd*, n *even*)...(54),

the sign of c being given by $\text{c} \equiv +1$, (mod 8) ....(54*a*);

whence $N = m^4 + 96m^2 n^2 + (4n)^4$ ....(55),

and, if $N = p$ (a prime), $\left(\dfrac{2}{p}\right)_{16} = (-1)^{\beta'}$

$= -1$, or $+1$, according as $\beta' = \frac{1}{16}\text{b}$

is *odd* or *even* ....(56).

Hence every solution (in integers) of the Pythagorean* triangle

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* This sub-case is specially interesting, as there are published tables of solutions: (1) *Neue und erweiterte Sammlung logarn. trigonn., &c., Tafeln*, by J. C. Schulze, [illegible], II., p. 308, Berlin, 1778; this gives solutions of $\text{a}^2 = \text{b}^2 + \text{c}^2$, *complete* up to

($a^2 = b^2 + c^2$), wherein b is of form 16β, yields a case, when $N = a^2 + b^2 = p$ (a prime), in which 2 is a 16-ic *non-residue* or *residue* according as $\frac{1}{16}b$ is *odd* or *even*.

The other two quadratic partitions are

$$N = (m^2+48n^2)^2 - 2(32n^2)^2 = e^2 - 2f^2 \quad \text{...........(57)},$$

$$N = 2(m^2+16n^2)^2 - (m^2-16n^2)^2 = 2g^2 - h^2 \quad \text{......(58)},$$

and N may also be expressed as the semi-sum of two fourth powers

$$N = \tfrac{1}{2}\left\{(m+4n)^4 + (m-4n)^4\right\} = \tfrac{1}{2}(X^4 + Y^4) \quad \text{.........(59)},$$

under which form it is discussed later (Art. 18).

Examples.—The lowest two solutions are evidently when $m = 1$, $n = 2$ or 4, and they both yield primes ($N = p$).

Ex. 1. $m = 1,\; n = 2,\; N = 4481 = p = 65^2 + 16^2 = 63^2 + 2 \cdot 16^2$,

and $$\left(\frac{2}{p}\right)_{16} = -1.$$

Ex. 2. $m = 1,\; n = 4,\; N = 67073 = p = 257^2 + 32^2 = 255^2 + 2 \cdot 32^2$,

and $$\left(\frac{2}{p}\right)_{16} = +1.$$

On account of the special interest of this case a *complete* list of all primes of this form < 9,000,000 (*i.e.*, within the range of the Factor Tables) is given (Table IV.); the values of a, b, c, d are also given (to show that b = d); these values have been checked off Sang's Tables (as far as they go); the values of X, Y are those entering into equation (59).

11. *Case* ii.

$$\mu^2 - 2\nu''^2 = +\kappa^2, \quad \text{or} \quad k = 16\kappa^2 \quad \text{................(61)}.$$

This is always possible, and its solution is

$$\pm\kappa = 2s^2 \sim t^2, \quad \mu = 2s^2 + t^2, \quad \nu'' = 2st \quad \text{...........(62)},$$

and the *two* general solutions are

$$\pm a = \kappa^2 m^2 \sim 16n^2, \quad c = \kappa^2 m^2 + 16n^2, \quad \theta = \tfrac{1}{2}mn \quad \text{......(63a)},$$

or $$\pm a = m^2 \sim 16\kappa^2 n^2, \quad c = m^2 + 16\kappa^2 n^2, \quad \theta = \tfrac{1}{2}mn \quad \text{......(63b)},$$

fixing the sign of a (in both cases) by $a \equiv +1$, (mod 8) ....(63c);

whence $$N = (\kappa m)^4 + 32(2\mu^2 - \kappa^2) m^2 n^2 + (4n)^4 \quad \text{...........(64a)},$$

or $$N = m^4 + 32(2\mu^2 - \kappa^2) m^2 n^2 + (4\kappa n)^4 \quad \text{...........(64b)}.$$

---

a = 1000, with a few beyond. (2) See *Edinburgh Trans.*, Vol. XXIII., p. 757, 1864; paper by E. Sang, "On the Theory of Commensurables"; this gives solutions of $a^2 = b^2 + c^2$, *complete* up to a = 1105. (3) See Hutton's edition of Ozanam's and Montucla's *Recreations in Mathematics and Natural Philosophy*, p. 43, 1814. Several rules for forming special classes of solutions and also a few examples are given.

Hence, if $N = p$ (a prime),

$$\left(\frac{2}{p}\right)_{16} = (-1)^{\frac{1}{2}n}, \quad [\text{for } (\mu+\nu) \text{ is } odd]$$

$$= -1, \text{ if } \tfrac{1}{2}n \text{ be } odd; \text{ or } +1, \text{ if } \tfrac{1}{2}n \text{ be } even \ldots\ldots (65).$$

12. *Sub-Case of* ii.

In (61) above, take $\mu = 3, \quad \nu'' = 2, \quad \kappa = 1$ ....(66),

giving $\nu = 8, \ \beta' = 3\theta, \ \delta = 8\theta, \ \text{b} = 48\theta, \ \text{d} = 32\theta, \ \theta = \tfrac{1}{2}mn$...(67).

Also, write $\epsilon = 16\theta$; then $N = \text{a}^2 + (3\epsilon)^2 = \text{c}^2 + 2\,(2\epsilon)^2$ ....(68).

Hence $\text{a}^2 + \epsilon^2 = \text{c}^2$..................................(69).

Hence every solution of the Pythagorean triangle (69) in which $\epsilon = 16\theta$ gives a number (N) of the required form; and,

when $N = p$, then $\beta' + \delta = \beta' + 8\theta$,

$$\left(\frac{2}{p}\right)_{16} = (-1)^{\beta'+\delta} = (-1)^{\beta'}$$

$= -1$, or $+1$, according as β' or $\frac{1}{16}$b is *odd* or *even* ...(70),

The general formulæ reduce to

$$\pm\text{a} = m^2 \sim (4n)^2, \ \text{c} = m^2 + (4n)^2, \ \text{b} = 24mn, \ \text{d} = 16mn \ldots (71),$$

fixing the sign of a by $\text{a} \equiv +1$, (mod 8)..............................(71*a*),

and $N = m^4 + 34\,(4mn)^2 + (4n)^4 = 16 \cdot 3\varpi + 1$, *always*......(72),

and note that (71) involves that *one* of $m, n = 3i$ ....(73),

and the third partition is

$$N = (m^2 + 16 \cdot 17n^2)^2 - 2\,(192n^2)^2 \ldots\ldots\ldots\ldots\ldots\ldots (74).$$

Examples.—The following five examples have been compiled with the aid of Shanks's table* of solutions of $\text{a}^2 + \epsilon^2 = \text{c}^2$; they are the only primes of this form ($\text{b} = 3\epsilon$, $\text{d} = 2\epsilon$) within the range of that table (c $\not>$ 1105).

2 a 16-ic Non-Residue.

p	$p-1$	a	b	c	d	
23,761	16 . 27 . 5 . 11	55	144	73	96	
351,361	128 . 9 . 5 . 61	575	144	577	96	
2,121,841	16 . 9 . 5 . 7 . 421	665	1296	793	864	

* Quoted in Art. 10.

2 a 16-ic Residue.						
p	$p-1$	a	b	c	d	
143,953	16 . 3 . 2999	247	288	265	192	
4,098,481	16.3.5.17,077	185	2016	697	1344	

13. *Application of Direct Computation Formulæ.*

Assuming any (mutually prime) values of μ, ν, the value of $k = 2(8\mu^2 - \nu^2)$ is to be calculated, and resolved into factors $k = k_1 k_2$ (one of which may be $= \pm 1$). Each such factorization gives a set of formulæ (37, 38, 40) for a, c, θ, β', b, δ, d, N with any assumed values of m, n (m, n *mutually prime*, also m *odd*, n *even*). As the first object is to ascertain whether N is a prime or not, it will probably be best either to compute N from the formula (40) *direct;* or else to compute *one* of the sets (a, θ, β', b) or (c, θ, δ, d), and thence compute N by the formula

$$N = \mathrm{a}^2 + \mathrm{b}^2 \quad \text{or} \quad = \mathrm{c}^2 + 2\mathrm{d}^2.$$

If any of these numbers turns out to be a prime ($N = p$), it certainly has 2 for an *octavic* residue [being of the proper quadratic forms (28, 29)], so that

$$\left(\frac{2}{p}\right)_{16} = \pm 1;$$

and the value of $\frac{1}{2}(\mu + \nu)\, n$, being odd or even, at once supplies the sextodecimic criterion.

[Thus by this process the difficulty of making the *two* quadratic partitions is avoided; the defect of the process is that, for given values of μ, ν, many numbers N may have to be computed before a prime value of N is found; in fact, it is not known whether prime values of N really exist for *all* values of μ, ν. But some such difficulty exists in most tentative processes.]

Examples.—The following table shows seven examples of primes computed from the general formulæ (37, 38, 40). They have been chosen as the lowest three primes for which $\left(\frac{2}{p}\right)_{16} = +1$ (or $e = 16$), and the lowest three primes for which $\left(\frac{2}{p}\right)_{16} = -1$ (or $e = 8$), and the lowest of the Pythagorean kind.

Assumptions.							Results—Quadratic Partitions.						Final Results.	
μ	ν	k	k_1	k_2	m	n	β'	δ	a	b	c	d	p	e
1	1	14	7	2	1	2	1	1	1	16	15	4	257	16
1	3	−2	−1	2	1	2	1	3	9	16	7	12	337	16
1	5	−34	−17	2	1	2	1	5	25	16	9	20	881	16
1	2	8	1	8	1	2	1	2	31	16	33	8	1217	8
2	3	46	23	2	1	2	2	3	15	32	31	12	1249	8
2	1	62	31	2	1	2	2	1	23	32	39	4	1553	8
1	4	−16	−1	16	1	2	1	4	65	16	63	16	4481	8

14. *Factors of Numbers of 16-ic Form.*

When a number N of the form (40) just investigated turns out to be *composite*, and has been separated into its factors, the residuary relation of the base 2 (and of other bases also) to those factors can often be *readily* determined.

Let $$N = N_1 . N_2, \quad \text{and} \quad N_1 < N_2 \quad\text{..................(75).}$$

Let $$N = \mathrm{a}^2+\mathrm{b}^2 = \mathrm{c}^2+\mathrm{d}^2 \quad\text{......................(76),}$$

$$N_1 = \mathrm{a}_1^2+\mathrm{b}_1^2 = \mathrm{c}_1^2+\mathrm{d}_1^2 \quad\text{......................(76a),}$$

$$N_2 = \mathrm{a}_2^2+\mathrm{b}_2^2 = \mathrm{c}_2^2+\mathrm{d}_2^2 \quad\text{......................(76b).}$$

Then a, b, c, d are supposed to have been formed along with N, and N_1, N_2 are supposed *known*. Now, it will often happen that the lesser factor N_1 is *so small* that its a_1, b_1, c_1, d_1 are comparatively easily determined. These being all known, the quadratic partitions of N_2 are given by the formulæ*

$$N_2 = \frac{N}{N_1} = \frac{\mathrm{a}^2+\mathrm{b}^2}{\mathrm{a}_1^2+\mathrm{b}_1^2} = \left(\frac{\mathrm{a}_1\mathrm{a}\pm\mathrm{b}_1\mathrm{b}}{\mathrm{a}_1^2+\mathrm{b}_1^2}\right)^2 + \left(\frac{\mathrm{b}_1\mathrm{a}\mp\mathrm{b}\mathrm{a}_1}{\mathrm{a}_1^2+\mathrm{b}_1^2}\right)^2 = \mathrm{a}_2^2+\mathrm{b}_2^2 \ldots(77a),$$

$$N_2 = \frac{N}{N_1} = \frac{\mathrm{c}^2+2\mathrm{d}^2}{\mathrm{c}_1^2+2\mathrm{d}_1^2} = \left(\frac{\mathrm{c}_1\mathrm{c}\pm 2\mathrm{d}_1\mathrm{d}}{\mathrm{c}_1^2+2\mathrm{d}_1^2}\right)^2 + 2\left(\frac{\mathrm{d}_1\mathrm{c}\mp\mathrm{d}\mathrm{c}_1}{\mathrm{c}_1^2+2\mathrm{d}_1^2}\right)^2 = \mathrm{c}_2^2+\mathrm{d}_2^2 \ldots(77b).$$

---

* These formulæ are taken from Mr. C. E. Bickmore's paper "On the Numerical Factors of (a^n-1)," in the *Messenger of Mathematics*, Vol. xxv., 1895, p. 10, Art. 8. The formulæ for the resolution of $N_2 = \mathrm{a}_2^2 + k\mathrm{b}_2^2$, when $N_2 = N \div N_1$, are quite similar; proof is there given that the algebraic portions of the formulæ are *arithmetical integers* for one of the two signs $\pm$ (either the upper or lower) in each case.

When N_1 is *prime*, its partitions (a_1, b_1, c_1, d_1) are *unique;* in consequence of the ambiguity of sign, *two* trials may be necessary to find a_1, b_1, and *two* trials to find c_1, d_1 (but not more than two in each case); the process is otherwise quite direct. When N_1 is *composite*, it admits of more than one partition of each type (a_1, b_1), (c_1, d_1); and each individual partition of N_1 gives rise to two formulæ for a partition of N_2, so that several trials may be necessary before a *correct* partition of N_2 is found. By the phrase "correct partition" is meant one in which both of a_2, b_2, and both of c_2, d_2 are *integers.*

If N_2 should be *composite*, the process may be repeated, until finally the quadratic partitions of the last remaining *prime* factor are determined. Thus the knowledge of the quadratic partitions of a large composite number N gives a direct clue to the similar quadratic partitions of its large factor when those of its other factor or factors are known.

When N_1, N_2 are both *prime*, it appears* that the base 2 is a residue of the *same even order*, viz., either 2, 4, 8, or 16 of *both* primes N_1, N_2; so that the determination of the (even) order for either prime suffices for the other, without the necessity of effecting the quadratic partitions of the second factor. This is conveniently expressed thus:

Let Q, B, O, S denote *primes*, such that

$$\left(\frac{2}{Q}\right)_2 = 1, \quad \left(\frac{2}{B}\right)_4 = 1, \quad \left(\frac{2}{O}\right)_8 = 1, \quad \left(\frac{2}{S}\right)_{16} = 1 \quad \ldots\ldots(78),$$

the suffix denoting the *maximum* residue-index of form

$$e = 2, \quad (e \not> 16).$$

$$\left.\begin{array}{l}\text{Then } N_1, N_2 \text{ are both } Q, \text{ or both } B, \text{ or both } O, \text{ or both } S \\ \quad \text{when } N \text{ is of form (40), and } (\beta' + \delta) \text{ is } \textit{even}\end{array}\right\} \ldots(79).$$

15. *Simple* 16-*ic Forms.*

The general formula (40) for N to be a prime of such form that 2 may be a 16-ic residue of it shows that all such primes must be of form

$$N = p = u \,.\, X^4 + v \,.\, X^2 Y^2 + w \,.\, Y^4 \ldots\ldots\ldots\ldots\ldots\ldots(80).$$

The formula (40) contains *four* arbitrary quantities (μ, ν, m, n), but the above form makes it seem probable that *three* (at most) should be sufficient. There are several simple forms of the above, which

* This is part of a more general theorem about the order of residuacity of the base with respect to the prime factors of numbers N of other quadratic forms than those here considered, which it is hoped to publish hereafter.

seem worth some detailed consideration. These are

$N = X^4 + Y^4$,	[see Arts. 16, 17]
$N = \frac{1}{2}(X^4 + Y^4)$,	[see Art. 18]
$N = X^4 + 4Y^4$,	[see Art. 19]
$N = (X^{12} + Y^{12}) \div (X^4 + Y^4)$,	[see Art. 20]
$N = (X^{20} + Y^{20}) \div (X^4 + Y^4)$,	[see Art. 21]
$N = (X^{4\varpi} + Y^{4\varpi}) \div (X^4 + Y^4)$,	[see Art. 22]

16. *Form* (X^4+Y^4).

In all cases

$$N = (X^2)^2+(Y^2)^2 = (X^2 \sim Y^2)^2+2(XY)^2$$
$$= (X^2+Y^2)^2-2(XY)^2 = 2(X^2-XY+Y^2)^2-(X \sim Y)^4 \quad \ldots (81).$$

Thus the four quadratic partitions

$$N = \mathrm{a}^2+\mathrm{b}^2 = \mathrm{c}^2+2\mathrm{d}^2 = \mathrm{e}^2-2\mathrm{f}^2 = 2\mathrm{g}^2-\mathrm{h}^2$$

are effected algebraically in simple forms, so that the numerical resolution can be effected *directly* (but it will be found not to be required for the present purpose).

Next, let N be an *odd* number: this involves that X, Y must be one odd, one even; take X *odd* and Y *even*.

Then $\quad X^4 = 16x+1, \quad Y^4 = 16y^4, \quad N = 16m+1,$ *always* ......(82),

and,
$$\left.\begin{array}{ll} \text{if } X \text{ or } Y = 3i, & N = 16.3m+1 \\ \text{if } X \text{ or } Y = 5i, & N = 16.5m+1 \equiv \epsilon 1^* \pmod{100} \\ \text{if } X \text{ and } Y \not\equiv 5i, & N \equiv \omega 7^* \pmod{100} \end{array}\right\} \ldots (83).$$

Also, if $X = \Omega.\omega$ (the product of two *odd* numbers),

Then
$$X^4 = \Omega^4.\omega^4 \equiv \omega^4, \quad [\text{mod } (2\omega)^4],$$
$$\equiv \Omega^4, \quad [\text{mod } (2\Omega)^4];$$

therefore
$$\left.\begin{array}{ll} N = \Omega^4\omega^4+Y^4 \equiv \omega^4+Y^4 & [\text{mod } (2\omega)^4] \\ \equiv \Omega^4+Y^4 & [\text{mod } (2\Omega)^4] \end{array}\right\} \ldots (84).$$

This shows that, if N be expressed in the scale whose radix is $r = 2\omega$ or 2Ω, the last *four* figures of all such numbers are the same for the same value of Y, *i.e.*, the same as those of (ω^4+Y^4) or of (Ω^4+Y^4) respectively.

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* Here $\epsilon 1$ and $\omega 7$ are to be read *arithmetically*; ω, ϵ being *odd* and *even* numbers respectively, denoting the *tens* digit of N (and 1 or 7 the units of N).

Ex.—Take $\omega = 5$, $N = (5^4\Omega^4 + Y^4)$. All such numbers end in the same *four* figures as $(5^4 + Y^4)$. *E.g.*, $5^4+8^4 = 4721$, $15^4+8^4 = 54721$, $25^4+8^4 = 394721$, &c.

Referring to (81), it is seen that

$$d = f; \quad \text{whence} \quad e^2 = c^2 + (2d)^2 \qquad (85).$$

Hence every solution of this Pythagorean triangle (in e, c, 2d) will lead to numbers of the form now discussed.

Next, let N be *prime*, say $N = p$.

Then, noting that $X^2 = (\text{odd number})^2 = 8a+1$, and $Y = 2y$ (being even), *always*,

(1) Since $p = 16m+1$, therefore $\left(\frac{2}{p}\right)_2 = +1$, *always.*

(2) Since $p = (8a+1)^2 + (4y^2)^2$, *always*, then, comparing with Gauss's condition (13) for quartic residues, $\beta_0 = y^2$, so that

$$\left(\frac{2}{p}\right)_4 = -1, \text{ if } y \text{ be } odd; \quad \left(\frac{2}{p}\right)_4 = +1, \text{ if } y \text{ be } even.$$

(3) Next, taking $y = 2\eta$ (an *even* number), so that $Y = 4\eta$, then

$$p = (8a+1)^2 + (16\eta^2)^2.$$

Comparing with (Bickmore's form of) Reuschle's condition (19) for ctavic residues, $\beta' = \eta^2$, so that

$$\left(\frac{2}{p}\right)_8 = +1, \textit{ always}, \text{ (when } Y = 4\eta).$$

Also, taking the partition

$$p = c^2 + 2d^2 = (X^2 - Y^2)^2 + 2(XY)^2,$$

here $\qquad d = 4\delta = XY = 4X\eta;$

therefore $\qquad \delta = X\eta;$

therefore $\beta' + \delta = \eta^2 + X\eta = \eta(X+\eta) =$ an *even* number ($\because X$ is *odd*);

therefore $\qquad \left(\frac{2}{p}\right)_{16} = +1.$

Summing up these results as to $N = (X^4 + Y^4) = p$:—

$$\left(\frac{2}{p}\right)_2 = +1, \textit{ always}; \quad \left(\frac{2}{p}\right)_4 = -1, \text{ when } Y = 2\omega \qquad (86),$$

$$\left(\frac{2}{p}\right)_{16} = \left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = +1, \text{ when } Y = 4\eta \qquad (87).$$

In this last case ($Y = 4\eta$), N may be written

$$N = X^4 + 2^8 . \eta^4;$$

hence 2 is a *sextodecimic* residue of all primes of form $p = X^4 + 2^8 . \eta^4$, and therefore also of all primes of form $p = X^4 + Y^8$, (Y being *even*).

It is interesting to compare this form $[X^4 + (4\eta)^4]$ with the general form (40) of all primes having 2 for a 16-ic residue. It will be found, after some considerable algebraic reductions (which it seems unnecessary to detail here), that the form (40) reduces to $N = (X^4 + Y^4)$ in one way only, viz., by making

$$\mu = \eta = \tfrac{1}{4} Y, \quad \nu = X \text{ ....(88).}$$

These give $\theta = \mu$, and the factors of k must be taken

$$k_1 = 8\mu^2 - \nu^2 = 8\eta^2 - X^2, \quad k_2 = 2 \text{ ....(89);}$$

whence, finally, $\quad m = \pm 1, \quad n = \pm 2\eta = \pm \tfrac{1}{2} Y$ ....(90),

and the sextodecimic criterion (42), $\tfrac{1}{2}(\mu + \nu)\, n =$ *even* number is satisfied.

Thus for *any* given values of μ, ν ($\mu = \eta = \tfrac{1}{4} Y, \ \nu = X$), the general formula (40) always gives $N = X^4 + (4\eta)^4$, by taking the values of k_1, k_2, m, n as above (so that the factorization of $k = k_1 . k_2$, and the values of m, n are here *not arbitrary*).

Examples.—The following are the lowest primes of form $\mu = X^4 + Y^4$, showing the connexion with the general formula (40).

Assumptions.		Results.			
$\mu = \eta$	$\nu = X$	k_1	k_2	n	p
1	1	7	2	2	257
1	3	− 1	2	2	337
1	5	−17	2	2	881

On account of the simplicity of the form $(X^4 + Y^4)$ of these primes, and also on account of their important relation to other bases (developed in next article), it has been thought worth while to give a *complete* list of all those within the range of the existing factor-tables, *i.e.*, of all $< 9{,}000{,}000$. These are given in Table III. This requires no explanation. The primes are seen to be distributed very irregularly, except that the number in each million decreases on the whole (as might be expected) as the numbers increase, as shown below—

16 under 100,000; 25 between 100,000 and 1,000,000.

Abstract										
Name of million ...	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	Total
Number of primes in million	41	16	9	8	7	9	3	9	7	109

The lowest is $p = 257 = 1^4 + 4^4$; and *one** (the highest known to the author) has been discovered beyond the range of the factor-table, viz.,

$$p = 10^8 + 81 = 3^4 + (4 \,.\, 25)^4.$$

Resolution into complex factors.—Writing $\rho = \sqrt[4]{-1} = (1 + \sqrt{-1}) \div \sqrt{2}$, then

$$N = X^4 + Y^4 = (X + Y\rho)(X - Y\rho)(X + Y\rho^3)(X - Y\rho^3) \quad \text{..............(91)},$$

17. *Relation to other Bases.*

The fact of 2 being known to be a 16-ic residue of primes of form

$$p = X^4 + Y^4 = X^4 + (4\eta)^4$$

often enables† the residue-index (e) of either X or K (where K is the odd factor in Y) to be determined for *even* orders ($e = 2, 4, 8$, &c.), and sometimes to a high order, in two cases:

(1) When either X or $K = 1$.

(2) When the residue-index e of either X or K is known.

The residue-index e depends on the power of 2 which enters into X, Y, p: thus, writing

$$X = 2^h \,.\, H \pm 1, \quad Y = 2^k \,.\, K, \quad p = 2^{\varpi} \,.\, \Omega + 1 \quad \text{.................(92)},$$

wherein H, Ω are *odd* (K may be odd or even)

$$\left.\begin{array}{lll} \text{and} & 2^h \not< 4, \quad 2^k \not< 4, \quad 2^{\varpi} \not< 16 \\ \text{or} & h \not< 2, \quad k \not< 2, \quad \varpi \not< 4 \end{array}\right\} \quad \text{.........................(93)}.$$

$$\text{Then} \qquad \left.\begin{array}{l} 2^{\varpi} = \text{the lesser of } 4 \,.\, 2^h,\ 2^{4k}, \text{ but } \not< 16 \\ \varpi = \text{the lesser of } (h+2),\ 4k \end{array}\right\} \quad \text{.....................(94)}.$$

Now $$p = X^4 + 2^{4k} \,.\, K^4;$$

therefore $$X^4 \equiv -2^{4k} \,.\, K^4, \quad (\text{mod } p).$$

Also this may be raised to any power $\frac{p-1}{4e}$, provided e be such that $\frac{p-1}{4e}$ be an integer. Then

$$X^{(p-1)/e} \equiv (-1)^{(p-1)/4e} \,.\, 2^{k \,.\, (p-1)/e} \,.\, K^{(p-1)/e} \quad \text{........................(95)}.$$

Now, let e be determined, such that

$$2^{k \,.\, (p-1)/e} \equiv 1, \quad \text{subject to } \frac{p-1}{4e} = \text{integer} \quad \text{........................(96)}.$$

Then $$X^{(p-1)/e} \equiv (-1)^{(p-1)/4e} \,.\, K^{(p-1)/e} \quad \text{..............................(97)}.$$

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* Given in a list of 99 primes > 100 millions in a "letter" by Mr. W. Davis, printed in the *Journal de Math. pures et appliqués*, 2e série, t. xi., 1866.

† This method is a development of one given on p. 24 of Mr. C. E. Bickmore's paper before quoted (since found to have been used by Euler; see *Nov. Com. Acad. Petr.*, t. I., p. 45).

Thus $$X^{(p-1)/e} \equiv -K^{(p-1)/e}, \text{ when } \frac{p-1}{4e} \text{ is } odd \text{ ......(97}a\text{)},$$

$$\equiv +K^{(p-1)/e}, \text{ when } \frac{p-1}{4e} \text{ is } even \text{ ......(97}b\text{)}.$$

Hence $\left(\frac{X}{p}\right)_e$ or $\left(\frac{K}{p}\right)_e = (-1)^{(p-1)/4e}$, when K or $X = 1$ respectively ...(98a).

Also $\left(\frac{X}{p}\right)_e$ or $\left(\frac{K}{p}\right)_e = (-1)^{(p-1)/4e}.(\pm 1)$, when $\left(\frac{2}{K}\right)_e$ or $\left(\frac{2}{X}\right)_e = \pm 1$ is known ....(98b).

These last (98a, b) are the required results, determining the residue-index e of X or K.

To determine e as above, write

$$e = 2^{\epsilon}, \text{ and suppose } k = 2_{\kappa}.\omega \text{ ......(99);}$$

then the exponent of 2 in (96) becomes

$$k.\frac{p-1}{e} = 2^{\kappa}.\omega.\frac{p-1}{2^{\epsilon}} = \omega.\frac{p-1}{2^{\epsilon-\kappa}} \text{ ......(100).}$$

The object being to determine the *maximum* residue-index (of even order) of X or K, it is required to find the maximum value of $e = 2^{\epsilon}$ satisfying the conditions (96). Noting that $p-1 = 2^{\varpi}.\Omega$, this is met by taking

$$\epsilon-\kappa = \text{the greatest of } 0, 1, 2, 3, 4; \text{ but } \epsilon+2 \not> \varpi \text{ ......(101),}$$

whence, max. ϵ = the greatest of κ, $\kappa+1$, $\kappa+2$, $\kappa+3$, $\kappa+4$; but $\not> \varpi-2$...(102a),

and $$\text{max. residue-index, } e = 2^{\max.\epsilon} \text{ ......(102}b\text{).}$$

If this maximum e gives $(-1)^{(p-1)/4e} = -1$ (as is likely), then

$$\left(\frac{X}{p}\right)_e \text{ or } \left(\frac{K}{p}\right)_e = -1, \text{ when } K \text{ or } X = 1 \text{ respectively ......(103),}$$

so that $$\left(\frac{X}{p}\right)_{\frac{1}{2}e} \text{ or } \left(\frac{K}{p}\right)_{\frac{1}{2}e} = +1 \text{ ......(103}a\text{),}$$

and $\frac{1}{2}e$ is the effective maximum residue-index of X or K in this case.

Examples.—Make $e = 2, 4, 8$, &c., in the fundamental congruence

(1) $e = 2$, $X^{\frac{1}{2}(p-1)} = +K^{\frac{1}{2}(p-1)}$, *always*......(109),

(2) $e = 4$, $X^{\frac{1}{4}(p-1)} = -K^{\frac{1}{4}(p-1)}$, if $p = 16\Omega+1$ } ......(110).
$\qquad = +K^{\frac{1}{4}(p-1)}$, if $p = 32i+1$ }

Ex. (1) shows that X, K are always both quadratic residues, or both quadratic non-residues.

Examples.—The following examples taken from Table III. of primes $p = X^4 + (4\eta)^4$

show *all* the cases of such primes in which the residue-index ($\frac{1}{2}e$) of the base (X or K) is not < 8 (one case of 64), so that, in fact,

$$\left(\frac{X}{p}\right)_{\frac{1}{2}e} \text{ or } \left(\frac{K}{p}\right)_{\frac{1}{2}e} = 1.$$

Such cases are difficult to find by any other method. *Many* cases of residue-index $\frac{1}{2}e = 4$ can be found from that table, but they were not thought worth detailing.

p	X	$Y = 2^k . K$	$\frac{p-1}{\Omega}$	Results.	
				X or K	$\frac{1}{2}e$
54,721	15	8 . 1	64	15	8
83,777	17	4 . 1	64	17	8
149,057	17	16 . 1	64	17	8
160,001	1	4 . 5	256	5	32
331,777	1	4 . 6	4,096	6	32
331,777	1	8 . 3	4,096	3	16
614,657	1	4 . 7	256	7	32
1,972,097	31	32	128	31	16
4,879,937	47	4	64	47	8
5,308,417	1	16 . 3	65,536	3	64
5,308,417	1	8 . 6	65,536	6	32
5,308,417	1	4 . 12	65,536	12	32

18. *Form* $\frac{1}{2}(X^4+Y^4)$.

In all cases

$$\left.\begin{aligned} N &= \left(\frac{X^2+Y^2}{2}\right)^2 + \left(\frac{X^2\sim Y^2}{2}\right)^2 = (XY)^2 + 2\left(\frac{X^2\sim Y^2}{2}\right)^2 \\ &= (X^2-XY+Y^2)^2 - 2\left(\frac{X^2+XY+Y^2}{2}\right)^2 = 2\left(\frac{X^2+Y^2}{2}\right)^2 - (XY)^2 \end{aligned}\right\} \quad \text{...............(111).}$$

Thus the four quadratic partitions are effected *algebraically.*

Now, take N an *odd* number; this involves X, Y *both odd.* This form will be found to possess many properties which are, in a certain sense, *complementary* to those of the form (X^4+Y^4), and it is interesting to compare them. It is easy to see that

$$\mathrm{a} = \tfrac{1}{2}(X^2+Y^2) = 4\alpha+1,\ \ \mathrm{b} = \tfrac{1}{2}(X^2-Y^2) = 4\beta;\ \ N = 8m+1,\ \textit{always} \quad \text{..................(112),}$$

$$\left.\begin{aligned} &\text{and, if } X \text{ and } Y \neq 3i, \text{ then } N = 8\,.\,3m+1 \\ &\text{if } X \text{ and } Y \neq 5i, \text{ then } N = 8\,.\,5m+1 \equiv \epsilon 1,^{*} \pmod{100} \\ &\text{if } X \text{ or } Y = 5i, \text{ then } N \equiv \omega 3,^{*} \pmod{100} \end{aligned}\right\} \text{...(113).}$$

Also, as in Art. 16, if $X = \Omega\omega$, (the product of two *odd* numbers), then

$$N = \tfrac{1}{2}(\Omega^4\omega^4 + Y^4) \equiv \tfrac{1}{2}(\omega^4 + Y^4), \quad [\text{mod } (2\omega)^3],$$
$$\equiv \tfrac{1}{2}(\Omega^4 + Y^4), \quad [\text{mod } (2\Omega)^3] \text{ ....(114),}$$

which shows that, if N be expressed in the scale whose radix is $r = 2\omega$ or 2Ω, the last *three* figures of all such numbers are the same for the same value of Y.

Ex.—Take $\omega = 5, \quad N = \tfrac{1}{2}(5^4\,.\,\Omega^4 + Y^4).$

All such numbers end in the same *three* figures as $\tfrac{1}{2}(5^4 + Y^4)$; *e.g.*,

$$\tfrac{1}{2}(5^4 + 3^4) = 353, \quad \tfrac{1}{2}(15^4 + 3^4) = 25{,}353, \text{ \&c.}$$

Next, supposing $X > Y$, and noting that one of $(X \pm Y)$ is necessarily divisible by 2, and the other by 4, it is permissible to write

$$m = \tfrac{1}{2}(X \pm Y), \quad n = \tfrac{1}{4}(X \mp Y), \text{ wherein } m \text{ is } odd \text{ ...(115),}$$

which gives $\quad X = m + 2n, \quad \pm Y = m \sim 2n$ ....(115*a*),

$$\mathrm{a} = \tfrac{1}{2}(X^2 + Y^2) = m^2 + (2n)^2, \quad \mathrm{b} = \tfrac{1}{2}(X^2 - Y^2) = 4mn = \mathrm{d} \text{...(116),}$$

$$N = m^4 + 24m^2n^2 + n^4 \text{ ....(117).}$$

Referring to (111), it is seen that

$$\mathrm{b} = \mathrm{d}, \quad \mathrm{a} = \mathrm{g}, \quad \mathrm{c} = \mathrm{h} \text{ ....(118);}$$

whence $\quad \mathrm{a}^2 = \mathrm{b}^2 + \mathrm{c}^2 = \mathrm{c}^2 + \mathrm{d}^2 = \mathrm{b}^2 + \mathrm{h}^2 = \mathrm{g}^2$...........(118*a*).

Thus every solution of any of these Pythagorean equations leads to numbers of the form now discussed. Thus also these numbers are the same as those discussed in Art. 10, and the final forms (55), (117) in m, n are the same† when $2n$ is written for n in (117).

Now, let N be prime, say $N = p$: then making $n = \omega, 2\omega, 4\omega$, &c., and attending to the criteria for 2-ic, 4-ic, 8-ic, and 16-ic residues (Arts. 3, 4), the following table may be drawn up, showing at a

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* Here $\epsilon 1$, $\omega 3$ are to be read *arithmetically*, ω, ϵ being *odd* and *even* digits, respectively.

† The apparent discrepancy between forms (55), (117) arises from the discussion in Art. 10 being confined to numbers $N = 16\varpi + 1$, which involves n being *even*, whereas those of this article include the form $N = 8\varpi + 1$.

glance the *maximum* residue-index (e), which makes $\left(\frac{2}{p}\right)_e = +1$, and consequently $\left(\frac{2}{p}\right)_{2e} = -1$ in most cases.

When $n = \frac{1}{4}(X \mp Y) =$	ω	2ω	4ω	8ν
Then a =	$4\omega+1$	$8\alpha+1$	$8\alpha+1$	$8\alpha+1$
b = d =	4ω	8ω	16ω	32ι
$p-1=$	8ϖ	16ϖ	16ϖ	16ϖ
$\left(\frac{2}{p}\right)_e = 1$, when $e =$	2	4	8	16
$\left(\frac{2}{p}\right)_{2e} = -1$, when $e =$	4	8	16	?

...(119).

It will be seen that $e = 2n \div \omega$ as far as $e = 16$ (beyond which the conditions as to residues are not* known).

Examples.—Table IV. contains a *complete* list of all the primes of this form < 9 million (*i.e.*, within the range of the Factor-Tables), when $n = \frac{1}{4}(X \mp Y) = 4\nu$, of which therefore 2 is an 8-ic or 16-ic residue, according as ν is odd or even.

19. *Form* $N = (X^4+4Y^4)$.

This number is always† composite, but its factors (say N_1, N_2) present some interest. Taking $N_1 < N_2$,

$$N_1 = X^2-2XY+2Y^2 = (X\sim Y)^2+Y^2 \text{ ....(120}a\text{)},$$

$$N_2 = X^2+2XY+2Y^2 = (X+Y)^2+Y^2 \text{ ....(120}b\text{)}.$$

Case i.—Taking X *odd*, Y *even* ($= 2y$), it is easy to see that when N_1, N_2 are both primes (say $= p_1, p_2$), then 2 may be a residue of *some even order* (2, 4, or 8) of both p_1, p_2, so that

(1) If $Y = 2\omega$, $\left(\frac{2}{p_1}\right)_2 = \left(\frac{2}{p_2}\right)_2 = -1$ ....(121a),

* This property will not extend beyond $e = 16$ without some additional condition to insure that $p-1 = 32\varpi$.

† This resolution is due to the late M. Aurifeuille; the case when $Y = 1$ was given by Sophie Germain.

(2) If $Y = 4\omega$, $\left(\frac{2}{p_1}\right)_2 = \left(\frac{2}{p_2}\right)_2 = +1$; $\left(\frac{2}{p_1}\right)_4 = \left(\frac{2}{p_2}\right)_4 = -1$ ....(121*b*),

(3) If $Y = 8y$, $\left(\frac{2}{p_1}\right)_4 = \left(\frac{2}{p_2}\right)_4 = +1$ ....(121*c*),

whence, writing $X = 4x \pm 1$ and comparing with (Bickmore's form of) Reuschle's Rule,* $\alpha = x \sim 2y$ for p_1, $\alpha = x + 2y$ for p_2, $\beta = y$; noting that p_1 and p_2 are each of form $(8n+1)$; therefore

$$\left.\begin{aligned}\left(\frac{2}{p_1}\right)_8 = \left(\frac{2}{p_2}\right)_8 &= -1, \quad \text{when } (x \mp y) \text{ is } \textit{odd} \\ &= +1, \quad \text{when } (x \mp y) \text{ is } \textit{even}\end{aligned}\right\} \ldots(121d).$$

It does not seem possible to pursue this to the 16th order *generally*, because the second quadratic partitions (of form $c^2 + 2d^2$) cannot be effected algebraically. There is one case, however, of some interest, viz., when

$$X \sim Y = U^2, \quad X + Y = V^2, \quad Y = W^2,$$

which give $X = \tfrac{1}{2}(V^2 + U^2)$, $Y = \tfrac{1}{2}(V^2 - U^2) = W^2$.

The general solution of these diophantine equations (in U, V, W) is

$$U = s^2 \sim 2t^2, \quad V = s^2 + 2t^2, \quad W = 2st, \ (s \text{ odd}, \ t \text{ even}) \ldots\ldots(122);$$

whence $X = s^4 + 4t^4$, $Y = (2st)^2$,

$$N_1 = (s^2 \sim 2t^2)^4 + (2st)^4, \quad N_2 = (s^2 + 2t^2)^4 + (2st)^4,$$

and, finally, $N = (s^4 + 4t^4)^4 + 4(2st)^8$ ....(123).

Thus, when the two factors N_1, N_2 of a number N of this last form are both prime, 2 is a 16-ic residue *of both* (because t is even); they thus furnish an example of the theorem of result (79), Art. 14.

Examples.—Cases in which *both* N_1, N_2 are prime are generally beyond the range of the large Factor-Tables; the only example within this range is given by

$s = 5$, $t = 2$; giving $U = 17$, $V = 33$, $W = 20$, $X = 689$, $Y = 400$;

whence $N_1 = p_1 = 243{,}521 = 17^4 + 20^4$ and $N_2 = p_2 = 1{,}345{,}921 = 33^4 + 20^4$, the factors of $N = 689^4 + 4 \cdot 400^4$,

and 2 is a 16-ic residue of both N_1, N_2.

Case ii. When X, Y are both *odd*, it may be easily seen that

$$\left(\frac{2}{p_1}\right)_2 = \mp 1, \quad \left(\frac{2}{p}\right)_2 = \pm 1, \quad \textit{always}.$$

* See footnote Art. 3, iii.

But, as the 16-ic character of 2 cannot be traced in a general manner, it is not worth while pursuing this case; it appears, in fact, that in this case the factors N_1, N_2 cannot both be of the form $\frac{1}{2}(x^4+y^4)$ discussed in Art. 18.

20. *Form* $N = (X^{12}+Y^{12}) \div (X^4+Y^4)$.

This form is of considerable interest. In all cases

$$N = X^8 - X^4Y^4 + Y^8 \qquad \text{(124)},$$

$$= (X^4 \sim Y^4)^2 + (XY)^4 \qquad = \mathrm{a}^2+\mathrm{b}^2 \qquad (124a),$$

$$= (X^4 - X^2Y^2 + Y^4)^2 + 2\{XY(X^2-Y^2)\}^2 = \mathrm{c}^2+2\mathrm{d}^2 \qquad (124b),$$

$$= (X^4 + X^2Y^2 + Y^4)^2 - 2\{XY(X^2+Y^2)\}^2 = \mathrm{e}^2-2\mathrm{f}^2 \qquad (124c),$$

$$= 2(X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4)^2 - (X^4 - 2X^3Y + X^2Y^2 - 2XY^3 + Y^4)^2 = 2\mathrm{g}^2 - \mathrm{h}^2 \qquad (124d),$$

$$= (X^4 - \tfrac{1}{2}Y^4)^2 + 3\,(\tfrac{1}{2}Y^4)^2 \qquad = A^2+3B^2, \text{ (when } Y = 2y) \qquad (124e),$$

$$= \left(\frac{X^4+Y^4}{2}\right)^2 + 3\left(\frac{X^4\sim Y^4}{2}\right)^2 \qquad = A^2+3B^2, \text{ (when } X, Y \text{ both odd)} \qquad (124f),$$

$$= \tfrac{1}{4}\{(2A)^2 + 27\,(\tfrac{2}{3}B)^2\} \qquad = \tfrac{1}{4}(L^2+27M^2), \text{ (when } B = 3\mathcal{E}) \qquad (124g),$$

$$= \tfrac{1}{4}\left\{(A\pm 3B)^2 + 27\left(\frac{A\mp B}{3}\right)^2\right\} = \tfrac{1}{4}(L^2+27M^2), \text{ (when* } A \text{ and } B \neq 3\mathcal{E}) \qquad (124h).$$

Thus the six quadratic partitions (124*a*–*h*) which are most often required† for the 2-ic, 3-ic, 4-ic, 8-ic, &c., residuary character of *each* of the bases 2, 3, 5, 6, 7, 10 are effected algebraically in simple forms, and the numerical work can be effected directly.

Taking N an *odd* number, which involves *one* of X, Y (say X) being also *odd*, it may be shown, as in Art. 16, that

$$N = 16.3.5.m+1 \equiv \epsilon 1, \quad (\text{mod } 100) \qquad (125),$$

and that, if

$$X = \Omega\omega,$$

then

$$\left.\begin{aligned} N &\equiv \omega^8 - \omega^4Y^4 + Y^8, \quad [\text{mod } (2\omega)^4] \\ &\equiv \Omega^8 - \Omega^4Y^4 + Y^8, \quad [\text{mod } (2\Omega)^4] \end{aligned}\right\} \qquad (126).$$

This shows that, if N be expressed in the scale whose radix is $r = 2\omega$

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* In result (124*h*) one of $(A\mp B)$ must be a multiple of 3, and M is to be taken = the *integer* value of $\frac{1}{3}(A\mp B)$, and the opposite sign is to be used for $L = (A\pm 3B)$; this removes the ambiguity of sign.

† See Bickmore's paper quoted, Arts. 13–17.

[redacted]

or 2Ω, the last *four* figures of all such numbers are the same for the same value of Y.

Example.—Take $\omega = 5$, $N = 5^8\Omega^8 - 5^4\Omega^4 Y^4 + Y^8$; all such numbers end in the same *four* figures as $(5^8 - 5^4 Y^4 + Y^8)$.

Case i. Taking X odd, Y even, it may be shown—by reasoning similar to that of Art. 16—that, when $N = p$ (a prime),

(1) $\left(\frac{2}{p}\right)_2 = +1$, always; and $\left(\frac{2}{p}\right)_4 = -1$, when $Y = 2\omega$...(127*a*),

(2) When $Y = 4\eta$, then $\left(\frac{2}{p}\right)_{16} = \left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = +1$......(127*b*).

Again, the partition (124*e*) shows, comparing with Jacobi's* rule for the cubic character of 2, that

$\left(\frac{2}{p}\right)_3 = 1$, when Y is a multiple of 6 (but not otherwise)...(128).

As the cubic condition does not interfere with the 2-ic, 4-ic, 8-ic, and 16-ic conditions, it follows that

When $Y = 6\omega$, then $\left(\frac{2}{p}\right)_6 = +1$, $\left(\frac{2}{p}\right)_{12} = -1$..............(128*a*),

When $Y = 12\eta$, then $\left(\frac{2}{p}\right)_{48} = \left(\frac{2}{p}\right)_{24} = \left(\frac{2}{p}\right)_{12} = \left(\frac{2}{p}\right)_6 = +1$ ....(128*b*).

Case ii.—Taking X, Y *both odd;* and taking $X > Y$, and writing (as in Art. 18)

$$X = m + 2n, \quad Y = m \sim 2n,$$

which give $\quad \mathrm{a} = (m^2 - 4n^2)^2, \quad \mathrm{b} = 16\,(m^2 + 4n^2)\,mn,$

$$\mathrm{d} = 8\,(m^2 - 4n^2)\,mn,$$

and it is easy to see that $N = 16\varpi + 1$ always; hence, attending to the criteria of Arts. 3, 4, it follows that (when $N = p$)

$\left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = \left(\frac{2}{p}\right)_2 = +1$, *always*.......................(129*a*),

$\left(\frac{2}{p}\right)_{16} = -1$, or $+1$, according as $n = \frac{1}{4}(X \sim Y)$ is *odd* or *even* ....(129*b*).

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* See Bickmore's paper already quoted, Art. 13 (2), p. 16. Results (124 *e*, *g*) show that 2 and 3 are both of them cubic residues of p, when $Y = 6y$ (but not otherwise).

Again, result (124f) shows (by Jacobi's rule for cubic residues) that
$\left(\frac{2}{p}\right)_3 = 1$, when $X \not\sim Y = 3i$, *i.e.*, when X and $Y \neq 3i$......(129c).

Hence, combining (129a, b, c),

$$\left(\frac{2}{p}\right)_{24} = \left(\frac{2}{p}\right)_{12} = \left(\frac{2}{p}\right)_6 = 1, \text{ when } X,\ Y \neq 3i \text{......(129}d\text{)},$$

$$\left(\frac{2}{p}\right)_{48} = -1, \text{ or } +1, \text{ when } \tfrac{1}{12}(X \not\sim Y) \text{ is } \textit{odd} \text{ or } \textit{even} \quad \text{...(129}e\text{)}.$$

Examples.—This form (124) is interesting as giving primes of which 2 is a 48-ic residue, when $Y = 12\eta$ and when $X \not\sim Y = 24\nu$. There are only *six* primes (shown below) within the range of the Factor-Tables (*i.e.*, < 9 million); the case of 2 a 48-ic residue is beyond this range.

2 a 16-ic Non-Residue. [* 2 is a 24-ic residue of these.]				2 a 16-ic Residue.			
p	$p-1$	X	Y	p	$p-1$	X	Y
6,481	16.81.5	3	1	51,361	32.3.5.107	3	4
390,001*	16.3.625.13	5	1	346,561	64.3.5.361	5	3
4,654,801*	16.27.25.431	7	5				
5,576,681	16.3.5.19.1223	7	3				

21. *Form* $N = (X^{20}+Y^{20}) \div (X^4+Y^4)$.

In all cases

$$N = X^{16}-X^{12}Y^4+X^8Y^8-X^4Y^{12}+Y^{16} \text{ ....(130)},$$
$$= (X^8-X^4Y^4+Y^8)^2+\left[X^2Y^2(X^4\sim Y^4)\right]^2 \qquad = \mathrm{a}^2+\mathrm{b}^2 \text{ ...(130}a\text{)},$$
$$= (X^8-X^6Y^2-X^4Y^4-X^2Y^6+Y^8)^2$$
$$+2\left[XY(X^6\sim Y^6)\right]^2 = \mathrm{c}^2+2\mathrm{d}^2 \text{...(130}b\text{)},$$
$$= (X^8+X^6Y^2-X^4Y^4+X^2Y^6+Y^8)^2$$
$$-2\left[XY(X^6+Y^6)\right]^2 = \mathrm{e}^2-2\mathrm{f}^2 \text{...(130}c\text{)},$$
$$= 2(\mathrm{e}-\mathrm{f})^2-(\mathrm{e}-2\mathrm{f})^2 \qquad = 2\mathrm{g}^2-\mathrm{h}^2\text{...(130}d\text{)}.$$

Case i.—Taking X *odd*, Y *even*, it is seen that $N = 16\varpi+1$ *always*; also, when $N = p$ (prime), it follows (as in Art. 20, Case i.) that

$$\left(\frac{2}{p}\right)_2 = +1, \textit{ always}; \text{ and, when } Y = 2\omega,\ \left(\frac{2}{p}\right)_4 = -1 \text{ ...(131}a\text{)},$$

$$\text{When } Y = 4\eta, \text{ then } \left(\frac{2}{p}\right)_{16} = \left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = +1 \text{ ......(131}b\text{)}.$$

Case ii.—Taking X, Y *both odd*, it follows (as in Art. 20, Case ii.) that $N = 16\varpi+1$, *always;* and that, when $N=p$, then

$$\left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = \left(\frac{2}{p}\right)_2 = +1, \textit{ always} \quad\ldots\ldots\ldots\ldots (132a),$$

$$\left(\frac{2}{p}\right)_{16} = -1, \text{ or } +1, \text{ according as } \tfrac{1}{4}(X \mp Y) \text{ is } \textit{odd} \text{ or } \textit{even} \quad\ldots (132b).$$

These primes have a further interest from the property* that 2 is a *quintic* residue of them when $Y=10y$ (Case i.), and when $(X \pm Y) = 10\eta$ (Case ii.); hence

When $Y=20y$, or $X \pm Y = 40y$, then

$$\left(\frac{2}{p}\right)_{80} = \left(\frac{2}{p}\right)_{40} = \left(\frac{2}{p}\right)_{20} = \left(\frac{2}{p}\right)_{10} = \left(\frac{2}{p}\right)_5 = +1 \quad\ldots\ldots (133).$$

Examples.—The numbers run so high as to render primes difficult of discovery (being all >9 millions); the only one† known to the author falls under Case i.,

$$p = (1^{20}+4^{20}) \div (1^4+4^4) = 4{,}278{,}255{,}361; \quad \text{here } Y = 4.$$

The (a, b), (c, d) partitions given by (130*a*, *b*) of course satisfy the 16-ic criterion (because $Y = 4$), and it is otherwise known† that 2 is a 16-ic residue of p [because p is a divisor of $(2^{40}+1)$]; so that this may be accepted as *part evidence* of the new 16-ic criterion.

22. *Form* $(X^{4\omega}+Y^{4\omega}) \div (X^4+Y^4)$.

Since the quadratic partitions of both numerator and denominator of this number are known [by (81) when X is *odd*, and Y *even;* and by (111) when both X, Y are *odd*], the similar partitions of the quotient can be found by the process of synthetic division given in Art. 14.

Case i.—Taking X *odd*, Y *even*, it follows that $N = 16m+1$ *always.* Also, when $N=p$ (a prime), it may be shown, as before, that

$$\left(\frac{2}{p}\right)_2 = +1, \textit{ always;} \text{ and, when } Y = 2\omega,\ \left(\frac{2}{p}\right)_4 = -1 \ldots (134a),$$

$$\text{When } Y = 4\eta,\ \left(\frac{2}{p}\right)_{16} = \left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = \left(\frac{2}{p}\right)_2 = +1 \ldots (134b).$$

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* See footnote (*) to Art. 22.

† This number is quoted as a *prime* divisor of $(2^{40}+1)$ in Mr. Ed. Lucas's paper "Sur la Série récurrente de Fermat," Rome, 1879 (p. 9).

Case ii.—Taking X, Y *both odd*, it may be shown, as before, that $N = 16\varpi + 1$; and that, when $N = p$, then

$$\left(\frac{2}{p}\right)_8 = \left(\frac{2}{p}\right)_4 = \left(\frac{2}{p}\right)_2 = +1 \quad \text{..........................}(134c),$$

$$\left(\frac{2}{p}\right)_{16} = -1, \text{ or } +1, \text{ according as } \tfrac{1}{4}(X \mp Y) \text{ is } \textit{odd} \text{ or } \textit{even} \quad \ldots(134d).$$

[It has not been thought worth while to show the details of this, as the algebraic reduction of the 16-ic criterion is somewhat tedious; and they may be inferred at once from the general theorem (79) alluded to at end of Art. 14, since the numerator and denominator of the fraction forming N are *both* of sextodecimic form when $Y = 4\eta$, and also when $(X \pm Y) = 8\nu$. Numerical examples are beyond the reach of the Factor-Tables when $\omega > 3$].

With these primes 2 is also* an ω-ic residue (if $\omega > 3$, and *prime*) when $Y = 2\omega y$ (Case i.) and also when $(X \pm Y) = 2\omega\eta$, (Case ii.); hence

When $Y = 4\omega y$, and when $(X \pm Y) = 8\omega\eta$; then, (if $\omega > 3$, and *prime*),

$$\left(\frac{2}{p}\right)_{16\omega} = \left(\frac{2}{p}\right)_{8\omega} = \left(\frac{2}{p}\right)_{4\omega} = \left(\frac{2}{p}\right)_{2\omega} = \left(\frac{2}{p}\right)_{\omega} = +1 \quad \ldots(134e).$$

23. *Mersenne's Numbers.*

These are numbers of form $N = (2^q - 1)$, where q is *prime*. The composite or prime character of these is difficult to discover, owing to their having no *algebraic* divisors. Nine new *composite* Mersenne's numbers have been discovered† (during

Exponent q	Divisor $p = 16q + 1$	$p = X^4 + Y^4$		Exponent q	Divisor $p = 16.3q + 1$	$p = X^4 + Y^4$	
		X	Y			X	Y
397	6,353	...	...	1,367	65,617	3	16
1,801	28,817	13	4	11,699	561,553	...	...
5,011	80,177	11	16	12,437	596,977	27	16
73,681	1,178,897	19	32				
279,211	4,467,377	43	32				
531,571	8,505,137	53	28				

* A recent discovery of Mr. C. E. Bickmore's as to primes of form
$p = (X^{\omega} \pm Y^{\omega}) \div (X \pm Y)$, where $\omega > 3$, and *prime*.

† These are the only cases at present known to the author of Mersenne's divisors of either of the forms $(16q + 1)$, $(16.3.q + 1)$.

the present research), as in table on p. 111 :—

6 with prime divisors of type $p = 16q+1$, (of which 2 is a 16-ic residue),

3 with prime divisors of type $p = 16 . 3q+1$, (of which 2 is a 48-ic residue);

seven of these are of type $p = X^4 + (4\eta)^4$, and there are no more prime Mersenne's divisors of the type $p = 16q+1 = X^4 + (4\eta)^4$ under 9 millions (as may be gathered from an examination of Table III.).

24. *Proof of the* 16-*ic Criterion.*

The proof to be offered of the new sextodecimic criterion is at present wholly an induction from numerous instances. As, however, the octavic test itself depends at present, (it is believed), wholly on a similar proof, a *general* proof of the new 16-ic criterion should, in fact, be preceded by a general proof of the 8-ic rule.

The first point is to collect a sufficiently large number of primes of form $p = 16m+1$ for which the residue-index of 2 is *known for certain* to be in some cases 8 (or 8ω), and in some cases 16 (or $16i$); *i.e.*, such that $\left(\frac{2}{p}\right)_8 = +1$ in all the cases, and $\left(\frac{2}{p}\right)_{16} = -1$ in some cases, and $= +1$ in some cases.

For this there are two published lists available.

(1) Reuschle's "Tables," already quoted, give the residue-index of 2 for *all* primes $< 5,000$, together with the quadratic partitions required; these include 9 primes for which $\left(\frac{2}{p}\right)_{16} = -1$, and 9 primes for which $\left(\frac{2}{p}\right)_{16} = +1$. [Several errata corrected among these.]

(2) Lucas's paper already quoted gives many of the prime divisors of $(2^n \pm 1)$ for many values of $n \not> 210$ (the table is of course very incomplete). Among these are 17 primes, between 5,000 and 21,000,000, of form $p = 16m+1$, such that $\left(\frac{2}{p}\right)_{16} = -1$ in 5 cases and $= +1$ in 12 cases. All of these have been utilized. This paper contains also about 14 primes $> 25,000,000$ (some of 12 figures) of form $16m+1$ for which the residue-index (8 or 16) is known; but the labour of effecting the quadratic partitions would have been so great that these are not here* utilized, with the exception of the large prime $(2^{40}+1) \div (2^8+1)$, quoted in Art. 21, the quadratic partitions of which are specially easy [being given by (130*a*, *b*)].

Besides these the author of this paper has computed himself for the present purpose the residue-index of *all* the primes of form $p = 16m+1$ between 5,000 and 10,000 (not given in Lucas's memoir) of which $\left(\frac{2}{p}\right)_{16} = \pm 1$; these give 10 cases

* Moreover, there is some risk about these very high numbers (when > 25 million) not being really prime; Lucas does not state how their prime character was determined.

of $\left(\frac{2}{p}\right)_{16} = -1$, and 2 cases of $\left(\frac{2}{p}\right)_{16} = +1$; and has also computed* the residue-index of even order of 55 larger primes (> 10,000, but < 9,000,000) of form $p = 16m+1$ to the extent of determining that $\left(\frac{2}{p}\right)_{16} = \pm 1$ (which is all that is required for the present purpose, *i.e.*, the labour of determining whether the residue-index was a *multiple* of 16, has not been specially undertaken); these comprise 25 cases of $\left(\frac{2}{p}\right)_{16} = -1$ and 30 cases of $\left(\frac{2}{p}\right)_{16} = +1$.

Also three cases for which $\left(\frac{2}{p}\right)_{16} = \pm 1$ were contributed to this paper by Mr. C. E. Bickmore.

The following is an abstract of the evidence offered—

From Reuschle's Tables:	9 cases of	$\left(\frac{2}{p}\right)_{16} = -1$,	9 of	$\left(\frac{2}{p}\right)_{16} = +1$;		total 18.
From Lucas's memoir:	5	,, ,,	13	,,	,,	total 18.
By the author:	35	,, ,,	32	,,	,,	total 67.
By Mr. C. E. Bickmore:	1	,, ,,	2	,,	,,	total 3.
Totals	50		56			106

This evidence is detailed in the Tables I. and II. Table I. contains all the primes (50) for which $\left(\frac{2}{p}\right)_{16} = -1$, and Table II. contains all those (56) for which $\left(\frac{2}{p}\right)_{16} = +1$. Both tables show for each prime (p), the elements (a, b), (c, d) of its two* quadratic partitions, and the corresponding values of $\beta' = \frac{1}{16}\text{b}$, $\delta = \frac{1}{4}\text{d}$; from which it will be seen that the 16-ic criterion is satisfied, viz., that

$\beta' + \delta$ is *odd* throughout Table I., and *even* throughout Table II.

The three columns headed $\frac{p-1}{\Omega}$, ξ, e are required to show that the residue-index is really 8 or 16 as stated in Table I. or II.; thus

The column of $\frac{p-1}{\Omega}$ shows the *complete* even factor (2^{ϖ}) contained in ($p-1$), where $p = (2^{\varpi}.\Omega+1)$.

The column of ξ shows the *minimum* exponent (ξ) for which $2^{\xi} \equiv +1 \pmod{p}$, *when known*, or else as many factors of ξ as are known (in the case of some of the higher primes); the unknown factors being indicated by "&c.", or by ω, (when known to be odd).

The column of $\frac{p-1}{\xi}$ shows the *maximum* residue-index, $\left(e = \frac{p-1}{\xi}\right)$ of 2 to modulus p, *when known*, or else as many factors (especially the even factors), as are known (in the case of the higher primes); the unknown factors being indicated by "&c.", or by ω (when known to be *odd*).

* The labour of computing the residue-indices for so many large primes (55), and of effecting the quadratic partitions for the whole number (about 75) not given in Reuschle's tables, was very great.

These three columns together show *distinctly* that—

(1) Throughout Table I. the maximum residue-index $\frac{p-1}{\xi}$ is either 8 or 8ϖ, and the rest of the even factors of $(p-1)$ will be found accounted for in the column of ξ; the product of the *even* factors in these two columns of ξ and $\frac{p-1}{\xi}$ being always $=\frac{p-1}{\Omega}$. Hence $\left(\frac{2}{p}\right)_{16}=-1$ throughout Table I.

(2) Throughout Table II., the maximum residue-index $\frac{p-1}{\xi}$ is either 16 or a multiple thereof; hence $\left(\frac{2}{p}\right)_{16}=+1$ throughout Table II.

Lastly, the column headed "Ref." shows a reference to the authority for the fact that the primes in question have 2 as either an 8-ic or a 16-ic residue; the reference is given by an initial letter, thus—

B for Bickmore, C for Cunningham, L for Lucas, R for Reuschle.

These two tables contain the whole* of the *evidence* of the new 16-ic criterion.

25. *Research of Primes for which* $\left(\frac{2}{p}\right)_{16}=-1$ *or* $+1$.

Most of the large primes marked C in the Tables I. and II., used for establishing the 16-ic criterion, were discovered by the following process†, which is interesting in itself.

A large series of numbers were computed of following *four* types:—

$$N=(2^x\sim a^e),\ (2^x+a^e);\quad (2^x.b^e-1),\ (2^x.b^e+1)\ldots\ldots(135),$$

for the following series of values of a^e, b^e, x,

$a^e=3^8, 3^{16}, 3^{32}; 5^8, 5^{16}; 7^8, 7^{16}; 11^8; b^e=3^8, 5^8, 7^8, 11^8;$

$x=$ *odd* and *even* values, from 1 to about 31, in both $(2^x \underset{+}{\sim} a^e)$;

$x=$ *even* values of form $x=2\omega$, from 2 to about 62, in (2^x+a^e), when $e=4e'$;

$x=$ *odd* and *even* values, from 1 to about 20 in $(2^x.b^e\mp 1)$.

$x=$ *even* values of form $x=2\omega$, from 2 to about 38, in $(2^x.b^e+1)$, when $e=4e'$.

It will be seen that these numbers run *very high*. All numbers of the four types within the range of the large Factor-Tables (*i.e.*, below 9 millions) were computed; the primes were noted, and the composite numbers were decomposed into their prime factors. Many such numbers were also computed far beyond the *square* of the 9 million

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* As these pages pass through the press, 12 more instances in support of the new criterion have been discovered (4 by Mr. Bickmore, 8 by the author), but too late to be here detailed (see foot of p. 117). This raises the number of primes in evidence to 118.

† Adapted from Mr. C. E. Bickmore's paper (p. 20, Art. 13) already quoted (since found to have been used by Euler; see *Nov. Comm. Acad. Petr.*, t. I., p. 45); the power of the process is considerably *extended* in the present paper by utilizing the chance of a and b being residues of orders 2, 4, 8.

limit, whenever it seemed possible to decompose* them, and they were then decomposed into prime factors. In this way a large stock† of primes of the above forms, and of prime factors of composite numbers of the above forms, was accumulated.

Now let p denote any prime of one of the above forms N, or any prime factor of a composite number N of one of those forms. With such primes, therefore, one of the following congruences exists

$$2^x \equiv \pm a^e, \quad \text{or} \quad 2^x . b^e \equiv \pm 1, \quad (\text{mod } p) \text{ ....(136).}$$

A selection is now to be made of those primes which are of any of the forms (no others are of any use for the present purpose)

$$p = ef+1, \quad p = \epsilon . ef+1, \quad p = 2\epsilon . ef+1 \text{ ....(137),}$$

where $\epsilon =$ the residue-index‡ of *even* order of a or b, so that

$$a^{(p-1)/\epsilon} \equiv +1, \; a^{(p-1)/2\epsilon} \equiv -1; \quad b^{(p-1)/\epsilon} \equiv +1, \; b^{(p-1)/2\epsilon} \equiv -1 \text{...(138).}$$

Now, according to these *three* forms (137) of primes, the fundamental congruences (136) may be raised to the powers

$$\text{i. } \frac{p-1}{e} = f, \quad \text{ii. } \frac{p-1}{\epsilon e} = f, \quad \text{iii. } \frac{p-1}{2\epsilon e} = f \; (= \text{integer in each case})$$
$$\text{...............(139),}$$

with the results

$$\text{i. } (2^x)^{(p-1)/e} \equiv (\pm 1)^f . a^{p-1}, \quad \text{or} \quad (2^x)^{(p-1)/e} . b^{p-1} \equiv (\pm 1)^f,$$

whence $$(2^x)^{(p-1)/e} \equiv (\pm 1)^f \text{ (in both cases) ....(140);}$$

$$\text{ii. } (2^x)^{(p-1)/\epsilon e} \equiv (\pm 1)^f . a^{(p-1)/\epsilon}, \quad \text{or} \quad (2^x)^{(p-1)/\epsilon e} . b^{(p-1)/\epsilon} \equiv (\pm 1)^f,$$

whence $$(2^x)^{(p-1)/\epsilon e} \equiv (\pm 1)^f \text{ (in both cases) ....(141);}$$

$$\text{iii. } (2^x)^{(p-1)/2\epsilon e} \equiv (\pm 1)^f . a^{(p-1)/2\epsilon}, \text{ or } (2^x)^{(p-1)/2\epsilon e} . b^{(p-1)/2\epsilon} \equiv \pm(-1)^f,$$

whence $$(2^x)^{(p-1)/2\epsilon e} \equiv -(\pm 1)^f \text{ (in both cases) ....(142).}$$

In each of these Cases i., ii., iii. *two* sub-cases must be distinguished, according as (1) $x = \omega$, (2) $x = 2\omega$; and the only case of any use in

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* The author has prepared a MS. "Canon Arithmeticus" for the bases 2, 3, 5, 7 on the pattern of Jacobi's, which enables the *small* factors of numbers of above types to be quickly found. Also all numbers of type $(2^{2\omega}+a^{4i})$ can be decomposed into *two* nearly equal factors $(2^{\omega} \mp 2^{\frac{1}{2}(\omega+1)} . a^i + a^{2i})$, a resolution due to the late Mons. Aurifeuille. By this means some numbers > $4\frac{1}{2}$ million billions, *e.g.* $(2^{62}+5^{16})$ were resolved into prime factors.

† The labour of doing this was, of course, very great.

‡ It is easy to determine when $\epsilon = 1, 2,$ or 4 for *small* bases a or $b = 3, 5, 7, 11$, &c. Simple rules for this have been given in Mr. C. E. Bickmore's paper already quoted, Arts. 13–17; see also Pepin's *Mémoire sur les lois de réciprocité relatives aux résidus de puissances*, Rome, 1878; pp. 41 to 69. For the case of $\epsilon = 8$ the author is indebted to a MS. communication from Mr. Bickmore, which it is hoped will be published shortly.

[As the rules for $\epsilon = 8$ for the bases 3, 5, 7, &c., have not yet been proved, it may be noted that they have been used (only in Table II. and) only to raise the residue-index (e) of 2 *above* 16, so that the present "evidence" of the new 16-ic criterion of 2 in no way depends on them.]

this connexion is when $\omega\,(=x$ or $\tfrac{1}{2}x)$ is *prime* to $(p-1)$. In this case only—

Sub-case (1). $x=\omega$, and prime to $(p-1)$; then x may be expunged from all the congruences, which thus give

$$\text{i. (1); } 2^{(p-1)/e}\equiv(\pm 1)^y,\quad \text{ii. (1); } 2^{(p-1)/\epsilon e}\equiv(\pm 1)^y,$$
$$\text{iii. (1); } 2^{(p-1)/2\epsilon e}\equiv-(\pm 1)^y \quad\ldots\ldots\ldots\ldots\ldots\ldots(143).$$

Sub-case (2). $x=2\omega$, and $\tfrac{1}{2}x$ prime to $(p-1)$; then $\tfrac{1}{2}x=\omega$ may be expunged from all the congruences, which thus give

$$\text{i. (2); } 2^{(p-1)/\frac{1}{2}e}\equiv(\pm 1)^y,\quad \text{ii. (2); } 2^{(p-1)/\frac{1}{2}\epsilon e}\equiv(\pm 1)^y,$$
$$\text{iii. (2); } 2^{(p-1)/\epsilon e}\equiv-(\pm 1)^y\ldots\ldots\ldots\ldots\ldots\ldots\ldots(144).$$

To obtain the *maximum* residue-index (E) of even order, this process should be pushed (if possible) to the extent of

$$\left.\begin{array}{l} 2^{(p-1)/2E}\equiv-1,\ \text{whence}\ 2^{(p-1)/E}\equiv+1,\\ \text{whence } E=\text{maximum residue-index required}\end{array}\right\}\ldots\ldots\ldots(145).$$

It is evident that this method is one of considerable power, as, taking $e=4,\ 8,\ 16,\ 32$, the maximum residue-index occasionally reaches 32* or even 64, with the help derived from a knowledge of $\epsilon=2,\ 4$, or 8 for the auxiliary base (a or b) used; and it also enables *large* primes (p) to be dealt with.

Numbers of the forms

$$N=a^e\sim 2^x.b^e,\quad\text{or}\quad a^e+2^x.b^e\ \ldots\ldots\ldots\ldots\ (146),$$

may also be treated in the same way as those of the forms (135). They are of course more laborious to compute than (135); moreover, the auxiliary bases a, b are seldom both of them residues of any *even* order >2 of the same prime (p), so that the process can seldom be pushed to finding residues of order higher than 8. By taking $e=8$; $a=5,\ 7$; $b=3,\ 5$; and x as before, a few additional instances in support of the 16-ic criterion were obtained, and are embodied in Tables I., II.

Example.—Take $N=(2^2+3^{32})=(2-2\,.\,3^8+3^{16})\,(2+2\,.\,3^8+3^{16})$ by Aurifeuille's decomposition, [see (120*a*), (120*b*)]; the larger factor $N_2=5\,.\,8{,}611{,}969$; also $8{,}611{,}969=p$, and $p-1=128\,.\,3\,.\,41\,.\,547=128\Omega$. Also $3^{\frac{1}{2}(p-1)}\equiv+1$, and $3^{\frac{1}{4}(p-1)}\equiv-1$, since $p=$† (2625^2+1312^2). Here $2^2\equiv-3^{32}$, whence

$$(2^2)^{\frac{1}{128}(p-1)}\equiv(-1)^{\Omega}\,.\,3^{\frac{1}{4}(p-1)}\,.\equiv(-1)\,(-1);\ \text{therefore}\ 2^{\frac{1}{64}(p-1)}\equiv+1.$$

---

* Table II. contains 45 cases of primes of form $p=(32\varpi+1)$; in 12 of these 2 is shown as a 16-ic (non-32-ic) residue, in 19 as a 32-ic residue; in the remaining 14 the analysis does not distinguish between 16-ic and 32-ic. Also, 10,657; 54,721; 67,073; 83,777 have been found to have 2 as a 32-ic, 16-ic, 16-ic (non-32-ic), 64-ic residue respectively. These will serve as a nucleus for verifying hereafter the (yet to be discovered) criterion of 32-ic residuacity.

† The (a, b) partition of this large prime (p) is effected thus: by (120*b*) $N_2=5\,.\,8{,}611{,}969=(3^8+1)^2+1^2$; $\therefore\ p=(1^2+6562^2)+(1^2+2^2)$; now apply Result (77*a*).

[The means at disposal here do not enable one to distinguish between the signs of $2^{\frac{1}{16}(p-1)} \equiv \pm 1$, so it is uncertain whether $e = 64$ is the *maximum* residue-index (of form $e = 2^\kappa$) or not.]

26. *Verification of Residue-index.*

As the labour of *verifying* the fact that $\left(\frac{2}{p}\right)_{16} = -1$ in Table I., and $= +1$ in Table II., would be very great (by any direct process) in the case of the larger primes (except when the minimum exponent ξ, giving $2^\xi \equiv +1$, is known to be small), the means has been given of making the verification easily by the method developed in Article 25, by adding a broad column headed "Verification" on the right of Tables I., II., sub-divided into six columns, giving the necessary data in the case of most of the primes (marked C) newly determined by the author, and of a few others.

The sub-column headed $(\mp)$ shows which of the signs $(\mp)$ belongs to the fundamental number N [see (135), (146)], from which the prime in question (p) is derived (as being a sub-multiple of N); of course the *signs are changed* in forming the congruences (136). The remaining sub-columns give the values of a, b, e, x, ϵ, needed with the prime p for the method of Art. 25; when *two* values of ϵ are shown, the first belongs to a, the second to b.

27. *Tables.*

Five tables of primes of which 2 is an 8-ic (but non-16-ic) or a 16-ic residue are here subjoined.

Tables I., II., III., IV., have been sufficiently explained in the text; see Arts. 24, 25, 26; 16, 18. Suffice it to say here in addition that Tables I. and II. contain a *complete* list of all primes under 10,000 of which 2 is an 8-ic (but non-16-ic) or a 16-ic residue, with details explained in Art. 24–26, besides many larger primes. Table V. contains the continuation of that list *complete* from 10,000 to 25,000, with the values of $(p-1)$, a, b, c, d for each.

Tables III. and IV. contain *complete* lists of all primes under 9 millions of two forms, viz., Table III. of primes of form $p = X^4 + (4\eta)^4$ of which 2 is a 16-ic residue (Art. 16), and Table IV. of primes of form $p = \frac{1}{2}(X^4 + Y^4)$ of which 2 is an 8-ic (but non-16-ic) or a 16-ic residue; see Arts. 10, 18.

Additional "Evidence-Primes"

[Discovered too late for insertion of details in Tables I., II.]

TABLE I.: 10,369 (B); 51,521 (C); 204,161 (C); 556,313 (B); 820,481 (C); 3,279,361 (C).

TABLE II.: 54,721 (C); 67,073 (C); 83,777 (C); 86,561 (B); 95,713 (B); 205,441 (C).

[*Errata.* *See* foot of p. 122.]

TABLE I.

Primes with 2 as 8-ic Residue and 16-ic Non-Residue.

p	$\frac{p-1}{\Omega}$	Quadratic Partitions.						Indices.		Ref.	Verification.					
		a	b	c	d	β′	δ	ξ	$\frac{p-1}{\xi}$		*a*	*b*	*e*	±	*x*	ε
1,217	64	31	16	33	8	1	2	8.19	8	R						
1,249	32	15	32	31	12	2	3	4.39	8	R						
1,553	16	23	32	39	4	2	1	2.97	8	R						
1,777	16	39	16	25	24	1	6	2.37	8.3	R						
2,833	16	23	48	41	24	3	6	2.59	8.3	R						
4,049	16	55	32	57	20	2	5	2.11.23	8	R						
4,273	16	57	32	41	36	2	9	2.267	8	R						
4,481	128	65	16	63	16	1	4	16.35	8	R	3	.	16	+	18	1
4,993	128	63	32	49	36	2	9	16.39	8	R						
5,297	16	71	16	57	32	1	8	2.331	8	C	5	.	8	~	17	1
6,449	16	7	80	57	40	5	10	2.403	8	C						
6,481	16	9	80	73	24	5	6	2.81.5	8	C						
6,689	32	17	80	81	8	5	2	4.209	8	C	3	.	8	+	7	1
7,121	16	55	64	57	44	4	11	2.5.89	8	C						
8,081	16	41	80	87	16	5	4	2.5.101	8	C						
8,609	32	47	80	81	32	5	8	4.269	8	C	3	.	8	+	11	1
9,137	16	71	64	87	28	4	7	2.571	8	C						
9,281	64	95	16	33	64	1	16	8.145	8	C						
9,649	16	57	80	71	48	5	12	2.201	8.3	C						
10,433	64	97	32	81	44	2	11	8.163	8	C	.	3	16	+	6	1
11,633	16	103	32	9	76	2	19	2.727	8	C						
13,121	64	95	64	111	20	4	5	8.41, &c.	8ω	C	.	3	8	~	1	1
14,753	32	47	112	81	64	7	16	4.461	8	C	3	.	8	+	13	1
15,121	16	105	64	89	60	4	15	2.27.5	8	L						
28,001	32	49	160	33	116	10	29	4.125	8.7	B	.	.	.	.	.	.
29,153	32	113	128	159	44	8	11	4.911	8	C	.	11	8	+	18	4
29,761	64	95	144	31	120	9	30	8.3.31ω	8ω	C	7	3	8	+	18	2,4
30,241	32	145	96	127	84	6	21	4.27.5	8.7	C	5	.	16	+	6	2
31,601	16	25	176	153	64	11	16	2.79ω	8ω	C	7	3	8	+	7	1,2
41,761	32	145	144	17	144	9	36	4.9.29ω	8ω	C	.	3	8	+	18	4
137,633	32	47	368	81	256	23	64	4ω	8ω	C	3	.	8	+	17	1
154,321	16	311	240	247	216	15	54	2.643ω	8.3ω	C	.	5	8	+	5	4
155,377	16	89	384	55	276	24	69	2.83	8.9.23	L						
246,241	32	15	496	143	336	31	82	4.27	8.3.5.19	L						
260,417	64	479	176	417	208	11	52	8.313ω	8ω	C	.	5	8	+	1	1
279,073	32	17	528	431	216	33	54	4.27	8.17.19	L						
319,201	32	399	400	527	144	25	36	4.3ω	8ω	C	.	5	8	+	18	4
390,113	32	143	608	225	412	38	103	4ω	8ω	C	5	.	8	~	9	1
600,577	512	769	96	625	324	6	81	64.3ω	8ω	C	5	3	8	+	5	1,4
1,151,041	64	479	960	529	660	60	165	8ω	8ω	C	7	.	16	+	6	1
1,153,921	128	961	480	1073	36	30	9	16ω	8ω	C	7	.	16	+	2	1
1,230,433	32	463	1008	625	648	63	162	4ω	8ω	C	5	3	8	+	7	1,4
1,230,881	32	785	784	687	616	49	154	4ω	8ω	C	.	7	8	+	18	4
2,103,713	32	977	1072	81	1024	67	256	4ω	8ω	C	3	.	8	+	21	1
3,033,169	16	905	1488	1481	648	93	162	2.29	8.6537	L						
3,356,641	32	1295	1296	1423	816	81	204	4ω	8ω	C	.	3	16	+	18	4
5,369,857	4096	1921	1296	2047	768	81	192	512ω	8ω	C	5	3	16	+	18	1,4
5,554,849	32	943	2160	2257	480	135	480	4ω	8ω	C	7	3	8	~	5	1,2
5,633,729	64	1823	1520	2271	488	95	122	8ω	8ω	C	7	.	8	~	17	1
5,764,289	64	1633	1760	849	1588	110	397	8.90067	8	C	7	.	8	~	9	1

Table II.
Primes with 2 as 16-ic Residue.

p	$\frac{p-1}{\Omega}$	Quadratic Partitions. a	b	c	d	β'	δ	Indices. ξ	$\frac{p-1}{\xi}$	Ref.	Verification. a	b	e	±	x	ϵ
257	256	1	16	15	4	1	1	16	16	R						
337	16	9	16	7	12	1	3	21	16	R						
881	16	25	16	9	20	1	5	55	16	R						
2,113	64	33	32	31	24	2	6	4 . 11	16 . 3	R						
2,593	32	17	48	1	36	3	9	81	32	R						
2,657	32	49	16	33	28	1	7	2 . 83	16	R						
4,177	16	9	64	55	24	4	6	87	16 . 3	R						
4,513	32	47	48	65	12	3	3	47	32 . 3	R	3	.	8	~	11	4
4,721	16	25	64	39	40	4	10	5 . 59	16	R			-			
6,353	16	73	32	9	56	2	14	397	16	C						
6,529	128	65	48	79	12	3	3	2 . 51	64	L	3	.	8	~	5	8
7,489	64	33	80	17	60	5	15	4 . 9 . 13	16	C	3	.	8	+	23	2
10,657	32	81	64	17	72	4	18	9 . 37, &c.	16, &c.	C	.	5	16	+	6	1
12,641	32	79	80	33	76	5	19	79, &c.	16, &c.	C	7	.	16	+	26	1
15,073	32	113	48	31	84	3	21	2 . 157	16 . 3	C						
15,809	64	97	80	81	68	5	17	247	64	C	5	.	8	+	17	4
18,593	32	47	128	111	56	8	14	83, &c.	16, &c.	C	3	.	16	+	30	1
21,713	16	73	128	9	104	8	26	23 . 59	16	C	.	7	8	+	1	1
25,121	32	145	64	111	80	4	20	157, &c.	16, &c.	C	3	.	16	+	26	1
25,601	1024	1	160	63	104	10	26	16 . 25	64	B						
37,537	32	81	176	193	12	11	3	17 . 23, &c.	16 . 3, &c.	C	5	.	16	+	62	1
41,729	256	127	160	129	112	10	28	2 . 163, &c.	16, &c.	C						
43,649	128	193	80	207	20	5	5	8 . 31, &c.	16, &c.	C	3	.	32	+	10	1
47,713	32	177	128	79	144	8	36	3 . 71, &c.	16, &c.	C	7	.	8	+	30	2
63,841	32	79	240	223	84	15	21	2 . 3 . 7 . 19	16 . 5	B						
65,537	256^2	1	256	255	16	16	4	32	2048	L						
71,473	16	263	48	151	156	3	39	1489	16 . 3	C	.	3	8	~	9	4
86,113	32	207	208	289	36	13	9	117	32 . 23	L						
92,737	64	289	96	287	72	6	18	63	64 . 23	L						
100,801	64	225	224	287	96	14	24	75	64 . 21	L						
104,417	32	271	176	225	164	11	41	251, &c.	16, &c.	C	5	3	8	~	13	1, 1
165,313	64	287	288	161	264	18	66	41, &c.	32 . 3, &c.	C	.	3	8	+	22	4
178,481	16	391	160	57	296	10	74	23	16 . 485	L						
209,953	32	273	368	1	324	23	81	6561	32	C	.	3	8	+	5	2
262,657	512	129	496	143	348	31	87	27	512 . 19	L						
318,817	32	561	64	305	336	4	84	243, &c.	16, &c.	C	5	.	16	+	26	1
476,513	32	463	512	15	488	32	122	14891, &c.	16, &c.	C	7	.	8	+	38	2
525,313	1024	513	512	575	312	32	78	4 . 19	256 . 27	L						
561,553	16	297	688	409	444	43	111	11699	16 . 3	C	3	.	8	~	27	2
589,681	16	665	384	359	480	24	120	81, &c.	16ω	C	3	.	16	~	3	4
662,401	128	575	576	449	480	36	120	8ω	16 . 3ω	C	.	3	8	+	26	4
839,809	128	447	800	1	648	50	162	8 . 6561	16	C	.	3	8	+	7	2
841,697	32	881	256	207	632	16	158	907, &c.	16, &c.	C	5	.	16	+	34	1
1,278,401	64	799	800	351	760	50	190	?	32, &c.	C	.	5	8	+	22	4
1,554,713	32	1137	512	881	624	32	156	5399, &c.	16, &c.	C	5	.	16	+	38	1
4,079,041	64	1121	1680	1967	324	105	81	4249, &c.	64ω	C	5	.	8	+	27	4
4,327,489	64	33	2080	2017	360	130	90	4 . 33	16 . 2049	L						
5,123,201	128	1601	1600	2049	680	100	170	8ω	16ω	C	.	5	8	+	26	4
5,182,913	64	2273	128	2079	656	8	164	?	32, &c.	C	7	.	16	+	30	2
5,920,513	256	2433	32	1921	1056	2	264	16ω	16ω	C	7	.	16	+	22	2
6,219,233	32	1423	2048	369	1744	128	436	?	16, &c.	C	5	.	16	+	46	1
8,611,969	128	2625	1312	2113	144	82	36	?	64, &c.	C	3	.	32	+	2	2
10,567,201	32	2415	2176	623	2256	136	564	75	32ω	L						
13,264,529	16	3625	352	489	2552	22	638	47	16ω	L						
20,394,401	32	4049	2000	4287	1004	125	251	53	32ω	L						
$\frac{2^{40}+1}{2^8+1}$	256	*	*	*	*			16 . 5	16ω	L						

* See Art. 21.

TABLE III.

Primes of Form $p = X^4 + Y^4 = X^4 + (4\eta)^4$.

[2 a 16-ic Residue of p; see Art. 16.]

p	$p-1$	X	Y	p	$p-1$	X	Y
257	256	1	4	1,959,457	32.3.20,411	23	36
337	16.3.7	3	4	1,972,097	128.7.31.71	31	32
881	16.5.11	5	4	2,034,161	16.5.47.541	37	20
2,657	32.83	7	4	2,070,241	32.3.5.19.227	25	36
4,177	16.9.29	3	8	2,378,977	32.3.24,781	39	16
4,721	16.5.59	5	8	2,473,441	32.3.5.5153	39	20
10,657	32.9.37	9	8	2,566,561	32.3.5.5347	9	40
14,897	16.49.19	11	4	2,690,321	16.5.33,629	19	40
28,817	16.1801	13	4	2,754,481	16.3.5.23.499	21	40
49,297	16.3.13.79	13	12	2,839,841	32.5.17,749	23	40
54,721	64.9.5.19	15	8	2,922,737	16.29.6299	37	32
65,537	256.256	1	16	3,157,537	32.3.31.1061	41	24
65,617	16.3.1367	3	16	3,362,017	32.3.7.5003	39	32
66,161	16.5.827	5	16	3,439,537	16.3.131.547	43	12
80,177	16.5011	11	16	3,553,777	16.9.24,679	37	36
83,777	64.7.11.17	17	4	3,578,801	16.25.23.389	43	20
134,417	16.31.271	19	8	3,750,577	16.3.78,137	43	24
149,057	64.17.137	17	16	3,874,337	32.41.2953	41	32
151,057	16.9.1049	19	12	3,942,577	16.9.11.19.131	21	44
160,001	256.625	1	20	4,100,881	16.3.5.7.2441	45	4
160,081	16.3.5.23.29	3	20	4,104,721	16.9.5.5701	45	8
166,561	32.3.5.347	9	20	4,279,537	16.9.113.263	27	44
243,521	64.5.761	17	20	4,467,377	16.279,211	43	32
260,017	16.3.5417	21	16	4,505,377	32.3.71.661	41	36
280,097	32.8753	23	4	4,715,281	16.27.5.37.59	45	28
283,937	32.19.467	23	8	4,879,937	64.76,249	47	4
331,777	64.64.81	1	24	5,039,681	64.5.15,749	47	20
334,177	32.3.13.37	7	24	5,211,457	64.3.27,143	47	24
346,417	16.3.7.1031	11	24	5,308,417	256.256.81	1	48
360,337	16.3.7507	13	24	5,309,041	16.3.5.11.2011	5	48
394,721	32.5.2467	25	8	5,385,761	32.5.41.821	41	40
411,361	32.3.5.857	25	12	5,391,937	64.9.11.23.37	17	48
462,097	16.9.3209	19	24	5,768,897	64.7.79.163	49	8
596,977	16.3.12,437	27	16	5,785,537	64.3.30,133	49	12
614,657	256.49.49	1	28	5,978,801	16.25.14,947	43	40
621,217	32.27.719	9	28	6,015,697	16.3.23.5449	29	48
643,217	16.7.5743	13	28	6,769,297	16.9.29.1621	51	8
728,017	16.3.29.523	29	12	6,925,201	16.25.3.29.199	51	20
744,977	16.101.461	19	28	7,166,897	16.11.43.947	43	44
867,281	16.5.37.293	29	20	7,326,257	16.7.65,413	11	52
944,257	128.3.2459	31	12	7,439,681	64.5.67.347	47	40
1,049,201	16.25.43.61	5	32	7,506,097	16.3.13.23.523	21	52
1,050,977	32.32,843	7	32	7,591,457	32.237,233	23	52
1,055,137	32.3.29.379	9	32	7,813,777	16.3.162,787	51	28
1,146,097	16.27.7.379	27	28	7,843,057	16.3.13.12,569	27	52
1,178,897	16.73,681	19	32	7,894,577	16.19.25,969	53	8
1,328,417	32.41,513	23	32	7,911,217	16.27.18,313	53	12
1,345,921	128.3.5.701	33	20	8,050,481	16.5.103.977	53	20
1,521,361	16.9.5.2113	35	12	8,222,257	16.27.7.2719	53	24
1,682,017	32.3.7.2503	7	36	8,324,801	64.25.61.823	49	40
1,763,137	64.9.3061	17	36	8,505,137	16.531,571	53	28
1,800,577	128.27.521	33	28	8,627,777	64.113.1193	47	44
1,809,937	16.9.12,569	19	36	8,812,241	16.5.59.1867	35	52
1,874,417	16.193.607	37	4	8,939,057	16.7.79,813	53	32
1,878,257	16.89.1319	37	8	10^8+81	16.9.5.138,889	3	100

TABLE IV.

Pythagorean Primes having 2 *as* 16-*ic Residue or Non-Residue* (see Arts. 10, 18).

$$p = \tfrac{1}{2}(X^4+Y^4) = a^2+b^2 = c^2+2d^2 = 16\varpi+1,\quad b = d = 16\beta,\quad a^2 = b^2+c^2,\quad X \pm Y = 16m.$$

2 a 16-ic Non-Residue (β odd, $X \pm Y = 16\omega$).						
p	$p-1$	X	Y	a	b = d	c
4,481	128.5.7	9	7	65	16	63
14,321	16.5.179	13	3	89	80	39
41,761	32.9.5.29	17	1	145	144	17
97,553	16.7.13.67	21	5	233	208	105
141,121	64.9.5.49	23	7	289	240	161
198,593	64.29.107	25	9	353	272	225
487,073	32.31.491	31	15	593	368	465
764,593	16.3.17.937	35	13	697	528	455
1,414,081	64.9.5.491	41	7	865	816	287
1,709,713	16.9.31.383	43	5	937	912	215
1,975,121	16.5.7.3527	43	27	1289	560	1161
2,901,601	32.9.25.13.31	47	31	1585	624	1457
3,032,801	32.25.17.3791	47	33	1649	560	1551
3,344,161	32.3.5.6967	49	31	1681	720	1519
3,736,241	16.5.46,703	51	29	1721	880	1479
4,132,913	16.7.36,901	51	35	1913	688	1785
6,690,881	64.5.7.29.103	57	41	2465	784	2337
7,768,081	16.9.5.10,789	59	43	2665	816	2537
8,967,073	32.3.93,407	65	17	2257	1968	1105

2 a 16-ic Residue (β even, $X \pm Y = 32m$).						
p	$p-1$	X	Y	a	b = d	c
67,073	512.131	17	15	257	32	255
104,561	16.5.1307	21	11	281	160	231
353,681	16.5.4421	29	3	425	416	87
750,353	16.23.2039	35	3	617	608	105
1,054,721	4096.5.103	33	31	1025	64	1023
1,416,161	32.5.53.167	41	9	881	800	369
2,481,601	64.3.25.11.47	47	17	1249	960	799
2,907,713	64.45,433	49	15	1313	1088	735
3,952,561	16.3.5.43.383	53	11	1465	1344	583
4,715,233	32.3.13.3779	55	23	1777	1248	1265
5,473,313	32.13.59.223	57	25	1937	1312	1425
5,654,641	16.3.5.23,561	53	43	2329	480	2279
5,988,193	32.3.49.19.67	55	41	2353	672	2255
6,058,993	16.3.126,229	59	5	1753	1728	295
6,324,401	16.25.97.163	59	27	2105	1376	1593
6,995,761	16.3.5.103.283	59	37	2425	1056	2183
8,925,313	128.3.11.2113	65	1	2113	2112	65

[redacted]

Table V.

Primes having 2 *as a* 16-*ic Residue or Non-Residue* (see Art. 27).

2 a 16-ic Non-Residue.						2 a 16-ic Residue.					
p	*p*−1	a	b	c	d	*p*	*p*−1	a	b	c	d
10,337	32.17.19	79	64	33	68	10,177	64.3.53	31	96	95	24
10,369	128.81	63	80	1	72	10,657	32.9.37	81	64	17	72
10,433	64.163	97	32	81	44	12,049	16.3.251	105	32	41	72
11,329	64.3.59	95	48	31	72	12,641	32.5.79	79	80	33	76
11,617	32.3.121	49	96	95	36	14,897	16.49.19	121	16	105	44
11,633	16.727	103	32	9	76	15,073	32.3.157	113	48	31	84
12,241	16.9.5.17	55	96	71	60	15,809	64.13.19	97	80	81	68
12,577	32.3.131	111	16	47	72	18,593	32.7.83	47	128	111	56
13,121	64.5.41	95	64	111	20	19,249	16.3.401	135	32	121	48
13,441	128.3.5.7	65	96	79	60	21,569	64.337	95	112	111	68
13,633	64.3.71	33	112	95	48	21,713	16.23.59	73	128	9	104
14,321	16.5.79	89	80	39	80	23,041	512.9.5	129	80	143	36
14,753	32.461	47	112	81	64	23,057	16.11.131	151	256	87	88
15,121	16.27.5.7	105	64	89	60						
15,569	16.7.139	55	112	9	88						
16,417	32.27.19	111	64	127	12						
16,433	16.13.79	7	128	105	52						
16,673	32.521	17	128	129	4						
17,137	16.9.7.17	89	96	55	84						
18,257	16.7.163	119	64	135	4						
18,433	2048.9	127	48	1	96						
18,481	16.3.5.7.11	135	16	7	96						
19,793	16.1237	137	32	135	28						
20,113	16.3.419	87	112	41	96						
20,353	128.3.53	63	128	79	84						
23,761	16.27.5.11	55	144	73	96						
23,857	16.3.7.71	121	96	23	108						

Errata in Col. Cunningham's Papers.

p. 54, l. 5, *for* "$2\frac{1}{16}_{p-1}$," *read* "$2^{\frac{1}{16}(p-1)}$."

,, l. 6, *for* "$2^{\frac{1}{16}(p-1)} =$," *read* "$2^{\frac{1}{16}(p-1)} \equiv$."

p. 87, formula 18, *for* "$(8\zeta \pm 1)^2$," *read* "$(8\theta \pm 1)^2$."

,, footnote †, *for* "1648," *read* "1658."

p. 89, footnote *. The rule as to c or d $= 3\kappa$, according as p $= 3\varpi - 1$, or $3\varpi + 1$, is not Euler's; it is due to Mr. C. E. Bickmore.

p. 96, formulæ (76), (76*a*), (76*b*), (77*b*), *insert* 2 *before* d^2, d_1^2, d_2^2, d_2^2.

p. 97, l. 2, *for* "(a_1, b_1, c_1, d_1)," *read* "(a_1, b_1), (c_1, d_1)."

,, l. 13 up, *for* "$e = 2$," *read* "$e = 2^e$."

p. 99, l. 11 up, *for* "ctavic," *read* "octavic."

p. 102, formula (99), *for* "$2^\kappa . \omega$," *read* "$2^\kappa . \varpi$."

p. 105, footnote †. The partition of each of Aurifeuille's two factors of $(X^4 + 4Y^4)$ into a *sum of squares* (120*a*), (120*b*) is due to Mr. Bickmore.

p. 108, l. 7 up, *for* "and it is easy to see that $N = 16\varpi + 1$," *read* "and, by (125), $N = 16 . 3\varpi + 1$."

The Spherical Catenary. *By* A. G. GREENHILL.

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The properties of the Spherical Catenary, the curve assumed by a chain wrapped on a globe or resting in a spherical bowl, have been investigated by

Minding, *Crelle*, 11 and 12;

Gudermann, *Crelle*, 33, "De curvis catenariis sphæricis dissertatio";

Biermann, "Problemata quædam mechanica functionum ellipticarum ope soluta." Dissertatio inauguralis, 1865;

Clebsch, *Crelle*, 57, "Ueber die Gleichgewicht eines biegsamen Fadens";

Fischer, "Die Kettenlinie auf der Kügel," Brill's *Catalogue of Mathematical Models*, No. 156;

Max Schlegel, *Jahresbericht der K. Wilhelms Gymnasium in Berlin*, 1884;

Appell, *Bulletin de la Société Mathématique de France*, III., 1884, *Traité de mécanique rationnelle*, I., p. 202.;

Routh, *Analytical Statics*, 1891;

Venske, "Behandlung einiger Aufgaben der Variationsrechnung." Inaugural-Dissertation, Göttingen, 1891.

Marcolongo, *Rendiconti della R. Accademia della Scienze Fisiche e Matematiche*, Napoli, 1892.

The object of the present paper is to introduce a special form of the elliptic integral of the third kind, required in the solution of this problem, and to discuss the particular cases which arise when this integral becomes *pseudo-elliptic*, in consequence of the parameter being made equal to an aliquot part of the periods.

In this manner the only elliptic transcendent which remains in the solution is the elliptic integral of the first kind; and, when by a special numerical choice of the constants this term can be made to disappear, the spherical catenary becomes a closed algebraical curve.

1. Suppose the chain is wrapped upon a terrestrial globe, suspended from its North Pole; then the general equation connecting ψ, the longitude, with z, the sine of the latitude (south), can be expressed by the integral

$$\psi = \int \frac{A\,dz}{(1-z^2)\sqrt{Z}} \qquad (1),$$

where
$$Z = (1-z^2)(h-z)^2 - A^2 \qquad (2),$$

and A, h are the arbitrary constants of the problem.

For, if T denotes the tension of the chain, in gravitation measure, w its weight per unit length,

$$T = w\,(h-z) \qquad (3),$$

where h denotes the depth below the centre of the sphere of the *directrix plane* (Routh, *Analytical Statics*, I., p. 357), the tension at any point being equal to the weight of the length of the chain which will reach the directrix plane hanging vertically downwards.

Again, the moment of the tension round the vertical diameter being constant,

$$Tr^2 \frac{d\psi}{ds} \text{ is constant } = wA, \text{ suppose} \qquad (4),$$

where s denotes the length of the chain measured from a fixed point, and r denotes the distance from the vertical diameter; so that

$$r^2 + z^2 = 1 \qquad (5),$$

if the radius of the sphere is taken as unity.

Taking equation (4), which holds for any system of forces which have no moment about the axis Oz, it may be transformed into

$$\frac{T^2r^2}{w^2A^2} = \frac{ds^2}{r^2d\psi^2} = 1 + \frac{dr^2+dz^2}{r^2d\psi^2} = 1 + \left(\frac{dr^2}{dz^2}+1\right)\frac{dz^2}{r^2d\psi^2},$$

or
$$\frac{d\psi^2}{dz^2} = \frac{\left(\frac{dr^2}{dz^2}+1\right)A^2}{r^2Z} \qquad (6),$$

where
$$Z = \frac{T^2r^2}{w^2} - A^2 \qquad (7),$$

equations suitable for any surface of revolution.

In the special case of the sphere given by equation (5),

$$\frac{dr^2}{dz^2}+1=\frac{1}{r^2} \qquad (8),$$

so that, in conjunction with the value of T in a field of gravity given by (3), equations (6) and (7) become

$$\frac{d\psi^2}{dz^2}=\frac{A^2}{r^4 Z}=\frac{A^2}{(1-z^2)^2 Z},$$

where $$Z=(1-z^2)(h-z)^2-A^2,$$

as in (1) and (2).

2. Also $$\frac{ds}{dz}=\frac{ds}{d\psi}\frac{d\psi}{dz}=\frac{Tr^2}{wA}\frac{\sqrt{\left(\frac{dr^2}{dz^2}+1\right)}A}{r\sqrt{Z}}$$

$$=\frac{Tr}{w}\frac{\sqrt{\left(\frac{dr^2}{dz^2}+1\right)}}{\sqrt{Z}} \qquad (9),$$

reducing for the spherical catenary to

$$\frac{ds}{dz}=\frac{h-z}{\sqrt{Z}} \qquad (10),$$

so that the curve is rectified by the integral

$$s=\int\frac{h-z}{\sqrt{Z}}\,dz \qquad (11).$$

3. If ϕ denotes the angle at which the curve crosses a parallel of latitude, on any surface of revolution,

$$\cos\phi=\frac{r\,d\psi}{ds}=\frac{wA}{Tr} \qquad (12),$$

so that, from (7), $$\sin\phi=\frac{w\sqrt{Z}}{Tr} \qquad (13),$$

$$\tan\phi=\frac{\sqrt{Z}}{A} \qquad (14).$$

The angle ϕ is thus a maximum when $dZ/dz=0$; this leads in the spherical catenary from equation (2) to

$$z=h, \quad \text{or} \quad 2z^2-hz-1=0 \qquad (15).$$

4. Denoting by R the pressure per unit length on the *outer* surface of the sphere, the equations of equilibrium of the chain may be written

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+Rx \quad = 0 \qquad (16),$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right)+Ry \quad = 0 \qquad (17),$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)+Rz+w = 0 \qquad (18),$$

where

$$x^2 + y^2 + z^2 = 1 \qquad (19),$$

$$x\frac{dx}{ds}+y\frac{dy}{ds}+z\frac{dz}{ds} = 0 \qquad (20),$$

$$x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}+z\frac{d^2z}{ds^2} = -1 \qquad (21),$$

so that, multiplying (16) by x, (17) by y, and (18) by z, and adding,

$$-T+R+wz = 0,$$

or

$$R = T-wz = w\,(h-2z) \qquad (22).$$

The pressure R thus changes sign at a depth below the centre greater than $\frac{1}{2}h$; the chain must then be supposed to rest on the interior of the sphere, if it cannot be made to adhere to the exterior surface.

5. Clebsch has shown, in *Crelle* 57, how the quartic Z in (2) can be exhibited as the product of two quadratic or four linear factors, of the form

$$Z = -(z^2+2kz\sin^2\epsilon+k^2\sin^2\epsilon-\cos^2\epsilon)(z^2-2kz\cos^2\epsilon+k^2\cos^2\epsilon-\sin^2\epsilon) \qquad (23),$$

$$= -\left\{z+k\sin^2\epsilon\pm\cos\epsilon\sqrt{(1-k^2\sin^2\epsilon)}\right\}$$
$$\times\left\{z-k\cos^2\epsilon\pm\sin\epsilon\sqrt{(1-k^2\cos^2\epsilon)}\right\} \qquad (24),$$

the arbitrary constants h and A being replaced by k and ϵ, such that

$$h = k\cos 2\epsilon \qquad (25),$$

$$A = \tfrac{1}{2}(1-k^2)\sin 2\epsilon \qquad (26),$$

$$A^2-h^2 = (\cos^2\epsilon-k^2\sin^2\epsilon)(\sin^2\epsilon-k^2\cos^2\epsilon) \qquad (27),$$

and thence the solution of the problem can be given by means of the Jacobian elliptic functions, the integral in equation (1) being composed of two elliptic integrals of the third kind.

By making

$$k \sin \epsilon = 1, \quad \text{or} \quad k \cos \epsilon = 1 \tag{28},$$

two of the roots of the quartic Z become equal, and the elliptic integrals degenerate into circular integrals; in this manner the model No. 156 in the mathematical collection of Brill, of Darmstadt, constructed by Herr Fischer, has been designed.

6. But it is the object of the present paper to bring out the connexion between the integral (1) and the standard form of the elliptic integral of the third kind, employed in my paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. xxv., expressed by the notation, slightly altered by the omission of μ,

$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(s-\sigma)-\sqrt{(-\Sigma)}}{(s-\sigma)\sqrt{S}}\,ds \tag{29},$$

where

$$M^2(s-\sigma) = \wp u - \wp v \tag{30},$$

and

$$M^{-6}\wp'^2 u = S = 4s\,(s+x)^2 - \{(1+y)\,s + xy\}^2 \tag{31},$$

$$M^{-6}\wp'^2 v = \Sigma = 4\sigma\,(\sigma+x)^2 - \{(1+y)\,\sigma + xy\}^2 \tag{32},$$

x and y being the quantities employed by Halphen (*F.E.*, I., p. 102).

It is our object also to utilize the *pseudo-elliptic* integrals for the construction of degenerate, algebraical, cases of the spherical catenary.

Putting

$$\psi - pu = \chi \tag{33},$$

where p is constant, and

$$u = \int \frac{dz}{\sqrt{Z}} \tag{34},$$

the associated elliptic integral of the first kind, so that

$$\chi = \int \frac{A - p\,(1-z^2)}{(1-z^2)\sqrt{Z}}\,dz \tag{35},$$

then it will be shown in the sequel (§ 8) that the integrals (1) and (35) can be made to depend upon the integral (29) by putting

$$A = M\,(y+1) \tag{36},$$

where

$$M^2 = -\frac{y+1}{2x} \tag{37},$$

and

$$A^2 - h^2 = 2y+1 \tag{38},$$

$$h^2 = -\frac{(y+1)^3}{2x} - 2y - 1 \tag{39}.$$

We then find that

$$p = \tfrac{1}{2}(M\rho + A) = \tfrac{1}{2}M(\rho + y + 1) \qquad (40),$$

and $$\sigma = 0 \qquad (41),$$

so that, when the integral (29) is pseudo-elliptic and the parameter v is an aliquot part, one μ^{th}, of a period, we may put

$$v = \frac{4\omega_3}{\mu} \qquad (42),$$

the parameter $$v = \frac{2\omega_3}{\mu} \qquad (43)$$

corresponding to $$\sigma = -x \qquad (44).$$

7. Writing equation (1) in the form

$$\psi = \psi_1 - \psi_2 \qquad (45),$$

where $$\psi_1 = \tfrac{1}{2}A\int\frac{dz}{(1-z)\sqrt{Z}} \qquad (46),$$

$$\psi_2 = \tfrac{1}{2}A\int\frac{dz}{(-1-z)\sqrt{Z}} \qquad (47),$$

shows that ψ is given by the difference of two elliptic integrals of the third kind, with Jacobian parameters v_1 and v_2, such that

$$u = v_1, \quad \text{when} \quad z = +1;$$

$$u = v_2, \quad \text{when} \quad z = -1;$$

and Legendre's theorem for the addition of these integrals shows that ψ can be made to depend upon an elliptic integral of the third kind with parameter

$$v = v_1 - v_2 \qquad (48).$$

The parameters v_1 and v_2, and therefore also v, are each of the form $f\omega_3$, fractions of the imaginary period ω_3; because the real roots of Z must lie between ± 1; and

$$z = \pm 1 \text{ makes } Z = -A^2.$$

It will also be found that

$$z = h \text{ corresponds to } u = \tfrac{1}{2}(v_1 + v_2).$$

8. Comparing the general quartic

$$Z = az^4+4bz^3+6cz^2+4dz+e \qquad (49)$$

and its invariants

$$g_2 = ae-4bd+3c^2 \qquad (50),$$

$$g_3 = ace+2bcd-ad^2-eb^2-c^3 \qquad (51),$$

with our quartic Z, as given in equation (2),

$$a=-1,\; b=\tfrac{1}{2}h,\; c=\tfrac{1}{6}(1-h^2),\; d=-\tfrac{1}{2}h,\; e=h^2-A^2 \qquad (52),$$

and we thus find

$$12g_2 = \quad (1-h^2)^2+12A^2 \qquad (53),$$

$$216g_3 = -(1-h^2)^3-18(1-h^2)A^2+54A^4 \qquad (54),$$

$$1728\Delta = \quad (12g_2)^3-(216g_3)^2$$

$$= 108A^4\{(1-h^2)^3-(8+20h^2-h^4)A^2+16A^4\} \qquad (55).$$

Next taking the cubic S of equation (31), and reducing it to the form in which the coefficient of t^2 is zero,

$$4t^3-\gamma_2 t-\gamma_3 \qquad (56),$$

by putting

$$s = t-\tfrac{1}{12}\{8x-(y+1)^2\} \qquad (57),$$

$$12\gamma_2 = \{(y+1)^2+4x\}^2-24x(y+1) \qquad (58),$$

$$216\gamma_3 = \{(y+1)^2+4x\}^3-36x(y+1)\{(y+1)^2+4x\}+216x^2 \qquad (59),$$

We can now make $g_2 = M^4\gamma_2,\quad g_3 = M^6\gamma_3$;

and therefore, from (34),

$$u = \int\frac{dz}{\sqrt{Z}} = \frac{1}{M}\int\frac{ds}{\sqrt{S}} \qquad (60),$$

on comparison of (53) and (58), (54) and (59), by taking

$$1-h^2 = -M^2\{(y+1)^2+4x\} \qquad (61),$$

$$A^2 = -2M^4x(y+1) \qquad (62),$$

$$A^4 = 4M^6x^2 \qquad (63),$$

provided that

$$M^2 = -\frac{y+1}{2x} \qquad (37);$$

and therefore

$$A^2 = -\frac{(y+1)^3}{2x} = M^2(y+1)^2 \qquad (36),$$

$$1-h^2 = \frac{(y+1)^3}{2x}+2(y+1) \qquad (64),$$

$$A^2-h^2 = 2y+1 \qquad (38).$$

9. Next, let u_1, u_2 denote the values of u corresponding to values z_1, z_2 of z; then, according to Weierstrass's important formula, first published in Biermann's thesis "Problemata quædam mechanica functionum ellipticarum ope soluta," 1865,

$$\wp(u_1 \pm u_2) = \frac{F(z_1, z_2) \mp \sqrt{Z_1}\sqrt{Z_2}}{2(z_1 - z_2)^2} \qquad (65),$$

where $F(z_1, z_2) = (az_1^2 + 2bz_1 + c)\, z_2^2 + 2(bz_1^2 + 2cz_1 + d)\, z_2 + cz_1^2 + 2dz_1 + e$

$$= (az_2^2 + 2bz_2 + c)\, z_1^2 + 2(bz_2^2 + 2cz_2 + d)\, z_1 + cz_2^2 + 2dz_2 + e \qquad (66).$$

Therefore, putting $z_1 = 1$, $z_2 = -1$ in our special form of Z,

$$F(1, -1) = a - 2c + e$$

$$= -\tfrac{4}{3}(1 - h^2) - A^2 \qquad (67),$$

$$\sqrt{Z_1} = \sqrt{Z_2} = Ai \qquad (68),$$

and

$$\wp(v_1 - v_2) = \frac{-\frac{4}{3}(1 - h^2) - A^2 - A^2}{8}$$

$$= -\tfrac{1}{6}(1 - h^2) - \tfrac{1}{4}A^2 \qquad (69),$$

$$\wp(v_1 + v_2) = -\tfrac{1}{6}(1 - h^2) \qquad (70).$$

We also find

$$i\wp'(v_1 - v_2) = \tfrac{1}{4}A(A^2 - h^2 - 1) \qquad (71),$$

$$i\wp'(v_1 + v_2) = \tfrac{1}{2}Ah \qquad (72).$$

It is convenient to denote $v_1 - v_2$ by v, and $v_1 + v_2$ by w; and now equations (36), (61), and (69) show that

$$\frac{12\wp v}{M^2} = -2\frac{1 - h^2}{M^2} - 3\frac{A^2}{M^2}$$

$$= 2(y+1)^2 + 8x - 3(y+1)^2$$

$$= 8x - (y+1)^2 \qquad (73).$$

But equation (57) shows that the relation between s, t, and $\wp u$ is

$$\frac{\wp u}{M^2} = t = s + \tfrac{1}{12}\{8x - (y+1)^2\} \qquad (74),$$

so that, as σ denotes in (29) the value of s corresponding to

$$u = v = v_1 - v_2,$$

therefore

$$\sigma = 0 \qquad (41).$$

Thus, if the integral in (29) is a *pseudo-elliptic* integral, we may put

$$v = \frac{4\omega_3}{\mu} \qquad (42),$$

Also, since $s = -x$ then corresponds to $u = \tfrac{1}{2}v$ (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 205), therefore

$$\wp\tfrac{1}{2}v = \tfrac{1}{12}(1-h^2) \qquad (75),$$

$$i\wp'\tfrac{1}{2}v = \tfrac{1}{2}A \qquad (76).$$

From (69) and (70),

$$\wp w - \wp v = \tfrac{1}{4}A^2 \qquad (78),$$

$$\wp w + \wp v = -\tfrac{1}{3}(1-h^2) - \tfrac{1}{4}A^2 \qquad (79).$$

10. If a factor $z-b$ of Z is known, then

$$A^2 = (1-b^2)(h-b)^2 \qquad (80),$$

and a well known formula of elliptic functions gives

$$\wp u = \frac{\{\tfrac{1}{6}(1-h^2)+b(h-b)\}\, z + \tfrac{5}{6}(1-h^2)\, b - h + 2b^2h - b^3}{2(z-b)} \qquad (81).$$

Then $z = h$ makes

$$\wp u = -\tfrac{1}{12}(5+h^2-6b^2), \quad \wp 2u = -\tfrac{1}{6}(1-h^2) \qquad (82),$$

so that $u = \tfrac{1}{2}w = \tfrac{1}{2}(v_1+v_2)$ corresponds to $z = h$ (83),

$$u = \tfrac{1}{2}v = \tfrac{1}{2}(v_1-v_2) \text{ corresponds to } z = h-b+\frac{1}{b} \qquad (84).$$

But, if c, the value of u which makes $z = \infty$, is given, when

$$\wp c = \tfrac{1}{12}(1-h^2) + \tfrac{1}{2}b(h-b) \qquad (85),$$

the quartic Z can be resolved into four linear factors, in the form

$$z - z_0 = \frac{i\wp' c}{\wp u - \wp c} \qquad (86),$$

$$z - z_1 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_1}{\wp c - e_1} \qquad (87),$$

$$z - z_2 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_2}{\wp c - e_2} \qquad (88),$$

$$z - z_3 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_3}{\wp c - e_3} \qquad (89),$$

$i\wp' c$ being real, because c is a fraction of the imaginary period ω_3.

[redacted]

Then
$$\sqrt{Z} = \frac{dz}{du} = -\frac{i\wp' c\, \wp' u}{(\wp u - \wp c)^2}$$
$$= i\wp(u+c) - i\wp(u-c) \qquad (90),$$

and integrating,

$$z - \tfrac{1}{2}h = -i\zeta(u+c) + i\zeta(u-c) + i\zeta 2c \qquad (91),$$
$$(z-\tfrac{1}{2}h)^2 = -\wp(u+c) - \wp(u-c) - \wp 2c \qquad (92).$$

Writing x for $z-\frac{1}{2}h$, and X for the corresponding value of z, then $6\wp 2c$ and $4i\wp' 2c$ are the coefficients of x^2 and x, so that

$$\wp 2c = -\tfrac{1}{12}(2+h^2) \qquad (93),$$
$$i\wp' 2c = \tfrac{1}{4}h \qquad (94).$$

Thus $c = \frac{1}{2}\omega_3$, if $h = 0$; and then $w = \frac{1}{2}\omega_3$, or $\omega_1 + \frac{1}{2}\omega_3$, as well; this is what Biermann calls the *parabolic* case of the spherical catenary.

Also
$$\wp'' 2c = \tfrac{1}{8}(1+2h^2) - \tfrac{1}{2}A^2 \qquad (95),$$
$$\wp 4c = -\frac{3-4h^2+4h^4}{48h^2} + \tfrac{1}{2}\frac{(1+2h^2)}{h^2}A^2 - \frac{A^4}{h^2} \qquad (96),$$

and c is the parameter required in the rectification of the catenary.

11. The values v_1 and v_2 of u make z assume the values ± 1 and $\sqrt{Z}$ the value Ai; therefore, from (90),

$$A = \frac{\wp' c\, \wp' v_1}{(\wp v_1 - \wp c)^2} = \frac{\wp' c\, \wp' v_2}{(\wp v_2 - \wp c)^2} \qquad (97),$$

and
$$1 - z = \frac{i\wp' c\,(\wp u - \wp v_1)}{(\wp v_1 - \wp c)(\wp u - \wp c)} \qquad (98),$$
$$-1 - z = \frac{i\wp' c\,(\wp u - \wp v_2)}{(\wp v_2 - \wp c)(\wp u - \wp c)} \qquad (99).$$

Therefore
$$\frac{d\psi_1 i}{du} = \frac{\frac{1}{2}iA}{1-z} = \frac{-\frac{1}{2}\wp' v_1 (\wp u - \wp c)}{(\wp v_1 - \wp c)(\wp u - \wp v_1)}$$
$$= \frac{-\frac{1}{2}\wp' v_1}{\wp v_1 - \wp c} + \frac{-\frac{1}{2}\wp' v_1}{\wp u - \wp v_1}$$
$$= -\tfrac{1}{2}\zeta(v_1 - c) - \tfrac{1}{2}\zeta(v_1 + c) + \zeta v_1$$
$$-\tfrac{1}{2}\zeta(u - v_1) + \tfrac{1}{2}\zeta(u + v_1) - \zeta v_1 \qquad (100),$$

and integrating,

$$\psi_1 i = -\tfrac{1}{2}\{\zeta(v_1 - c) + \zeta(v_1 + c)\}u + \tfrac{1}{2}\log\frac{\sigma(u+v_1)}{\sigma(u-v_1)} \qquad (101).$$

Similarly,

$$\psi_2 i = -\tfrac{1}{2}\{\zeta(v_2-c)+\zeta(v_2+c)\}u+\tfrac{1}{2}\log\frac{\sigma(u+v_2)}{\sigma(u-v_2)} \qquad (102);$$

and therefore

$$\psi i = -\tfrac{1}{2}Ru+\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u-v_2)}{\sigma(u-v_1)\,\sigma(u+v_2)} \qquad (103),$$

where $\quad R = \zeta(v_1-c)+\zeta(v_1+c)-\zeta(v_2-c)-\zeta(v_2+c) \qquad (104).$

12. But the formula

$$\frac{\sigma(u+v_1)\,\sigma(u+v_2)\,\sigma(v_1+v_2)}{\sigma(u+v_1+v_2)\,\sigma u\,\sigma v_1\,\sigma v_2} = \begin{vmatrix} 1, & \wp u, & \wp^2 u \\ 1, & \wp v_1, & \wp^2 v_1 \\ 1, & \wp v_2, & \wp^2 v_2 \end{vmatrix} \div \begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v_1, & \wp' v_1 \\ 1, & \wp v_2, & \wp' v_2 \end{vmatrix}$$

$$= \zeta(u+v_1+v_2)-\zeta u-\zeta v_1-\zeta v_2 \qquad (105)$$

shows that, changing v_2 into $-v_2$, and writing v for v_1-v_2,

$$\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}\,\Omega \qquad (106),$$

where Ω is a rational algebraical function of $\wp u$ and $\wp' u$, and therefore also of z and $\surd Z$, of the form

$$\Omega = \frac{C+Bi\,\wp' u}{C-Bi\,\wp' u} \qquad (107),$$

so that $\quad \tfrac{1}{2}\log\Omega = i\tan^{-1}\frac{B}{C}\,\wp' u \qquad (108).$

Equation (29), expressed by elliptic functions of u and v, gives

$$M^2(s-\sigma) = \wp u-\wp v \qquad (109)$$

$$I(v) = \tfrac{1}{2}\rho Mu-iu\zeta v+\tfrac{1}{2}i\log\frac{\sigma(u+v)}{\sigma(u-v)} \qquad (110),$$

so that, with $\quad v = v_1-v_2,$

$$\tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} = -u\zeta v+\tfrac{1}{2}\rho Miu-iI(v) \qquad (111),$$

and thus $\quad \psi = -(i\zeta v-\tfrac{1}{2}Ri-\tfrac{1}{2}\rho M)\,u-I(v)+\tan^{-1}\frac{B}{C}\,\wp' u \qquad (112).$

13. Now

$$
\begin{aligned}
i\zeta v - \tfrac{1}{2}Ri = \; & \tfrac{1}{2}i\,\{\zeta(v_1-v_2)-\zeta(v_1-c)-\zeta(v_2-c)\} \\
& +\tfrac{1}{2}i\,\{\zeta(v_1-v_2)-\zeta(v_1+c)-\zeta(v_2+c)\} \\
= \; & \tfrac{1}{2}\sqrt{\{-\wp(v_1-v_2)-\wp(v_1-c)-\wp(v_2-c)\}} \\
& +\tfrac{1}{2}\sqrt{\{-\wp(v_1-v_2)-\wp(v_1+c)-\wp(v_2+c)\}} \qquad (113).
\end{aligned}
$$

But, taking $z=\pm 1$, $z_2=\infty$, in equation (65),

$$\wp(v_1\pm c)=\frac{-1+h+\frac{1}{6}(1-h^2)\pm A}{2} \qquad (114),$$

$$\wp(v_2\pm c)=\frac{-1-h+\frac{1}{6}(1-h^2)\pm A}{2} \qquad (115);$$

and therefore

$$
\begin{aligned}
-\wp(v_1-v_2)-\wp(v_1-c)-\wp(v_2-c) &= 1+A+\tfrac{1}{4}A^2 \\
&= (1+\tfrac{1}{2}A)^2 \qquad (116), \\
-\wp(v_1-v_2)-\wp(v_1+c)-\wp(v_2+c) &= 1-A+\tfrac{1}{4}A^2 \\
&= (1-\tfrac{1}{2}A)^2 \qquad (117),
\end{aligned}
$$

and taking the square roots of opposite signs, equation (113) gives

$$i\zeta v-\tfrac{1}{2}Ri=-\tfrac{1}{2}A \qquad (118),$$

so that

$$\psi=\tfrac{1}{2}(A+\rho M)\,u-I(v)+\tan^{-1}\frac{B}{C}\,\wp' u \qquad (119),$$

or, as stated at the outset,

$$\psi=pu+\chi \qquad (33),$$

where

$$p=\tfrac{1}{2}(A+\rho M) \qquad (40),$$

and then

$$\chi=-I(v)+\tan^{-1}\frac{B}{C}\,\wp' u \qquad (120).$$

One great difficulty in these calculations is the determination of the proper sign to employ when a square root is taken, or with an imaginary quantity; in most cases this can be settled only by a verification, or by an appeal to a special case.

14. When $I(v)$ is pseudo-elliptic, then $\mu\chi$ is an angle such that $\sin\mu\chi$ and $\cos\mu\chi$ are algebraical functions of z and $\sqrt{Z}$; and now, putting

$$P = \mu p = \tfrac{1}{2}\mu\,(A+M\rho) \qquad (121),$$

$$Q = \mu q = \tfrac{1}{2}\mu\,(A-M\rho) \qquad (122),$$

so that

$$P+Q = \mu A \qquad (123),$$

then

$$\mu\chi = \mu\psi - Pu \qquad (124),$$

and the equation of the catenary can be written in either of the forms

$$r^\mu \cos\mu\chi = Hz^\mu + H_1 z^{\mu-1} + \ldots + H_\mu \qquad (125),$$

$$r^\mu \sin\mu\chi = (Lz^{\mu-2} + L_1 z^{\mu-3} + \ldots + L_{\mu-2})\sqrt{Z} \qquad (126),$$

where

$$r^2 + z^2 = 1 \qquad (5),$$

both leading to the differential relation, equivalent to (1),

$$\mu\frac{d\chi}{dz} = \frac{Pz^2+Q}{(1-z^2)\sqrt{Z}} \qquad (127),$$

or

$$\mu\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2 + Q \qquad (128).$$

15. Squaring and adding (125) and (126), and equating coefficients, leads to

$$H^2 - L^2 = (-1)^\mu$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots,$$

$$H_\mu^2 - (2y+1)\,L_{\mu-2}^2 = 1 \qquad (129),$$

and to other relations, theoretically sufficient, in conjunction with the differentiations of (125) and (126), leading to (128), to determine the other coefficients H and L in terms of P, Q, A, h, and ρ; all functions of a single parameter, when once the pseudo-elliptic form of integral (29) for an assigned order μ has been introduced.

But for values of μ above 6 the complication of this method became so formidable that it was absolutely necessary to seek for some other method of determining the leading coefficients H and L; this was effected by a consideration of the form assumed when $z = \infty$, in a manner to be explained in the sequel (§ 30).

and $3 I(\;) = \frac{1}{2}\int \frac{-s-3c}{s\sqrt{S}}\,ds$

$$= \cos^{-1}\frac{s+c}{2s^{\frac{3}{2}}} = \sin$$

where $$S = 4s^3-(s+c)^2$$

Then, as in (58) and (59),

$$12\gamma_2 = 1+24c$$
$$216\gamma_3 = 1+36c+216c^2$$

so that we must take, as in (61), (62), (63)

$$1-h^2 = -M^2$$
$$A^2 = -2M^4c$$
$$A^2 = \quad 4M^6c^2$$

and therefore $$M^2 = A^2 = -\frac{1}{2c}$$

$$1-h^2 = -A^2, \quad \text{or} \quad h^2 = A$$

and $$Z = -z^4+2hz^3+(1-h^2$$

Equation (131), on comparison with (29

take $$3\rho = -1;$$

and thus

$$P = \tfrac{3}{2}(A+M\rho) = A$$

17. Differentiating (143) logarithmically,

$$\frac{-3z}{1-z^2}-3\tan 3\chi\frac{d\chi}{dz}=\frac{3Hz^2+2H_1z+H_2}{Hz^3+H_1z^2+H_2z+H_3},$$

or $$3\,\frac{Lz+L_1}{Hz^3+H_1z^2+H_2z+H_3}\sqrt{Z}\,\frac{d\chi}{dz}=\frac{3z}{z^2-1}-\frac{3Hz^2+2H_1z+H_2}{Hz^3+H_1z^2+H_2z+H_3},$$

or $$3\,(Lz+L_1)(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz}$$

$$=-3z\,(Hz^3+H_1z^2+H_2z+H_3)+(3Hz^2+2H_1z+H_2)(z^2-1)$$

$$=-H_1z^3-(3H+2H_2)\,z^2-(2H_1+3H_3)\,z-H_2 \qquad (146),$$

and this must

$$=(Lz+L_1)\,A\,(z^2+2) \qquad (147);$$

and therefore, equating coefficients

$$0-H_1=AL \qquad (148),$$

$$-3H-2H_2=AL_1 \qquad (149),$$

$$-2H_1-3H_3=2AL \qquad (150),$$

$$-H_2-0=2AL_1 \qquad (151).$$

Differentiating (144) logarithmically, we find in a similar manner

$$3\,(Hz^3+H_1z^2+H_2z+H_3)(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz},$$

or $$3\,(Hz^3+H_1z^2+H_2z+H_3)\,A\,(z^2+2)$$

$$=3z\,(Lz+L_1)\,Z-L\,(z^2-1)\,Z-(Lz+L_1)(z^2-1)\,\tfrac{1}{2}\frac{dZ}{dz}$$

$$=(2Lz^2+3L_1z+L)\,\{-z^4+2hz^3+(1-h^2)\,z^2-2hz+1\}$$

$$-(Lz+L_1)(z^2-1)\{-2z^3+3hz^2+(1-h^2)\,z-h\} \qquad (152),$$

and therefore, equating coefficients,

$$3AH=hL-L_1 \qquad (153),$$

... ...

$$6AH_3=L-hL_1 \qquad (154).$$

From these equations we find

$$H=A,\quad H_1=-Ah,\quad H_2=-2A,\quad H_3=0,$$

$$L=h,\quad L_1=1; \qquad (155);$$

so that the equation of the catenary may be written

$$(1-z^2)^{\frac{3}{2}} e^{3\chi i} = A(z^3-hz^2-2z)+i(hz+1)\sqrt{Z} \qquad (156),$$

with $\qquad 3\chi = 3\psi - Au$, and $h = \sqrt{(A^2+1)}$,

and now squaring and adding equations (143) and (144) will lead to a verification, when the conditions obtained above are satisfied.

Equation (139) gives, in conjunction with Clebsch's notation of (25), (26), (27),

$$\cos 2\epsilon = \frac{\sqrt{(k^4-2k^2+5)}}{k^2+1}, \quad \sin 2\epsilon = 2\frac{\sqrt{(k^2-1)}}{k^2+1},$$

$$h^2 = \frac{k^6-2k^4+5k^2}{(k^2+1)^2},$$

$$A^2 = \frac{(k^2-1)^3}{(k^2+1)^2},$$

$$Z = \left\{-z^2+\frac{k^2+1+\sqrt{(k^4-2k^2+5)}}{k^2+1}kz-\frac{k^2-1+\sqrt{(k^4-2k^2+5)}}{2}\right\}$$
$$\left\{z^2+\frac{k^2+1-\sqrt{(k^4-2k^2+5)}}{k^2+1}kz+\frac{k^2-1-\sqrt{(k^4-2k^2+5)}}{2}\right\}.$$

$$\mu = 4.$$

18. Here $y=0$, and $v=\omega_3$;

but, from (38), $\qquad A^2 = h^2+1 \qquad (157),$

$$Z = -z^4+2hz^3+(1-h^2)z^2-2hz-1$$
$$= -(z^2-hz-1)^2-z^2 \qquad (158),$$

so that $\sqrt{Z}$ is imaginary, and the catenary also.

$$\mu = 5.$$

19. Here we must put

$$x = y = -c,$$

suppose (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 213); and now

$$M^2 = \frac{1+y}{-2x} = \frac{1-c}{2c} \qquad (159),$$

$$A^2 = \frac{(1-c)^3}{2c} \qquad (160),$$

$$h^2 = A^2 - 2y - 1$$

$$= \frac{(1-c)^3}{2c} - 1 + 2c$$

$$= \frac{1 - 5c + 7c^2 - c^3}{2c} \qquad (161).$$

Now, with $$v = \tfrac{4}{5}\omega_3,$$

we may write (29) in the form

$$5I(v) = \tfrac{1}{2}\int \frac{(1+3c)\,s - 5c^2}{s\sqrt{S}}\,ds$$

$$= \tan^{-1} \frac{(s-c^2)\sqrt{S}}{(1+3c)\,s^2 - (2c^2+c^3)\,s + c^4} \qquad (162),$$

where $$S = 4s\,(s-c)^2 - \{(1-c)\,s + c^2\}^2 \qquad (163),$$

$$5\rho = 1 + 3c \qquad (164),$$

so that $$P = \tfrac{5}{2}(A + M\rho) = M(3-c) \qquad (165),$$

$$Q = \tfrac{5}{2}(A - M\rho) = 2M(1-2c) \qquad (166),$$

$$P + Q = 5M(1-c) = 5A \qquad (167).$$

The equation of the catenary is now of either of the forms

$$(1-z^2)^{\frac{5}{2}} \cos 5\chi = Hz^5 + H_1 z^4 + H_2 z^3 + H_3 z^2 + H_4 z + H_5 \qquad (168),$$

$$(1-z^2)^{\frac{5}{2}} \sin 5\chi = (Lz^3 + L_1 z^2 + L_2 z + L_3)\sqrt{Z} \qquad (169),$$

leading to the relation

$$5(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2 + Q \qquad (170),$$

also $$L^2 - H^2 = 1 \qquad (171).$$

20. A straightforward verification of (170) by logarithmic differentiation of (168) and (169) leads to the following values of the coefficients

$$H = \frac{2 - 5c + c^2}{c} M \qquad (172),$$

$$H_1 = \frac{(3-c)(2-c)}{c} Mh \qquad (173),$$

$$H_2 = \frac{3-15c+20c^2-4c^3}{c^3} M \tag{174}$$

$$H_3 = -\frac{1-9c+4c^2}{c^3} Mh \tag{175}$$

$$H_4 = -2\frac{(1-2c)^2}{c^3} M \tag{176}$$

$$H_5 = -\frac{2}{c} Mh \tag{177}$$

$$L = -\frac{2-c}{c} h \tag{178}$$

$$L_1 = -\frac{2-6c+3c^2}{c^2} \tag{179}$$

$$L_2 = \frac{1-2c}{c^2} h \tag{180}$$

$$L_3 = \frac{1-2c}{c^2} \tag{181}$$

The verification is rather long, but the work is given herewith, as a type of the calculations required.

Differentiating (168) logarithmically,

$$\frac{-5z}{1-z^2} - 5\tan 5\chi \frac{d\chi}{dz} = \frac{5Hz^4+4H_1z^3+\dots}{Hz^5+H_1z^4+\dots} \tag{182}$$

that is, $\quad 5\tan 5\chi \dfrac{d\chi}{dz}$ or $5\dfrac{Lz^3+L_1z^2+\dots}{Hz^5+H_1z^4+\dots}\sqrt{Z}\dfrac{d\chi}{dz}$

$$= \frac{-5z}{1-z^2} - \frac{5Hz^4+4H_1z^3+\dots}{Hz^5+H_1z^4+\dots} \tag{183}$$

$$5(Lz^3+L_1z^2+\dots)(1-z^2)\sqrt{Z}\frac{d\chi}{dz} \quad\text{or}\quad (Lz^3+L_1z^2+\dots)(Pz^2+Q)$$

$$= -5z(Hz^5+H_1z^4+\dots)+(z^2-1)(5Hz^4+4H_1z^3+\dots) \tag{184}$$

and therefore, equating coefficients of z^6, z^5, ...,

$$0+PL = 0 - H_1 \tag{185}$$

$$0+PL_1 = -5H-2H_2 \tag{186}$$

$$QL+PL_2 = -4H_1-3H_3 \tag{187}$$

$$QL_1+PL_3=-3H_2-4H_4 \qquad (188),$$

$$QL_2+\ 0\ =-2H_3-5H_5 \qquad (189),$$

$$QL_3+\ 0\ =-\ H_4-\ 0 \qquad (190).$$

Differentiating (169) logarithmically,

$$\frac{-5z}{1-z^2}-5\cot 5\chi\frac{d\chi}{dz}=\frac{3Lz^2+2L_1z+L_2}{Lz^3+L_1z^2+L_2z+L_3}+\frac{-2z^3+3hz^2+(1-h^2)z-h}{Z} \qquad (191),$$

$$5\,\frac{Hz^5+H_1z^4+\dots}{Lz^3+L_1z^2+\dots}\,\frac{1}{\sqrt{Z}}\,\frac{d\chi}{dz}=\frac{5z}{1-z^2}+\frac{3Lz^2+\dots}{Lz^3+\dots}+\frac{-2z^3+3hz^2+\dots}{Z} \qquad (192),$$

$$5\,(Hz^5+H_1z^4+\dots)(1-z^2)\sqrt{Z}\frac{d\chi}{dz}\quad\text{or}\quad(Hz^5+H_1z^4+\dots)(Pz^2+Q)$$

$$=\left\{5Lz^4+5L_1z^3+5L_2z^2+5L_3z-(z^2-1)(3Lz^2+2L_1z+L_2)\right\}Z$$

$$-(z^2-1)(Lz^3+L_1z^2+L_2z+L_3)\left\{-2z^3+3hz^2+(1-h^2)z-h\right\} \qquad (193),$$

and equating coefficients,

$$0\ +PH\ =hL-L_1 \qquad (194),$$

$$0\ +PH_1=-(4+h^2)L+3hL_1-2L_2 \qquad (195),$$

$$QH\ +PH_2=6hL-2(1+h^2)L_1+5hL_2-3L_3 \qquad (196),$$

$$QH_1+PH_3=\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \qquad (197),$$

$$QH_2+PH_4=\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \qquad (198),$$

$$QH_3+PH_5=\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \qquad (199),$$

$$QH_4+\ 0\ =-2(A^2-h^2)L_1-3hL_2-(5A^2-4h^2-1)L_3 \qquad (200),$$

$$QH_5+\ 0\ =-(A^2-h^2)L_2-hL_3 \qquad (201).$$

Put

$$\frac{H}{L}=x \qquad (202),$$

so that, when x is found, the values of H and L can be inferred from (171),

$$H=\frac{x}{\sqrt{(1-x^2)}} \qquad (203),$$

$$L=\frac{1}{\sqrt{(1-x^2)}} \qquad (204).$$

[redacted]

21. Then, from (185), $\dfrac{H_1}{L} = -P$ (205),

from (194), $\dfrac{L_1}{L} = h - Px$ (206),

from (186), $2\dfrac{H_2}{L} = -5x - P\dfrac{L_1}{L} = (P^2-5)\,x - Ph$ (207),

from (195), $2\dfrac{L_2}{L} = -4 - h^2 + 3h\dfrac{L_1}{L} - P\dfrac{H_1}{L}$

$= -3Phx + P^2 + 2h^2 - 4$ (208),

from (187), $3\dfrac{H_3}{L} = -4\dfrac{H_1}{L} - Q - P\dfrac{L_2}{L}$

$= \tfrac{3}{2}P^2hx - \tfrac{1}{2}P^3 + 6P - Q - Ph^2$ (209),

from (196), $3\dfrac{L_3}{L} = 6h - 2\,(1+h^2)\dfrac{L_1}{L} + 5h\dfrac{L_2}{L} - Qx - P\dfrac{H_2}{L}$

$= 6h - 2\,(1+h^2)(h-Px)$

$+ \tfrac{5}{2}h\,(-3Phx + P^2 + 2h^2 - 4)$

$- Qx - \tfrac{1}{2}(P^3 - 5P)\,x + \tfrac{1}{2}P^2h$ (210),

from (188), $4\dfrac{H_4}{L} = -3\dfrac{H_2}{L} - Q\dfrac{L_1}{L} - P\dfrac{L_3}{L}$

$= -\tfrac{3}{2}(P^2-5)\,x - Q\,(h-Px) - P\dfrac{L_3}{L}$ (211),

and from (190), $\dfrac{H_4}{L} + Q\dfrac{L_3}{L} = 0$ (212),

so that $-9\,(P^2-5)\,x - 6Q\,(h-Px)$

$+ (4Q-P)\left[h\left\{12 - 4(1+h^2) + 5\,(P^2+2h^2-4) + P^2\right\}\right.$

$\left.- x\left\{P^3 - 5P - 4(1+h^2)\,P + 15Ph^2\right\}\right] = 0$ (213),

the equation to determine x; thence

$$x = \frac{(4Q-P)(6P^2+6h^2-12) - 6Q}{9\,(P^2-5) - 6PQ + (4Q-P)(P^2-9+11h^2)\,P}\,h \quad (214),$$

where, from the values of M, h, P, Q, ... given in (159)—(167), with

$$4Q - P = 5\,(1-3c)\,M \quad (215),$$

we find, after reduction and cancelling a common factor,

$$5 - 14c + 6c^2,$$

of the numerator and denominator,

$$x = \frac{2-5c+c^2}{2-c}\,\frac{M}{h} \tag{216}$$

so that, from (203) and (204),

$$H = \frac{2-5c+c^2}{c} M \tag{172}$$

$$L = \frac{2-c}{c} h \tag{178}$$

and thence the values of H_1, H_2, H_3, H_4, H_5, L_1, L_2, L_3 are readily inferred from equations (185)—(201).

Putting $z = \pm 1$ in (184), we obtain

$$(\ \ L+L_1+L_2+L_3)(P+Q) = -5\,(\ \ H+H_1+\ldots+H_5) \tag{179}$$

$$(-L+L_1-L_2+L_3)(P+Q) = \ \ 5\,(-H+H_1-\ldots+H_5) \tag{180}$$

so that, in conjunction with (167),

$$A\,(L_1+L_3)+H\ +H_2+H_4 = 0 \tag{181}$$

$$A\,(L\ +L_2)+H_1+H_3+H_5 = 0 \tag{182}$$

useful as verifications; and the same can be obtained by putting $z = \pm 1$ in (193).

$$\mu = 6.$$

22. The equation of the catenary can be reduced to the same form as for $\mu = 3$, namely, combining (143) and (144),

$$(1-z^2)^{\frac{3}{2}}\, e^{3\chi i} = Hz^3+H_1z^2+H_2z+H_3+i\,(Lz+L_1)\sqrt{Z} \tag{217}$$

Referring to the *Proc. Lond. Math. Soc.*, Vol. xxv., p. 216,

$$\gamma_6 = 0 \tag{218}$$

or

$$y-x-y^2 = 0 \tag{219}$$

is satisfied by taking

$$y = -c, \quad x = -c-c^2 \tag{220}$$

and then

$$M^2 = \frac{1-c}{2c\,(1+c)} \tag{221}$$

$$A^2 = \frac{(1-c)^3}{2c\,(1+c)} \tag{222}$$

$$h^2 = \frac{(1-3c)(1-2c-c^2)}{2c\,(1+c)} \tag{223}$$

The pseudo-elliptic form of (29) can now be written, with

$$v = \tfrac{2}{3}\omega_3 \qquad (224),$$

$$3I(v) = \tfrac{1}{2}\int \frac{(1+3c)\,s - 3c^2(1+c)}{s\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1+3c)\,s - c^2(1+c)}{2s^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} \qquad (225),$$

and thus

$$3\rho = 1+3c \qquad (226),$$

$$P = \tfrac{3}{2}(A+M\rho) = 2M \qquad (227),$$

$$Q = \tfrac{3}{2}(A-M\rho) = (1-3c)\,M \qquad (228),$$

$$P+Q = 3A \qquad (229),$$

and the differential relation to be satisfied by (217) is

$$3(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \qquad (230).$$

23. Differentiating (217) logarithmically and equating coefficients, as in the case of $\mu = 3$, the resulting equations are

$$0 - H_1 = PL \qquad (231),$$

$$-3H - 2H_2 = PL_1 \qquad (232),$$

$$-2H_1 - 3H_3 = QL \qquad (233),$$

$$-H_2 - 0 = QL_1 \qquad (234),$$

$$PH = hL - L_1 \qquad (235),$$

$$PH_1 = \dots \quad \dots \qquad (236),$$

$$QH + PH_2 = \dots \quad \dots \qquad (237),$$

$$QH_1 + PH_3 = \dots \quad \dots \qquad (238),$$

$$QH_2 + 0 = \dots \quad \dots \qquad (239),$$

$$QH_3 + 0 = \dots \quad \dots \qquad (240).$$

From (231) and (233), eliminating H_1,

$$3H_3 = (2P-Q)\,L,$$

or

$$H_3 = (1+c)\,ML \qquad (241).$$

From (232) and (234), eliminating H_2,

$$3H = (2Q-P)\,L_1,$$

or
$$H = -2cML_1 \qquad (242).$$

Substituting in (235),

$$\frac{hL}{L_1} = 1+\frac{PH}{L_1} = 1-2cPM$$
$$= 1-4cM^2 = 1-2\frac{1-c}{1+c} = -\frac{1-3c}{1+c} \qquad (243);$$

and therefore
$$\frac{H}{L} = -2cM\frac{L_1}{L} = 2c\frac{1+c}{1-3c}Mh$$
$$= 2c\frac{1+c}{1-3c}\,\frac{\sqrt{\{(1-c)(1-3c)(1-2c-c^2)\}}}{2c\,(1+c)}$$
$$= \sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{1-3c}\right\}} \qquad (244).$$

Also, by squaring and adding (143) and (144),

$$L^2-H^2 = 1 \qquad (245),$$

and thence

$$H^2 = \frac{H^2}{L^2-H^2} = \frac{(1-c)(1-2c-c^2)}{1-3c-(1-c)(1-2c-c^2)} = \frac{(1-c)(1-2c-c^2)}{-c^2\,(1+c)},$$

$$H = \frac{1}{c}\sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{-1-c}\right\}} \qquad (246),$$

$$L = \frac{1}{c}\sqrt{\left(\frac{1-3c}{-1-c}\right)} \qquad (247),$$

$$H_1 = -PL = -2ML = -\frac{1}{c\,(1+c)}\sqrt{\left\{\frac{2\,(1-c)(1-3c)}{-c}\right\}} \qquad (248),$$

$$L_1 = -\frac{H}{2cM} = -\frac{1}{c}\sqrt{\left(\frac{1-2c-c^2}{-2c}\right)} \qquad (249),$$

$$H_2 = -QL_1 = \frac{1-3c}{2c^2}\sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{-1-c}\right\}} \qquad (250),$$

$$H_3 = (1+c)\,ML = \frac{1}{c}\sqrt{\left\{\frac{(1-c)(1-3c)}{-2c}\right\}} \qquad (251).$$

The verifications obtained by putting $z = \pm 1$ in (146) are

$$AL+H_1+H_3 = 0 \qquad (252),$$
$$AL_1+H+H_2 = 0 \qquad (253),$$

and these are found to be satisfied.

24. When μ is even, the cubic S can be resolved into factors; and therefore Z also can be factorized.

Proceeding in Clebsch's manner (*Crelle*, 57, p. 105), and writing k for Clebsch's ρ, we put

$$Z = -\left(z^2 - hz + \frac{k^2-1}{2}\right)^2 + \left(kz - \frac{h}{k}\,\frac{k^2+1}{2}\right)^2$$

$$= -z^4 + 2hz^3 + (1-h^2)\,z^2 - 2hz$$

$$-\left(\frac{k^2-1}{2}\right)^2 + \frac{h^2}{k^2}\left(\frac{k^2+1}{2}\right)^2 \qquad (254),$$

and this is the case provided that

$$A^2 - h^2 = \left(\frac{k^2-1}{2}\right)^2 - \frac{h^2}{k^2}\left(\frac{k^2+1}{2}\right)^2$$

or

$$(k^2-1)^2\,(k^2-h^2) - 4A^2k^2 = 0 \qquad (255),$$

a cubic equation for k^2; and Z then breaks up into the quadratic factors Z_1 and Z_2, where

$$Z_1 = -z^2 + (h+k)\,z - \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \qquad (256),$$

$$Z_2 = \quad z^2 - (h-k)\,z + \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \qquad (257).$$

But, putting, with Halphen's notation of x and y,

$$k^2 = 4M^2s + h^2 - A^2 = -2\,\frac{1+y}{x}\,s - 1 - 2y \qquad (258),$$

$$k^2 - 1 = -2\,(1+y)\,\frac{s+x}{x} \qquad (259),$$

$$k^2 - h^2 = 4M^2s - A^2 = M^2\left\{4s - (1+y)^2\right\} \qquad (260),$$

the cubic equation (255) for k^2 becomes transformed into

$$4s\,(s+x)^2 - \left\{(1+y)\,s + xy\right\}^2 = 0, \quad \text{or} \quad S = 0 \qquad (31);$$

and therefore the quartic Z can be resolved when we know a root, [redacted] s_3, of equation (31).

25. Clebsch's $\sin 2\epsilon$ or x (which we change into ξ to avoid confusion of notation) is connected with k by the relation

$$\cos^2 2\epsilon = 1-\xi^2 = \frac{h^2}{k^2} \tag{261}$$,

and it is connected with the s above in (258)—(260) by the relation

$$\xi = M(-2\sqrt{s}+1+y) = -2M\sqrt{s}+A \tag{262}$$.

For Clebsch's m is our h, and his b is our A; so that his resolvent cubic (*Crelle*, 57, p. 105)

$$(x-2b)(1-x^2)-m^2x = 0 \tag{263}$$

becomes
$$(\xi-2A)(1-\xi^2)-h^2\xi = 0,$$

or
$$\xi^3-2A\xi^2+(1-h^2)\,\xi+2A = 0 \tag{264}$$,

and, putting
$$\xi = M(-2t+1+y) \tag{265}$$,

this reduces (264) to

$$2t^3-(1+y)\,t^2+2xt-xy = 0 \tag{266}$$,

or, putting $t = \sqrt{s}$, to

$$4s\,(s+x)^2-\{(1+y)\,s+xy\}^2 = 0 \tag{31}$$.

26. In the case of $\mu = 6$, we find that

$$s_3 = y^2 = c^2 \tag{267}$$,

and then
$$k^2 = -1+2c+2\frac{1-c}{c+c^2}c^2 = \frac{1-3c}{-1-c} \tag{268}$$,

$$\frac{k^2-1}{2} = \frac{1-c}{-1-c} \tag{269}$$,

$$\frac{k^2+1}{2} = \frac{2c}{1+c} \tag{270}$$,

$$\frac{h^2}{k^2} = \frac{1-2c-c^2}{-2c} \tag{271}$$,

and the quadratic factors Z_1 and Z_2 of Z are thus determined.

Putting
$$Z_1 = (z_1-z)(z-z_0) \tag{272}$$,

so that $z = z_1$ and $z = z_0$ give parallels of latitude between which a branch of the catenary lies, then, as z grows from z_0 to z_1, the

[redacted]

variable u may be taken to grow from 0 to ω_1, the real period of the elliptic functions, such that

$$\omega_1 = \int_{z_0}^{z_1} \frac{dz}{\sqrt{Z}} \qquad (273).$$

If K denotes the corresponding quarter-period of the associated Jacobian elliptic functions, then (Klein, *Math. Ann.*, XIV., p. 118)

$$\sqrt[4]{(12g_2)}\,\omega_1 = \sqrt[4]{(1-\kappa^2\kappa'^2)}\,2K \qquad (274).$$

At the same time $\mu\chi$ grows from 0 to $\frac{1}{2}\pi$, or $\mu\psi$ from 0 to $P\omega_1+\frac{1}{2}\pi$; so that, if $P\omega_1$ can be made an exact multiple of $\frac{1}{2}\pi$, the catenary will close in upon itself, and form a closed curve.

27. The simplest mode of effecting this closure is to make P vanish; but, in the case of $\mu = 6$, this requires A to vanish also, and the catenary degenerates into a vertical great circle.

Calculating the invariants g_2 and g_3 of the quartic Z in this case of $\mu = 6$, we find, from (53) and (54),

$$12g_2 = \frac{(1-c)^2\,(1+3c)(1+9c+3c^2+3c^3)}{4c^2\,(1+c)^2} \qquad (275),$$

$$216g_3 = \frac{(1-c)^3\,(1+6c-3c^2)(1+12c+30c^2+36c^3+9c^4)}{8c^3\,(1+c)^3} \qquad (276)$$

$$\begin{aligned} S &= 4s\,(s-c-c^2)^2-\{(1-c)\,s+c^2+c^3\}^2 \\ &= \{4s^2-(1+c)(1+5c)\,s+c^2\,(1+c)^2\}\,(s-c^2) \\ &= 4\,(s-s_1)(s-s_2)(s-s_3) \qquad (277), \end{aligned}$$

suppose; and, with $s_1 > s_3 > s_2$,

$$\kappa^2\kappa'^2 = \frac{(s_1-s_3)(s_3-s_2)}{(s_1-s_2)^2} = \frac{16c^3}{(1+c)^3\,(1+9c)} \qquad (278),$$

$$1-\kappa^2\kappa'^2 = \frac{(1+3c)(1+9c+3c^2+3c^3)}{(1+c)^3\,(1+9c)} \qquad 279),$$

and thus, from (274),

$$\frac{\omega_1}{2K} = \sqrt[4]{\left\{\frac{4c^2}{(1-c)^2\,(1+c)(1+9c)}\right\}} \qquad (280),$$

$$P\omega_1 = 2M\omega_1 = \frac{4K}{\sqrt[4]{\{(1+c)^3\,(1+9c)\}}} = \frac{2K\sqrt{(\kappa\kappa')}}{c^{\frac{3}{4}}} \qquad (281).$$

To construct a closed catenary we must search, by a tentative process, for values of c which will make

$$\frac{2K\sqrt{(\kappa\kappa')}}{\frac{1}{2}\pi c^{\frac{1}{2}}} = 1, 2, 3, \dots \qquad (282),$$

where $\kappa\kappa'$ is given by (278).

The formula given by Klein (*Math. Ann.*, XIV., p. 119),

$$12g_2\left(\frac{\omega_1}{\pi}\right)^4 = 1+240\left(\frac{q^2}{1-q^2}+\frac{2^3q^4}{1-q^4}+\dots\right)$$

$$\simeq 1+\frac{240q^2}{1-q^2} \quad \text{or} \quad 1+240q^2 \qquad (283),$$

will perhaps assist in obtaining a first approximation; or otherwise the curve which is the graph of (282) must be plotted, preferably with logarithmic coordinates.

28. With negative discriminant

$$\Delta = g_2^3-27g_3^2 = \frac{(1-c)^6(1+9c)}{64(1+c)^3} \qquad (284),$$

two of the roots of the quartic Z and also of the cubic S, are imaginary; and now

$$4\kappa^2\kappa'^2 = \frac{(1+c)^3(1+9c)}{64c^3} \qquad (285),$$

the reciprocal of the preceding value of $4\kappa^2\kappa'^2$ in (278); and

$$\kappa^2 = \tfrac{1}{2}-\frac{1+6c-3c^2}{16(-c)^{\frac{3}{2}}} \qquad (286),$$

$$\kappa'^2 = \tfrac{1}{2}+\frac{1+6c-3c^3}{16(-c)^{\frac{3}{2}}} \qquad (287),$$

so that c is now negative, or $y = -c$ positive.

With negative discriminant, suppose

$$Z_2 = (z-m)^2+n^2 \qquad (288),$$

and Z_1 as before in (272); then, from (256) and (257),

$$z_0+z_1 = h+k \qquad (289),$$

$$z_0z_1 = \frac{k^2-1}{2}+\frac{h}{k}\,\frac{k^3+1}{2} \qquad (290),$$

$$2m = h - k \tag{291}$$

$$m^2+n^2 = \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \tag{292}$$

so that, in the general case,

$$\{(z_0-m)^2+n^2\}\{(z_1-m)^2+n^2\} = (k^2-1)^2+\left(1-\frac{h^2}{k^2}\right)(k^4-1) \tag{293}$$

In the special case of $\mu = 6$, from (269), (270), and (271), with $c = -y$,

$$\{(z_0-m)^2+n^2\}\{(z_1-m)^2+n^2\} = 4\left(\frac{1+y}{1-y}\right)^2 - 4\,\frac{(1+y)^2}{1-y}$$

$$= 4y\left(\frac{1+y}{1-y}\right)^2 \tag{294}$$

and then

$$P\omega_1 = 2M\int_{z_0}^{z_1}\frac{dz}{\surd Z} = 2M\,\frac{2K}{\surd\{(z_0-m)^2+n^2\,.\,(z_1-m)^2+n^2\}}$$

$$= \frac{2K}{y}\sqrt{\left(\frac{1-y}{1+y}\right)} \tag{295}$$

But since $(1-y)(1-9y)$ is now negative, or $1 > y > \frac{1}{9}$, M^2 is negative, and the catenary is imaginary.

$$\mu = 7.$$

29. Here the first relation to be satisfied is

$$\gamma_7 = 0 \tag{296}$$

or

$$(y-x)\,x - y^3 = 0 \tag{297}$$

the equation of a unicursal cubic, in which we can put

$$x = -c\,(1+c)^2 \tag{298}$$

$$y = -c\,(1+c) \tag{299}$$

and now

$$M^2 = \frac{1-c-c^2}{2c\,(1+c)^2} \tag{300}$$

$$A^2 = \frac{(1-c-c^2)^3}{2c\,(1+c)^2} \tag{301}$$

$$A^2 - h^2 = 1 - 2c - 2c^3 \tag{302}$$

$$h^2 = \frac{1-5c+0+15c^3+12c^4+c^5-c^6}{2c\,(1+c)^2} \qquad (303).$$

The integral (29) is now (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 226), with

$$v = \tfrac{4}{7}\omega_3 \qquad (304),$$

$$7I\,(v) = \tfrac{1}{2}\int \frac{(3+9c+5c^2)\,s+7c^2\,(1+c)^3}{s\sqrt{S}}\,ds$$

$$= \tan^{-1}\frac{(s^2+Cs+D)\sqrt{S}}{7\rho s^3+\sigma s^2+\tau s+\phi} \qquad (305),$$

where

$$7\rho = 3+9c+5c^2 \qquad (306),$$

$$\sigma = -(1+c)^3\,(1+6c+13c^2+5c^3) \qquad (307),$$

$$\tau = c^2\,(1+c)^6\,(2+6c+c^2) \qquad (308),$$

$$\phi = -c^4\,(1+c)^9 \qquad (309),$$

$$C = -(1+c)^3\,(1+3c) \qquad (310),$$

$$D = c^2\,(1+c)^6 \qquad (311),$$

$$P = \tfrac{7}{2}\,(A+M\rho) = (5+c-c^2)\,M \qquad (312),$$

$$Q = \tfrac{7}{2}\,(A-M\rho) = 2\,(1-4c-3c^2)\,M \qquad (313),$$

$$P+Q = 7A \qquad (314).$$

The equation of the catenary is now of the form

$$(1-z^2)^{\frac{7}{2}}\,e^{7\chi i} = Hz^7+H_1z^6+\ldots+H_7+i\,(Lz^5+L_1z^4+\ldots+L_5)\sqrt{Z} \qquad (315),$$

but when it was attempted to employ the differential relation

$$7\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \qquad (316)$$

for the determination of the H's and L's, in the manner illustrated above, the complication became so formidable that another method had to be sought for to determine the leading H and L, connected by the relation

$$L^2-H^2 = 1 \qquad (317),$$

upon which the other coefficients depend.

[redacted]

30. The clue was obtained by noticing that the value $z=\infty$ makes, in the general case, in (125) and (126),

$$\tan\mu\chi = \frac{Li}{H} \qquad (318),$$

or

$$\mu\chi i = i\tan^{-1}\frac{Li}{H} = \tfrac{1}{2}\log\frac{L+H}{L-H} = \log(L+H) \qquad (319),$$

$$L+H = e^{\mu\chi i},\quad \frac{L+H}{L-H} = e^{2\mu\chi i} \qquad (320).$$

By means of the formulas

$$\wp(u+v_1)-\wp(u+v_2) = -\frac{\sigma(2u+v_1+v_2)\,\sigma(v_1-v_2)}{\sigma^2(u+v_1)\,\sigma^2(u+v_2)} \qquad (321),$$

$$\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \frac{\sigma(2u-v_1-v_2)}{\sigma(2u+v_1+v_2)}\,\frac{\sigma^2(u+v_1)\,\sigma^2(u+v_2)}{\sigma^2(u-v_1)\,\sigma^2(u-v_2)} \qquad (322),$$

equation (103) may be replaced by

$$\psi i = -\tfrac{1}{2}Ru + \tfrac{1}{4}\log\frac{\sigma(2u+v)}{\sigma(2u-v)} + \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} \qquad (323),$$

and now, supposing that

$$M^2 s = \wp 2u - \wp v \qquad (324),$$

in equation (29), then

$$\psi i = -\tfrac{1}{2}Ru + u\zeta v + \tfrac{1}{2}\rho Mui - \tfrac{1}{2}iI(v) + \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} \qquad (325),$$

or

$$\mu\chi i = \tfrac{1}{4}\log\left\{\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)}\right\}^{\mu} - \tfrac{1}{2}\mu i I(v) \qquad (326).$$

With the special value of $z=\infty$ and $u=c$, suppose S becomes C, and in (29),

$$\mu I(v) = \tan^{-1}\frac{G}{F}\sqrt{C}$$

$$= \tfrac{1}{2}i\log\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}} \qquad (327).$$

Then

$$\mu\chi i,\ \text{or}\ \tfrac{1}{2}\log\frac{L+H}{L-H} = \tfrac{1}{4}\log\left\{\frac{\wp(v_1-c)-\wp(v_2-c)}{\wp(v_1+c)-\wp(v_2+c)}\right\}^{\mu}$$

$$+\tfrac{1}{4}\log\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}} \qquad (328).$$

But, from (114) and (115),

$$\wp(v_1-c)-\wp(v_2-c) = h+A \qquad (329),$$

$$\wp(v_1+c)-\wp(v_2+c) = h-A \qquad (330),$$

so that

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{1}{2}\mu} \left(\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}}\right)^{\frac{1}{2}} \qquad (331),$$

$$L+H = \frac{(h+A)^{\frac{1}{2}\mu}\left\{F-G\sqrt{(-C)}\right\}^{\frac{1}{2}}}{(h^2-A^2)^{\frac{1}{2}\mu}(F^2+G^2C)^{\frac{1}{2}}} \qquad (332).$$

31. If H denotes the Hessian of the general quartic in (49),

$$\wp 2u = -\frac{H}{Z} \qquad (333),$$

by Hermite's transformation; also, from (49) and (52),

$$H = -\left\{\tfrac{1}{6}(1-h^2)+\tfrac{1}{4}h^2\right\}x^4+\dots \qquad (334),$$

so that, taking $z=\infty$ and $u=c$,

$$\wp 2c = -\tfrac{1}{6}(1-h^2)-\tfrac{1}{4}h^2 \qquad (335),$$

and from (69),

$$\wp 2c-\wp v = \tfrac{1}{4}(A^2-h^2) \qquad (336),$$

so that the corresponding value of s is given by

$$s = \tfrac{1}{4}\frac{A^2-h^2}{M^2} = -x\frac{1+2y}{2+2y} \qquad (337),$$

$$s+x = \frac{x}{2+2y} \qquad (338),$$

$$(1+y)s+xy = -\tfrac{1}{2}x \qquad (339),$$

and

$$C = -\frac{x^2h^2}{4A^2}, \quad \sqrt{(-C)} = \frac{xh}{2A} \qquad (340).$$

32. In a similar manner we may employ the special value $z=0$, when

$$Z = \sqrt{(h^2-A^2)} = \sqrt{(-1-2y)} \qquad (341),$$

to determine the ratio of the final coefficients H_μ and $L_{\mu-2}$.

Now, from (65), denoting by e the value of u corresponding to $z=0$,

$$\wp(v_1 \pm e) = \frac{c+2d+e \pm A\sqrt{(1+2y)}}{2} \qquad (342),$$

$$\wp(v_2 \pm e) = \frac{c-2d+e \mp A\sqrt{(1+2y)}}{2} \qquad (343),$$

so that $\wp(v_1-e)-\wp(v_2-e) = -h-A\sqrt{(1+2y)}$ (344),

$$\wp(v_1+e)-\wp(v_2+e) = -h+A\sqrt{(1+2y)} \qquad (345).$$

Also (noting the two meanings of the letters c and e)

$$\wp 2e = -\frac{ce-d^2}{e} = -\tfrac{1}{6}(1-h^2)-\frac{\frac{1}{4}h^2}{1+2y} \qquad (346),$$

$$\wp 2e - \wp v = \tfrac{1}{4}A^2 - \frac{\frac{1}{4}h^2}{1+2y} \qquad (347),$$

and the value of s corresponding to $z=0$ is given by

$$s = \tfrac{1}{4}\left(A^2-\frac{h^2}{1+2y}\right)\frac{-2x}{1+y} = \frac{-x(1+2y)+y(1+y)^3}{2(1+y)(1+2y)} \qquad (348),$$

$$s+x = \tfrac{1}{2}x\,\frac{1+2y}{1+y}+\frac{y(1+y)^2}{2(1+2y)} \qquad (349),$$

$$(1+y)s+xy = -\tfrac{1}{2}x(1-2y)+\frac{y(1+y)^3}{2(1+2y)} \qquad (350),$$

and, if E denotes the corresponding value of S, we shall find, after reduction,

$$S = -\frac{\{x(1+2y)^2-y(1+y)^3\}^2}{(1+2y)^3}\,\frac{h^2}{A^2} \qquad (351).$$

Thence, as in equations (331), (332), we shall find

$$\frac{H_\mu+L_{\mu-2}\sqrt{(1+2y)}}{H_\mu-L_{\mu-2}\sqrt{(1+2y)}} = \left\{\frac{h+A\sqrt{(1+2y)}}{h-A\sqrt{(1+2y)}}\right\}^{\frac{1}{2}\mu}\left\{\frac{F-G\sqrt{(-E)}}{F+G\sqrt{(-E)}}\right\}^{\frac{1}{2}} \qquad (352),$$

or, since $H_\mu^2-L_{\mu-2}^2(1+2y) = 1$ (353),

$$H_\mu+L_{\mu-2}\sqrt{(1+2y)} = \frac{\{h+A\sqrt{(1+2y)}\}^{\frac{1}{2}\mu}\{F-G\sqrt{(-E)}\}^{\frac{1}{2}}}{\{h^2-A^2(1+2y)\}^{\frac{1}{2}\mu}(F^2+G^2E)^{\frac{1}{2}}} \qquad (354).$$

33. As a preliminary test of these new methods, we apply them to the cases of $\mu = 3$ and $\mu = 5$, already worked out independently.

Thus, for instance, with $\mu = 3$, we find $z = \infty$ makes

$$s = -\tfrac{1}{2}x, \quad \surd(-C) = \frac{xh}{2A} \qquad (355),$$

$$F = s + x = \tfrac{1}{2}x, \quad G = 1 \qquad (356);$$

and therefore
$$\frac{F - G\surd(-C)}{F + G\surd(-C)} = \frac{h-A}{h+A} \qquad (357),$$

so that, in (331),
$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{3}{2}} \left(\frac{h-A}{h+A}\right)^{\frac{1}{2}}$$

$$= \frac{h+A}{h-A} \qquad (358),$$

thus giving
$$H = A, \quad \text{and} \quad L = h,$$
as before, in (155).

34. With $\mu = 5$, and taking equation (162),

$$\frac{F - G\surd(-C)}{F + G\surd(-C)} = \frac{(1+3c)\,s^2 - (2c^2+c^3)\,s + c^4 - (s-c^2)\surd(-C)}{(1+3c)\,s^2 - (2c^2+c^3)\,s + c^4 + (s-c^2)\surd(-C)} \qquad (359),$$

with
$$s = \frac{c-2c^2}{2-2c}, \quad \surd(-C) = -\frac{ch}{2A} \qquad (360),$$

so that, from (160) and (161),

$$\frac{h+A}{h-A} = \frac{\surd(1-5c+7c^2-c^3) + (1-c)\surd(1-c)}{\surd(1-5c+7c^2-c^3) - (1-c)\surd(1-c)} \qquad (361),$$

$$\frac{F - G\surd(-C)}{F + G\surd(-C)}$$

$$= \frac{(1-5c+6c^2+2c^3)\surd(1-c) + (1-4c+2c^2)\surd(1-5c+7c^2-c^3)}{(1-5c+6c^2+2c^3)\surd(1-c) - (1-4c+2c^2)\surd(1-5c+7c^2-c^3)} \qquad (362),$$

and thus equation (331) gives

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^2 \left\{\frac{h+A}{h-A}\,\frac{F - G\surd(-C)}{F + G\surd(-C)}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{h+A}{h-A}\right)^2 \frac{(1-3c)\surd(1-c) + \surd(1-5c+7c^2-c^3)}{(1-3c)\surd(1-c) - \surd(1-5c+7c^2-c^3)}$$

$$= \frac{(2-c)\surd(1-5c+7c^2-c^3) + (2-5c+c^2)\surd(1-c)}{(2-c)\surd(1-5c+7c^2-c^3) - (2-5c+c^2)\surd(1-c)} \qquad (363),$$

$$\frac{H}{L} = \frac{(2-5c+c^2)\sqrt{(1-c)}}{(2-c)\sqrt{(1-5c+7c^2-c^3)}} \qquad (364),$$

so that

$$H = \frac{2-5c+c^2}{c} M \qquad (172),$$

$$L = \frac{2-c}{c} h \qquad (178),$$

as before.

35. The preceding verifications for $\mu = 3$ and $\mu = 5$ having served to settle the doubtful signs in the expressions, we now resume the case of

$$\mu = 7,$$

employing this new procedure.

With $z = \infty$, equations (298)–(303) show that

$$s = -x\frac{1-2y}{2+2y} = \tfrac{1}{2}c\,(1+c)^3\,\frac{1-2c-2c^2}{1-c-c^2} \qquad (365),$$

$$\sqrt{(-C)} = \frac{xh}{2A} = -\tfrac{1}{2}\,\frac{c\,(1+c)}{1-c-c^2}\,\frac{h}{M} \qquad (366),$$

$$F = (3+9c+5c^2)\,s^3-(1+c)^3(1+6c+13c^2+5c^3)\,s^2 + c^2\,(1+c)^6\,(2+6c+c^2)\,s-c^4\,(1+c)^9 \qquad (367),$$

$$G = s^2-(1+c)^3(1+3c)\,s+c^2\,(1+c)^6 \qquad (368),$$

and proceeding as before, we shall find, after considerable reduction,

$$\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}}$$

$$= \left\{\frac{(1+c)(1-4c+c^2+3c^3) + \sqrt{(1-c-c^2)}\,\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}}{(1+c)(1-4c+c^2+3c^3) - \sqrt{(1-c-c^2)}\,\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}}\right\}^2 \qquad (369),$$

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^3 \left\{\frac{h+A}{h-A}\,\frac{F-G\surd(-C)}{F+G\surd(-C)}\right\}^{\frac{1}{2}} \qquad (370),$$

which reduces to

$$\frac{L+H}{L-H} = \frac{\begin{array}{r}(4-2c-5c^2+0+c^4)\surd(1-5c+0+15c^3+12c^4+c^5-c^6)\\ +(4-10c-15c^2+5c^3+10c^4+c^5-c^6)\surd(1-c-c^2)\end{array}}{\begin{array}{r}(4-2c-5c^2+0+c^4)\surd(1-5c+0+15c^3+12c^4+c^5-c^6)\\ -(4-10c-15c^2+5c^3+10c^4+c^5-c^6)\surd(1-c-c^2)\end{array}}$$

(371),

or
$$\frac{H}{L} = \frac{4-10c-15c^2+5c^3+10c^4+c^5-c^6}{4-2c-5c^2+0+c^4}\,\frac{M}{h} \qquad (372).$$

Also
$$L^2-H^2 = 1$$

and $(4-2c-5c^2+0+c^4)\,h^2-(4-10c-15c^2+5c^3+10c^4+c^5-c^6)^2M^2$

$$= c^4(1+c)^4 \qquad (373),$$

so that
$$H = \frac{4-10c-15c^2+5c^3+10c^4+c^5-c^6}{c^2(1+c)^2}M \qquad (374),$$

$$L = \frac{4-2c-5c^2+0+c^4}{c^2(1+c)^2}h \qquad (375).$$

36. The leading coefficients H and L being now determined, the remainder are readily found from the identities obtained by the logarithmic differentiation of (315) and a comparison with (316), namely,

$$0+PL = 0-H_1 \qquad (376),$$

$$0+PL_1 = -7H-2H_2 \qquad (377),$$

$$QL+PL_2 = -6H_1-3H_3 \qquad (378),$$

$$QL_1+PL_3 = -5H_2-4H_4 \qquad (379),$$

$$QL_2+PL_4 = -4H_3-5H_5 \qquad (380),$$

$$QL_3+PL_5 = -3H_4-6H_6 \qquad (381),$$

$$QL_4+0 = -2H_5-7H_7 \qquad (382),$$

$$QL_5+0 = -H_6-0 \qquad (383),$$

as in (185) to (190), and

[redacted]

$$0 + PH = hL - L_1 \qquad (384),$$

$$0 + PH_1 = -(6+h^2)\,L + 3hL_1 - 2L_2 \qquad (385),$$

$$QH + PH_2 = 10hL - 2\,(2+h^2)\,L_1 + 5hL_2 - 3L_3 \qquad (386),$$

$$QH_1 + PH_3 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (387),$$

$$QH_2 + PH_4 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (388),$$

$$QH_3 + PH_5 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (389),$$

$$QH_4 + PH_6 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (390),$$

$$QH_5 + PH_7 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (391),$$

$$QH_6 + \ 0 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (392),$$

$$QH_7 + \ 0 = \quad \dots \quad \dots \quad \dots \quad \dots \qquad (393).$$

I am indebted to Mr. T. I. Dewar for the calculation of these coefficients, and for a general verification of the work; his results are

$$H_1 = -PL = -\frac{(5+c-c^2)(4-2c-5c^2+0+c^4)}{c^3\,(1+c)^2}\,Mh \qquad (394),$$

$$H_2 = \frac{20-65c-93c^2+190c^3+307c^4+20c^5-151c^6-63c^7+8c^8+6c^9}{c^3\,(1+c)^4}\,M \qquad (395),$$

$$H_3 = \frac{-20+35c+139c^2+28c^3-167c^4-134c^5-7c^6+23c^7+c^8}{c^3\,(1+c)^4}\,Mh \qquad (396),$$

$$H_4 = \frac{5-45c+52c^2+217c^3-80c^4-459c^5-276c^6+59c^7+99c^8+24c^9}{c^4\,(1+c)^4}\,M \qquad (397),$$

$$H_5 = \frac{-1+19c-33c^2-46c^3+21c^4+35c^5+9c^6}{c^4\,(1+c)^2}\,Mh \qquad (398),$$

$$H_6 = -QL_5 = -\frac{2\,(1-4c-3c^2)(1-4c+0+6c^3+3c^4)}{c^4\,(1+c)^2}\,M \qquad (399),$$

$$H_7 = 2\,\frac{-1+2c+c^2}{c^3}\,Mh \qquad (400),$$

$$L_1 = \frac{-8+22c+34c^2-35c^3-60c^4-10c^5+14c^6+5c^7}{c^3\,(1+c)^4} \qquad (401),$$

$$L_2 = \frac{12-c-36c^2-21c^3+16c^4+17c^5+4c^6}{c^3\,(1+c)^4}\,h \qquad (402),$$

$$L_3 = \frac{-4+20c-3c^2-64c^3-19c^4+60c^5+51c^6+12c^7}{c^4(1+c)^4} \qquad (403),$$

$$L_4 = \frac{1-8c+4c^2+10c^3+3c^4}{c^4(1+c)^4} \qquad (404),$$

$$L_5 = \frac{1-4c+0+6c^3+3c^4}{c^4(1+c)^2} \qquad (405),$$

and the verifications obtained by putting $z = \pm 1$,

$$A(L+L_2+L_4)+H_1+H_3+H_5+H_7 = 0 \qquad (406),$$

$$A(L_1+L_3+L_5)+H+H_2+H_4+H_6 = 0 \qquad (407),$$

are found to be satisfied.

Try putting $P = 0$; then

$$c^2-c-5 = 0, \quad c = \tfrac{1}{2}-\tfrac{1}{2}\surd(21) \qquad (408),$$

taking the negative root, as this makes

$$1-c-c^2 \text{ and } 1-5c+0+15c^3+12c^4+c^5-c^6 \text{ negative,}$$

and therefore M^2 and h^2 positive; and now, from (299)–(303),

$$x = \frac{-39+9\surd(21)}{2}, \quad y = -6+\surd(21),$$

$$M^2 = \frac{1+\surd(21)}{30},$$

$$A^2 = \frac{-82+18\surd(21)}{15}, \quad h^2 = \frac{83-12\surd(21)}{15}.$$

$$\mu = 8.$$

37. In this case the parameter

$$v = \tfrac{1}{2}\omega_3 \qquad (409),$$

and the equation of the catenary will reduce to either of the forms

$$(1-z^2)\cos 2\chi = (Hz-H_1)\surd Z_1 \qquad (410),$$

$$(1-z^2)\sin 2\chi = (Lz-L_1)\surd Z_2 \qquad (411),$$

leading to the differential relation

$$2(1-z^2)\surd Z\frac{d\chi}{dz} = Pz^2+Q \qquad (412),$$

where Z_1, Z_2, the quadratic factors of the quartic Z, are given in (256), (257).

The pseudo-elliptic form of (29) to be employed here must be taken from p. 229 of the article on "Pseudo-Elliptic Integrals," and, referring to p. 226, replacing the z employed there by $\frac{1}{2}(1+c)$, we must take

$$x = -\tfrac{1}{2}c\,(1+c) \tag{413}$$

$$y = -c\,\frac{1+c}{1-c} \tag{414}$$

$$1+y = \frac{1-2c-c^2}{1-c} \tag{415}$$

$$M^2 = \frac{1-2c-c^2}{c\,(1-c^2)} \tag{416}$$

$$A^2 = \frac{(1-2c-c^2)^3}{c\,(1+c)(1-c)^3} \tag{417}$$

$$h^2 = \frac{(1-c-c^2)(1-6c+8c^2+6c^3-c^4)}{c\,(1+c)(1-c)^3} \tag{418}$$

$$s_3 = \tfrac{1}{4}(1+c)^2 \tag{419}$$

$$k^2 = 4M^2s_3-1-2y = \frac{1-c-c^2}{c} \tag{420}$$

$$\frac{h^2}{k^2} = \frac{1-6c+8c^2+6c^3-c^4}{(1+c)(1-c)^3} \tag{421}$$

$$8\rho = \frac{2c^3}{1-c} \tag{422}$$

$$P = \tfrac{1}{2}(2A+M\rho) = \frac{1-2c}{1-c}\,M \tag{423}$$

$$Q = \tfrac{1}{2}(2A-M\rho) = \frac{1-2c-2c^2}{1-c}\,M \tag{424}$$

$$P+Q = 2A \tag{425}$$

The coefficients H, H_1, L, L_1 can now be determined in a straightforward manner by differentiation and verification; in this way we find

$$L^2-H^2 = 1 \tag{426}$$

$$L^2+H^2 = \frac{2-3c-c^2}{c\,(1+c)}\,\frac{h}{k} \tag{427}$$

$$\frac{H}{H_1} = -\frac{(1+c)(1-3c)}{(1-c)^2}\,\frac{L_1}{L} \tag{428}$$,

$$\frac{L_1}{L} = -\frac{1-c}{2c\,(1+c)(1-3c)}\,\frac{1-4c-c^2+(1-c^2)\,\dfrac{h}{k}}{M} \tag{429}$$,

$$\frac{L}{H_1} = \frac{1}{2c\,(1-c)}\,\frac{1-4c-c^2+(1-c^2)\,\dfrac{h}{k}}{M} \tag{430}$$.

If we try to make this catenary a purely algebraical curve, by putting $P = 0$, $c = \frac{1}{2}$, we find

$$M^2 = -\tfrac{2}{3} \tag{431}$$,

so that the catenary becomes imaginary.

$$\mu = 9.$$

38. This case has been worked out by Mr. T. I. Dewar, making use of the second method of § 30 for the determination of the leading coefficients H and L; the numerical calculations were extremely laborious, and the leading steps only are indicated here; the results satisfy the tests of accuracy that have been applied so far.

The equations are now of the form

$$(1-z^2)^{\frac{9}{2}} \cos 9\chi = Hz^9+H_1z^8+\ldots+H_9 \tag{432}$$,

$$(1-z^2)^{\frac{9}{2}} \sin 9\chi = (Lz^7+L_1z^6+\ldots+L_7)\sqrt{Z} \tag{433}$$,

leading to the relation

$$9\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \tag{434}$$.

Referring to "Pseudo-Elliptic Integrals," p. 232, we take

$$x = p^2\,(1-p)(1-p+p^2) \tag{435}$$,

$$y = p^2\,(1-p) \tag{436}$$,

$$1+y = 1+0+p^2-p^3 \tag{437}$$,

and the pseudo-elliptic form employed for (29) is

$$v = \tfrac{4}{9}\omega_3 \tag{438}$$

$$I(v) = \tfrac{1}{2}\int \frac{\rho s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{9}\tan^{-1}\frac{s^3+Cs^2+Ds+E}{9\rho s^4+\sigma s^3+\tau s^2+Ts+V}\sqrt{S} = \tfrac{1}{9}\tan^{-1}\frac{G}{F}\sqrt{S} \tag{439}$$

where $9\rho = 1+0-3p^2+7p^3$ (440),

$$\sigma = -p^4(4-11p+11p^2+4p^3-17p^4+14p^5) \tag{441}$$

$$\tau = p^8(1-p+p^2)(6-23p+37p^2-24p^3-2p^4+7p^5) \tag{442}$$

$$T = -p^{12}(1-p)^2(1-p+p^2)^2(4-9p+6p^2+p^3) \tag{443}$$

$$V = p^{16}(1-p)^4(1-p+p^2)^3 \tag{444}$$

$$C = -3p^4(1-2p+2p^2) \tag{445}$$

$$D = p^8(1-p+p^2)(3-7p+5p^2) \tag{446}$$

$$E = -p^{12}(1-p)^2(1-p+p^2)^2 \tag{447}$$

$$P = \tfrac{1}{2}(9A+M\rho) = (5+0+3p^2-p^3)\,M \tag{448}$$

$$Q = \tfrac{1}{2}(9A-M\rho) = 2\,(2+0+3p^2-4p^3)\,M \tag{449}$$

$$P+Q = 9A \tag{450}$$

39. When $z=\infty$,

$$s = -x\frac{1-2y}{2+2y} = -\frac{p^2(1-p)(1-p+p^2)(1+0+2p^2-p^3)}{2\,(1+0+p^2-p^3)} \tag{451}$$

$$\sqrt{(-S)} = \frac{x}{2+2y}\,\frac{h}{M}$$

$$= \frac{p^2(1-p)(1-p+p^2)}{2\,(1+0+p^2-p^3)}$$

$$\times\sqrt{\left(\frac{-1+0-5p^2+7p^3-11p^4+20p^5-20p^6+15p^7-7p^8+p^9}{-1+0+p^2-p^3}\right)} \tag{452}$$

and

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^4\left\{\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-S)}}{F+G\sqrt{(-S)}}\right\}^{\frac{1}{2}} \tag{453}$$

the expression requiring the enormous algebraical labour for its reduction.

Mr. Dewar first calculated

$$\frac{h+A}{h-A}\,\frac{F-G\surd(-S)}{F+G\surd(-S)} \tag{454},$$

and found that it was a perfect square; its root was then multiplied by

$$\frac{h+A}{h-A} \tag{455}$$

four times in succession, large common factors making their appearance each time in the numerator and denominator, and thus he found finally that, using detached coefficients,

$$\frac{L+H}{L-H}=\frac{\begin{array}{r}(4-0+14-16+21-26+19-13+6-p^9)\surd(-1+0-5+7-11+20-20+15-7+p^9)\\ +(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\surd(-1+0+p^2-p^3)\end{array}}{\begin{array}{r}(4-0+14-16+21-26+19-13+6-p^9)\surd(-1+0-5+7-11+20-20+15-7+p^9)\\ -(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\quad(-1+0+p^2-p)\end{array}} \tag{456},$$

$$\frac{H}{L}=\frac{\begin{array}{r}4-0+22-28+55-96+114-124+101-63+31-9+p^{12}\surd(-1+0+p^2-p^3)\\ -1+0-5+7-11+20-20+15-7+p^9\end{array}}{4-0+14-16+21-26+19-13+6-p^9}$$

$$=\frac{4-0+22-28+55-96+114-124+101-63+31-9+p^{12}}{4-0+14-16+21-26+19-13+6-p^9}\;\frac{M}{h} \tag{457},$$

and, since $$L^2-H^2=1,$$

$$(4-0+14-16+21-26+19-13-6-p^9)^2h^2$$
$$-(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\,M^2=p^8(1-p)^6(1-p+p^2)^2 \tag{458}$$

therefore, finally,

$$H=\frac{4-0+22-28+55-96+114-124+101-63+31-9+p^{12}}{p^4(1-p)^3(1-p+p^2)}M \tag{459},$$

$$L=\frac{4-0+14-16+21-26+19-13+6-p^9}{p^4(1-p)^3(1-p+p^2)}h \tag{460}.$$

40. The determination of the remaining coefficients is now comparatively an easy matter, and Mr. Dewar finds

$$H_1=-PL=-\frac{(5-0+3-1)(4-10+14-16+21-26+19-13+6-p^9)}{p^4(1-p)^3(1-p+p^2)}Mh \tag{461},$$

$$H_2=\frac{\begin{array}{r}-20+0-165+190-626+1169-1946+3141-4161+5063\\ -5262+4610-3453+2092-1011+371-84+8p^{17}\end{array}}{p^6(1-p)^4(1-p+p^2)^2}M \tag{462},$$

$$H_3 = \frac{20-10+175-210+642-1123+1842-2619+3043-3177+2738-2010+1247-608+237-65+8p^{16}}{p^6(1-p)^4(1-p+p^2)^2} Mh \quad (463),$$

$$H_4 = \frac{5+5+90-45+489-877+1296-3946+6401-10002+13532-16302+17394-15762+12194-7806+3974-1558+406-48p^{19}}{p^8(1-p)^5(1-p+p^2)^2} M \quad (464),$$

$$H_5 = \frac{-1-2-39-8-220+283-575+1206-1582+2080-2194+1852-1378+772-330+114-20p^{16}}{p^8(1-p)^5(1-p+p^2)} Mh \quad (465),$$

$$H_6 = \frac{-4-4-64+40-288+636-1076+2320-3580+4980-6515+7091-6698+5294-3258+1551-512+80p^{17}}{p^8(1-p)^6(1-p+p^2)} M \quad (466),$$

$$H_7 = \frac{12-0+68-120+160-400+524-520+517-348+157-65+16p^{12}}{p^6(1-p)^6} Mh \quad (467),$$

$$H_8 = -QL_7 = \frac{2(2-0+3-4)(2-0+8-14+16-30+34-26+15-4p^9)}{p^6(1-p)^6} M \quad (468),$$

$$H_9 = -\frac{2(1-1+1)(2-0+4-8+4-2+p^6)}{p^4(1-p)^5} Mh \quad (469),$$

$$L_1 = \frac{8-0+54-60+164-284+389-542+575-544+440-274+139-48+7p^{14}}{p^6(1-p)^4(1-p+p^2)^2} \quad (470),$$

$$L_2 = 3\,\frac{-4+0-23+22-57+82-109+132-120+100-64+30-11+2p^{13}}{p^6(1-p)^4(1-p+p^2)^2} h \quad (471),$$

$$L_3 = \frac{-4-4-48+16-185+299-541+1009-1365+1807-2000+1860-1516+974-487+175-30p^{16}}{p^8(1-p)^5(1-p+p^2)^2} \quad (472),$$

$$L_4 = \frac{1+1+23-15+97-161+286-436+560-654+598-470+294-130+46-10p^{15}}{p^8(1-p)^5(1-p+p^2)^2} h \quad (473),$$

$$L_5 = \frac{3+3+30-15+87-192+246-468+618-660+672-516+306-138+30p^{14}}{p^8(1-p)^6(1-p+p^2)} \quad (474),$$

$$L_6 = \frac{-6+0-16+30-20+42-46+22-11+4p^9}{p^6(1-p)^6} h \quad (475),$$

$$L_7 = \frac{-2+0-8+14-16+30-34+26-15+4p^9}{p^6(1-p)^6} \quad (476).$$

Mr. Dewar has also performed the verification of showing that

$$A(L+L_2+L_4+L_6)+H_1+H_3+H_5+H_7+H_9 = 0 \quad (477),$$

$$A(L_1+L_3+L_5+L_7)+H+H_2+H_4+H_6+H_8 = 0 \quad (478).$$

$$\mu = 10.$$

41. This investigation was interesting as affording the first case of a purely algebraical Spherical Catenary, shown in the stereographic and stereoscopic projections of the accompanying diagrams (pp. 170, 171), drawn to scale and in perspective by Mr. T. I. Dewar.

The equations to be satisfied are

$$(1-z^2)^{\frac{1}{2}}\, e^{\chi i} = (Hz^5+H_1z^4+\ldots+H_5)+i\,(Lz^3+L_1z^2+\ldots+L_3)\sqrt{Z} \quad (479),$$

leading to
$$5\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \quad (480),$$

as in the case of $\mu = 5$.

Referring to the *Proc. Lond. Math. Soc.*, Vol. xxv., p. 235, the relation

$$\gamma_{10} = 0 \quad (481),$$

the equation of a unicursal quintic curve, is satisfied by

$$x = \frac{-a\,(1+a)}{(1-a)(1-a-a^2)^2} \quad (482),$$

$$y = \frac{-a\,(1+a)}{(1-a)(1-a-a^2)} \quad (483),$$

$$1+y = \frac{1-3a-a^2+a^3}{(1-a)(1-a-a^2)} \quad (484),$$

and thence

$$M^2 = \frac{(1-a-a^2)(1-3a-a^2+a^3)}{2a\,(1+a)} \quad (485),$$

$$A^2 = \frac{(1-3a-a^2+a^3)^3}{2a\,(1+a)(1-a)^2\,(1-a-a^2)} \quad (486),$$

$$h^2 = \frac{(1-5a-a^2+a^3)(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}{2a\,(1+a)(1-a)^2\,(1-a-a^2)} \quad (487).$$

The pseudo-elliptic integral to employ for (29) is

$$v = \tfrac{2}{5}\omega_3,$$

$$I(v) = \tfrac{1}{2}\int \frac{\rho s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{5}\tan^{-1}\frac{s-C}{5\rho s^2-\sigma s+\tau}\sqrt{S} = \tfrac{1}{5}\tan^{-1}\frac{G\sqrt{S}}{F} \quad (488),$$

where $$5\rho = \frac{3+3a+a^2-a^3}{(1-a)(1-a-a^2)} \tag{489}$$

$$\sigma = \frac{3\,(1+a)^2\,(1+a+2a^2-a^3)}{(1-a)^3\,(1-a-a^2)^3} \tag{490}$$

$$\tau = \frac{a^2\,(1+a)^4}{(1-a)^4\,(1-a-a^2)^5} \tag{491}$$

$$C = \frac{(1+a)^2}{(1-a)^2\,(1-a-a^2)^2} \tag{492}$$

Thence $$P = \tfrac{5}{8}\,(A+M\rho) = 2\,\frac{2-a}{1-a}\,M \tag{493}$$

$$Q = \tfrac{5}{8}\,(A-M\rho) = \frac{1-9a-3a^2+3a^3}{(1-a)(1-a-a^2)}\,M \tag{494}$$

$$P+Q = 5A \tag{495}$$

We take $$s_3 = \frac{a^2}{(1-a)^2\,(1-a-a^2)^2} \tag{496}$$

$$k^2 = 4M^2 s_3 - 1 - 2y = \frac{-1+5a+a^2-a^3}{(1+a)(1-a)^2} \tag{497}$$

$$\frac{h^2}{k^2} = \frac{-1+6a-3a^2-8a^3+3a^4+2a^5-a^6}{2a\,(1-a-a^2)} \tag{498}$$

42. Corresponding to $z = \infty$,

$$s = -x\,\frac{1+2y}{2+2y} = \frac{a\,(1+a)(1-4a-2a^2+a^3)}{2\,(1-a)(1-a-a^2)(1-3a-a^2+a^3)} \tag{499}$$

$$\sqrt{(-S)} = \frac{xh}{2A} = -\frac{a\,(1+a)}{2\,(1-a)(1-a-a^2)^2}\,\frac{h}{A} \tag{500}$$

Working with these values, Mr. Dewar found, after a long calculation,

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^2 \left\{\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-S)}}{F+G\sqrt{(-S)}}\right\}^{\frac{1}{2}}$$

$$= \frac{(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)} + (2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}}{(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)} - (2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}} \tag{501}$$

$$\frac{H}{L}=\frac{(2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}}{(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)}} \qquad (502);$$

and therefore

$$H=\frac{2-4a-a^2+a^3}{a^2(1+a)(1-a)}\sqrt{\left\{\frac{(-1+3a+a^2-a^3)(1-6a-3a^2+8a^3-3a^4-2a^5+a^6)}{1+a}\right\}} \qquad (503),$$

$$L=\frac{(1-a-a^2)(2-6a+0+3a^3-a^4)}{a^2(1+a)(1-a)}\sqrt{\left(\frac{-1+5a+a^2-a^3}{1+a}\right)} \qquad (504).$$

Thence the other coefficients follow, and writing

α for $-1+3a+a^2-a^3$,

β for $-1+5a+a^2-a^3$,

γ for $-1+a+a^2$,

δ for $1-6a+3a^2+8a^3-3a^4-2a^5+a^6$,

$$H_1=-\frac{2(2-a)(1-a-a^2)(2-6a+0+3a^3-a^4)}{a^2(1+a)^2(1-a)^2}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \qquad (505),$$

$$H_2=\frac{12-60a+32a^2+91a^3-23a^4-30a^5+8a^6+3a^7-a^8}{2a^3(1+a)^2(1-a)}\sqrt{\left(\frac{a\delta}{1+a}\right)} \qquad (506),$$

$$H_3=\frac{-4+28a-28a^2-64a^3+37a^4+36a^5-18a^6-6a^7+3a^8}{a^3(1+a)^3(1-a)}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \qquad (507),$$

$$H_4=-QL_3=\frac{(1-9a-3a^2+3a^3)(1-3a-2a^2)}{2a^4(1+a)^2}\sqrt{\left(\frac{a\delta}{1+a}\right)} \qquad (508),$$

$$H_5=\frac{1-4a-a^2+2a^3}{a^3(1+a)^2}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \qquad (509),$$

$$L_1=\frac{-6+16a+10a^2-7a^3-2a^4+a^5}{a^3(1+a)^2}\sqrt{\left(\frac{\gamma\delta}{2a}\right)} \qquad (510),$$

$$L_2=\frac{(1-a-a^2)(3-9a-6a^2+8a^3+2a^4-2a^5)}{a^3(1+a)^2}\sqrt{\left(\frac{\beta}{1+a}\right)} \qquad (511),$$

$$L_3=\frac{-(1-a)(1-3a-2a^2)}{a^3(1+a)^2}\sqrt{\left(\frac{\gamma\delta}{2a}\right)} \qquad (512),$$

and these values verify the equations

$$A(L+L_2)+H_1+H_3+H_5=0 \qquad (513),$$

$$A(L_1+L_3)+H+H_2+H_4=0 \qquad (514).$$

43. Here we can make P vanish by taking

$$a = 2,$$

and this gives real numerical values to the coefficients, namely,

$$H = \frac{1}{6}\sqrt{\left(\frac{17}{3}\right)}, \quad H_1 = 0, \quad H_2 = -\frac{5}{12}\sqrt{\left(\frac{17}{3}\right)}, \quad H_3 = -\frac{25}{108},$$

$$H_4 = \frac{65}{288}\sqrt{\left(\frac{17}{3}\right)}, \quad H_5 = \frac{25}{144};$$

$$L = -\frac{5}{6}\sqrt{\frac{5}{3}}, \quad L_1 = -\frac{5}{36}\sqrt{(85)}, \quad L_2 = \frac{35}{72}\sqrt{\frac{5}{3}}, \quad L_3 = \frac{13}{144}\sqrt{(85)},$$

$$h = \frac{1}{2}\sqrt{\left(\frac{17}{3}\right)}, \quad A = -\frac{1}{2\sqrt{(15)}};$$

and the equation of the catenary can be written in either of the forms

$$r^5 \cos 5\psi = \frac{48\sqrt{(51)}\,z^5 - 120\sqrt{(51)}\,z^3 - 200z^2 + 62\sqrt{(51)}\,z + 150}{864} \qquad (515),$$

$$r^5 \sin 5\psi = \frac{-120\sqrt{(15)}\,z^3 - 60\sqrt{(85)}\,z^2 + 70\sqrt{(15)}\,z - 39\sqrt{(85)}}{432}$$

$$\times\sqrt{\left\{(1-z^2)\left[\frac{1}{2}\sqrt{\left(\frac{17}{3}\right)} - z\right]^2 - \frac{1}{60}\right\}} \qquad (516),$$

so that its projection on the equatorial plane is a closed algebraical curve, with pentagonal symmetry.

With $\qquad a = 2, \quad k^2 = \frac{5}{3},$

$$k = -\tfrac{1}{3}\sqrt{(15)}, \quad \frac{h}{k} = -\frac{\sqrt{(85)}}{10} \qquad (517),$$

$$Z_1 = -z^2 + \frac{\sqrt{(51)} - 2\sqrt{(15)}}{6}\,z - \tfrac{1}{3} + \frac{2\sqrt{(85)}}{15} \qquad (518),$$

$$Z_2 = z^2 + \frac{\sqrt{(51)} + 2\sqrt{(15)}}{6}\,z - \tfrac{1}{3} - \frac{2\sqrt{(85)}}{15} \qquad (519).$$

The roots of $Z_2 = 0$ are imaginary, but the roots of $Z_1 = 0$ are

$$z_0 = -0\cdot9982585 \qquad (520),$$

$$z_1 = 0\cdot8975022 \qquad (521),$$

giving the limits of latitude between which the catenary lies, namely,

86° 37′ 5″ N. and 63° 49′ 54″ S.

The curve crosses the twenty prime meridians, at intervals of 18°, at the points corresponding to the roots h_1, h_2, h_3, h_4, of the quintic

$$Hz^5+H_1z^4+\ldots+H_5=0 \qquad (522),$$

and to the roots, l_1, l_2, l_3, of the cubic

$$Lz^3+L_1z^2+L_2z+L_3=0 \qquad (523).$$

These roots have been calculated by Mr. Dewar, employing Horner's method, and the results are tabulated in the following columns; it will be observed that, near the upper N. pole, the roots are crowded together, and the disentanglement of these roots caused some trouble.

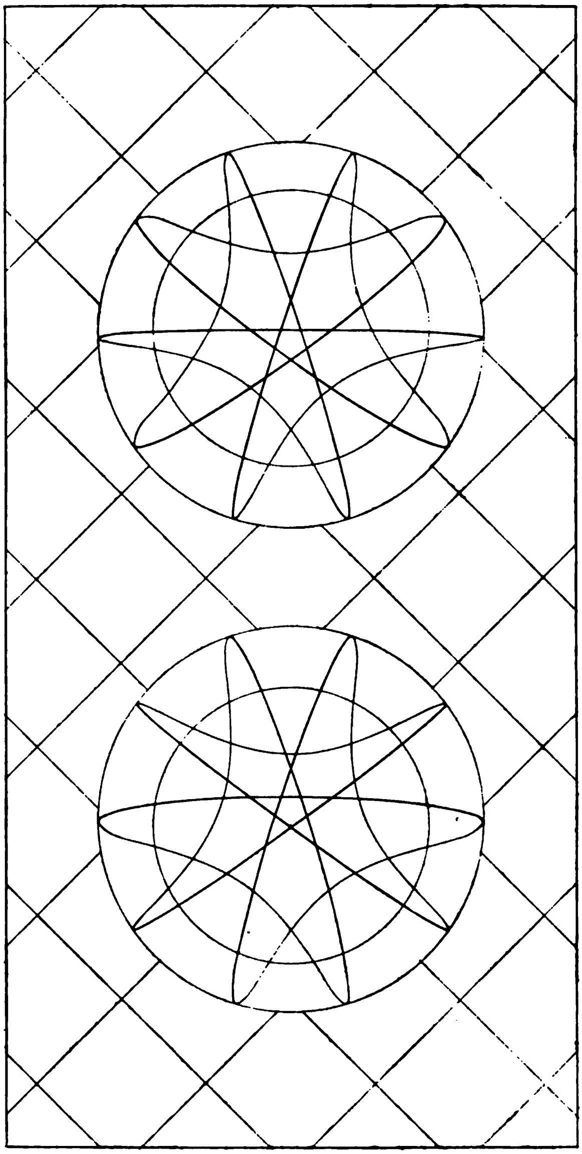
Some intermediate points were also calculated, for the purpose of plotting the curve with accuracy on a globe; and, denoting the latitude (South) by λ, the following table embodies the numerical results:—

z	sin λ	cos λ	tan ½ (90° − λ)	λ	ψ
z_0	−0·9982585	0·0589918	−0·0295	86° 37′ 5″ N.	0° 0′ 0″
h_1	9980756	0619599	0311	86 26 42	18 0 0
l_1	9973452	0728175	0365	85 49 27	36 0 0
h_2	9949993	0998841	0501	84 16 3	54 0 0
l_2	9824577	1864859	0941	79 15 8	72 0 0
h_3	3436408	9391010	6989	20 5 56 N.	90 0 0
0	0·	1·	1·	0 0 0	91 59 50
½h	+0·5951190	0·8036694	0·5038	36 31 12 S.	98 45 0
l_3	7895648	6136251	3430	52 8 42	108 0 0
h_4	8805984	4738620	2520	61 42 53	126 0 0
z_1	8975022	4410099	2324	63 49 54	144 0 0
h_5	1·4561185				

The curve crosses the equator at an angle 83° 46′ 24″, and it makes a maximum angle, 86° 56′ 45″, with the parallel of latitude 28° 0′ 31″ N.

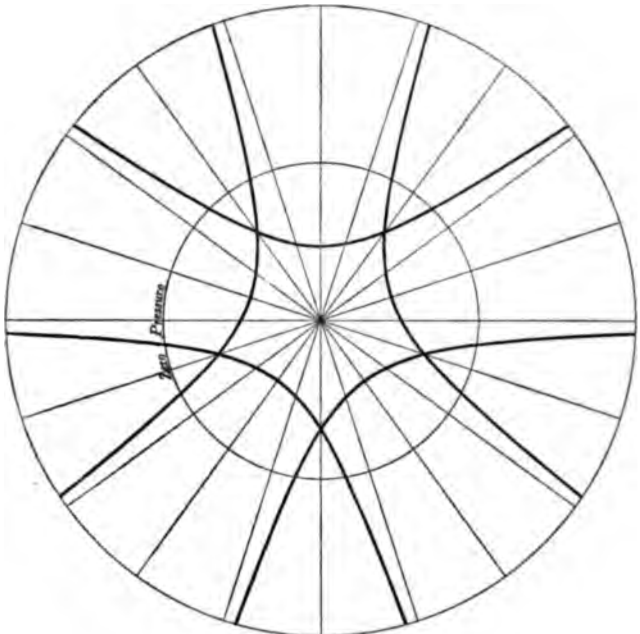
Below the depth ½h from the centre, that is, in latitude greater than 36° 31′ 12″ S., the pressure changes sign, and the chain must be supposed to rest on the inside of a spherical surface.

[redacted]

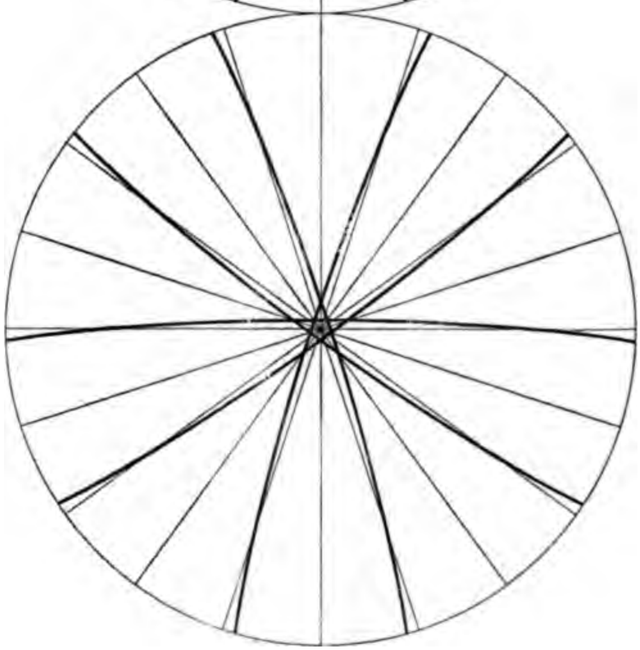


Stereoscopic View of Catenary.

Lower Hemisphere.



Upper Hemisphere.



The stereoscopic photograph can be made to show the solid figure, in the absence of a stereoscope, either by fixing the eyes on a distant object, and then raising the card into the line of sight at the distance of distinct vision; or else by holding the card at a distance of about the arm's length, and focussing the eyes on a point half way; a smaller image is now formed, and the tesselated background, representing the directrix plane, is now seen as a network in front of the solid sphere.

$$\mu = 12.$$

44. Here the parameter of (29) is

$$v = \tfrac{1}{3}\omega_3 \qquad (524),$$

and the equation of the catenary may be written

$$(1-z^2)^{\frac{3}{2}} e^{3\chi i} = (Hz^2+H_1z+H_2)\sqrt{Z_2}+i\,(Lz^2+L_1z+L_2)\sqrt{Z_1} \qquad (525),$$

leading to

$$(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = pz^2+q \qquad (526).$$

Referring to "Pseudo-Elliptic Integrals," p. 248,

$$x = -\frac{a(1+a)(1+a^2)(1+a+a^2)}{(1-a)^2} \qquad (527),$$

$$y = -\frac{a\,(1+a)(1+a+a^2)}{1-a} \qquad (528),$$

$$1+y = \frac{1-2a-2a^2-2a^3-a^4}{1-a} \qquad (529),$$

$$M^2 = \frac{(1-a)(1-2a-2a^2-2a^3-a^4)}{2a\,(1+a)(1+a^2)(1+a+a^2)} \qquad (530),$$

Clebsch's x or ξ is given by (265), so that, with

$$t = -\frac{a\,(1+a+a^2)}{1-a} \qquad (531),$$

$$\xi = M\,(1+a)(1+a^2) \qquad (532),$$

and

$$\frac{h^2}{k^2} = 1-\xi^2 = -\frac{1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8}{2a\,(1+a+a^2)} \qquad (533).$$

But

$$h^2 = A^2 - 1 - 2y$$

$$= \frac{(1-4a-4a^2-4a^3-a^4)(1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8)}{2a\,(1-a^4)(1+a+a^2)} \quad (534),$$

so that

$$k^2 = -\frac{1-4a-4a^2-4a^3-a^4}{1-a^4} \quad (535),$$

and the same value of k^2 is obtained from formula (258) with

$$s = t^2 = \frac{a^2\,(1+a+a^2)^2}{(1-a)^2} \quad (536),$$

so that

$$\frac{k^2-1}{2} = -\frac{1-2a-2a^2-2a^3-a^4}{1-a^4} \quad (537),$$

$$\frac{k^2+1}{2} = 2a\frac{1+a+a^2}{1-a^4} \quad (538).$$

45. The chief difficulty here is the determination of the appropriate value of ρ to employ in (29), which we distinguish as $\rho\,(\frac{1}{3}\omega_3)$; but this is given by the general formula

$$\rho\left(\frac{4\omega_3}{\mu}\right) + 2\rho\left(\frac{2\omega_3}{\mu}\right) = -(1+y) \quad (539),$$

and with $\mu = 12$,

$$12\rho\,(\tfrac{1}{6}\omega_3) = -2\,(5+3a+3a^2+a^3) \quad (540),$$

(*Proc. Lond. Math. Soc.*, Vol. xxv., p. 251); so that

$$12\rho\,(\tfrac{1}{3}\omega_3) = 4\,(5+3a+3a^2+a^3) - 12\frac{1-2a-2a^2-2a^3-a^4}{1-a}$$

$$= 8\frac{(1+a+a^2)^2}{1-a} \quad (541),$$

Next

$$p = \tfrac{1}{2}\,(A+M\rho) = \tfrac{1}{2}M\left\{\frac{1-2a-2a^2-2a^3-a^4}{1-a} + \tfrac{2}{3}\,\frac{(1+a+a^2)^2}{1-a}\right\}$$

$$= \tfrac{1}{6}M\,(5+3a+3a^2+a^3) \quad (542),$$

$$q = \tfrac{1}{2}\,(A-M\rho) = \tfrac{1}{6}M\frac{1-10a-12a^2-10a^3-5a^4}{1-a} \quad (543),$$

$$p+q = A \quad (544).$$

The integral (29) now assumes the pseudo-elliptic form

$$I\left(\tfrac{1}{3}\omega_3\right) = \tfrac{1}{2}\int^{\frac{2}{3}} \frac{\dfrac{(1+a+a^2)^2}{1-a} - 3\dfrac{a^2(1+a)^2(1+a^2)(1+a+a^2)^2}{1-a}}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{3}\tan^{-1}\frac{F\sqrt{(s-s_3)}}{G\sqrt{\{4(s-s_1)(s-s_2)\}}} \qquad (545),$$

where $$F = 2s - \frac{(1+a)^2(1+a^2)(1+a+a^2)^2}{(1-a)^2} \qquad (546),$$

$$G = \frac{(1+a+a^2)^2}{1-a} \qquad (547).$$

46. Proceeding to the value $z=\infty$,

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{1}{2}} \left\{\frac{F\sqrt{(s-s_3)} - G\sqrt{(-4\,.\,s-s_1\,.\,s-s_2)}}{F\sqrt{(s-s_3)} + G\sqrt{(-4\,.\,s-s_1\,.\,s-s_2)}}\right\}^{\frac{1}{3}} \qquad (548),$$

where

$$s = \frac{1+2y}{2+2y} = \frac{a(1+a)(1+a^2)(1+a+a^2)(1-3a-4a^2-4a^3-2a^4)}{2(1-a)^2(1-2a-2a^2-2a^3-a^4)} \qquad (549).$$

Writing α for $1-2a-2a^2-2a^3-a^4$ (550),

β for $1-4a-4a^2-4a^3-a^4$ (551),

γ for $1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8$ (552),

so that $$M^2 = \frac{(1-a)\,\alpha}{2a(1+a)(1+a^2)(1+a+a^2)} \qquad (553),$$

$$A^2 = \frac{\alpha^3}{2a(1-a^4)(1+a+a^2)} \qquad (554),$$

$$h^2 = \frac{\beta\gamma}{2a(1-a^4)(1+a+a^2)} \qquad (555),$$

$$\frac{A^2}{h^2} = \frac{\alpha^3}{\beta\gamma} \qquad (556),$$

$$\frac{h+A}{h-A} = \frac{\sqrt{(\beta\gamma)}+\alpha\sqrt{\alpha}}{\sqrt{(\beta\gamma)}-\alpha\sqrt{\alpha}} \qquad (557).$$

Then $$s-s_3 = \frac{a(1+a+a^2)}{2(1-a)^2}\,\frac{\beta}{a} \qquad (558),$$

$$-4(s-s_1)(s-s_2) = \frac{a(1+a)^2(1+a^2)^2(1+a+a^2)\gamma}{2(1-a)^2a^2} \qquad (559),$$

$$\frac{F\sqrt{(s-s_3)}}{G\sqrt{(-4\,.\,s-s_1\,.\,s-s_2)}} = \frac{1-a-a^2-5a^3-7a^4-6a^5-4a^6-a^7}{(1-a)^3}\sqrt{\left(\frac{\beta}{a\gamma}\right)} \qquad (560),$$

and finally, after considerable reduction, Mr. Dewar finds

$$\frac{L+H}{L-H} = \sqrt{\left\{\frac{m\sqrt{\gamma}+n\sqrt{(a\beta)}}{m\sqrt{\gamma}-n\sqrt{(a\beta)}}\right\}} \qquad (561),$$

where $$m = 4-10a-3a^2-3a^3+9a^4+12a^5+9a^6+5a^7+a^8 \qquad (562),$$

$$n = (1-a)(4-2a-5a^2-10a^3-11a^4-7a^5-4a^6-a^7) \qquad (563),$$

and thence the coefficients H and L can be inferred.

Putting $$L = \cosh\lambda, \quad H = \sinh\lambda \qquad (564),$$

then $$e^{4\lambda} = \frac{m\sqrt{\gamma}+n\sqrt{(a\beta)}}{m\sqrt{\gamma}-n\sqrt{(a\beta)}} \qquad (565),$$

and $$\tanh 2\lambda = \frac{n\sqrt{(a\beta)}}{m\sqrt{\gamma}} \qquad (566),$$

$$\cosh 2\lambda = \frac{m\sqrt{\gamma}}{l\sqrt{\delta}}, \quad \sinh 2\lambda = \frac{n\sqrt{(a\beta)}}{l\sqrt{\delta}} \qquad (567),$$

where $$l = a^2(1+a)^2(1+a^2)(1+a+a^2) \qquad (568),$$

$$\delta = -2a(1+a+a^2) \qquad (569),$$

and $$L^2 = \tfrac{1}{2}(\cosh 2\lambda+1) = \frac{m\sqrt{\gamma}+l\sqrt{\delta}}{2l\sqrt{\delta}} \qquad (570),$$

$$H^2 = \tfrac{1}{2}(\cosh 2\lambda-1) = \frac{m\sqrt{\gamma}-l\sqrt{\delta}}{2l\sqrt{\delta}} \qquad (571),$$

$$L = \frac{\sqrt{[m\sqrt{\gamma}+n\sqrt{(\alpha\beta)}]}+\sqrt{[m\sqrt{\gamma}-n\sqrt{(\alpha\beta)}]}}{2\sqrt{l}\,\sqrt[4]{\delta}} \qquad (572),$$

$$H = \frac{\sqrt{[m\sqrt{\gamma}+n\sqrt{(\alpha\beta)}]}-\sqrt{[m\sqrt{\gamma}-n\sqrt{(\alpha\beta)}]}}{2\sqrt{l}\,\sqrt[4]{\delta}} \qquad (573).$$

[redacted]

These values of H and L have been checked by Mr. Dewar by a straightforward verification, but the work was very long.

This catenary will be an algebraical curve if we can make $P = 0$, and then

$$4+(1+a)^3 = 0, \quad a = -1-\sqrt[3]{4} \qquad (574);$$

this makes

$$k^2 = 1-\tfrac{1}{2}\sqrt[3]{4} \qquad (575),$$

but

$$a = -\tfrac{2}{3}(\sqrt[3]{2}+1),$$

so that the catenary is imaginary.

47. It will be noticed now that, in general,

$$P = \tfrac{1}{2}\mu M \rho\left(\frac{2\omega_3}{\mu}\right) \qquad (576),$$

so that an algebraical catenary requires as a first condition that

$$\rho\left(\frac{2\omega_3}{\mu}\right) = 0 \qquad (577);$$

afterwards we must examine the reality of the catenary by finding out if M^2, A^2, h^2, k^2, ... are positive.

Taking, for example, the case of

$$\mu = 14$$

(*Proc. Lond. Math. Soc.*, Vol. xxv., p. 257), and the formula (p. 206)

$$\mu\rho\,\wp' v = \tfrac{1}{2}(q_1 q_2 + q_2 q_3 + \ldots + q_{\mu-4} q_{\mu-3}) - (\mu-2)\,\wp'' v,$$

or

$$-\mu\rho x = \tfrac{1}{2}(q_1 q_2 + \ldots) - (\mu-2)\, x\,(1+y) \qquad (578),$$

where

$$q_{r-1} = 2(s_r - s_1) = -2x^{\frac{1}{2}}\frac{\gamma_{r-1}\gamma_{r+1}}{\gamma_r^2} \qquad (579),$$

$$q_r q_{r-1} = 4x^{\frac{1}{2}}\frac{\gamma_{r-1}\gamma_{r+2}}{\gamma_r \gamma_{r+1}} \qquad (580),$$

we find (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 199)

$$\tfrac{1}{4}q_1q_2 = xy \qquad (581),$$

$$\tfrac{1}{4}q_2q_3 = x - \frac{x^2}{y} \qquad (582),$$

$$\tfrac{1}{4}q_3q_4 = \frac{x^2}{y} - \frac{x^2y}{y-x} \qquad (583),$$

$$\tfrac{1}{4}q_4q_5 = \frac{x^2y}{y-x} - \frac{xy^3}{y-x-y^2} \qquad (584),$$

$$\tfrac{1}{4}q_5q_6 = \frac{xy^3}{y-x-y^2} - \frac{xy\{(y-x)(y-2x)+y^3\}}{xy-x^2-y^3} \qquad (585),$$

$$\tfrac{1}{4}q_6q_7 = \frac{xy\{(y-x)(y-2x)+y^3\}}{xy-x^2-y^3} - x\frac{(y-x)^3+y^2(y-x)^2-2xy^2(y-x-y^2)}{y\{x(y-x-y^2)-(y-x)^2\}} \qquad (586),$$

$$\tfrac{1}{4}q_7q_8 = x\frac{(y-x)^3+y^2(y-x)^2-2xy^2(y-x-y^2)}{y\{x(y-x-y^2)-(y-x)^2\}} - \frac{N_9}{y^3(y-x-y^2)-(y-x)^3} \qquad (587).$$

where $N_9 = x\{(y-x)^3-3xy(y-x)^2+2y^6\}$.

48. With $\mu = 14$, $q_{12} = 0$ (Abel), and

$$q_4 = q_1, \quad q_{10} = q_3, \quad q_9 = q_3, \quad q_8 = q_4, \quad q_7 = q_5 \qquad (588),$$

$$-14\rho x = q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6-12x(1+y)$$

$$= 4x(1+y)-4xy\frac{(y-x)(y-2x)+y^3}{xy-x^2-y^3}-12x(1+y) \qquad (589),$$

$$\tfrac{7}{2}\rho = 2(1+y)+y\frac{(y-x)(y-2x)+y^3}{xy-x^2-y^3}$$

$$= y+2+\frac{y(y-x)^2}{xy-x^2-y^3}$$

$$= y+2+c(z-p) \qquad (590),$$

and, with the values of p, z, y given on p. 257, *Proc. Lond. Math. Soc.*, Vol. xxv., this reduces to

$$7\rho = \frac{4+4c+3c^2+2c^3+c\sqrt{C}}{1+c} \qquad (591),$$

where $C = c(1+2c)(4+5c+2c^2)$ (592).

Then ρ, and therefore P, vanishes if

$$(4+4c+3c^2+2c^3)^2-c^3(1+2c)(4+5c+2c^2) = 0,$$

or $$(1+c)(4+4c+6c^2+3c^3)=0,$$

or $$(c+1)\left\{(3c+2)^3+28\right\}=0 \qquad (593),$$

$$3c+2=-\sqrt[3]{(28)}.$$

With $$a=\sqrt[3]{(28)}=2^{\frac{2}{3}}.7^{\frac{1}{3}},$$

$$b=\sqrt[3]{(98)}=2^{\frac{1}{3}}.7^{\frac{2}{3}},$$

we find $$\sqrt{C}=\frac{14+5a-2b}{9},$$

and $$p=\frac{a}{7} \text{ or } \frac{28+2b+7a}{21} \qquad (594);$$

but these values make M^2 negative, and the catenary is imaginary.

49. So also, with

$$\mu=18, \quad v=\tfrac{2}{9}\omega_3,$$

$$q_{16}=0, \quad q_{15}=q_1, \quad q_{14}=q_2, \quad q_{10}=q_8 \qquad (595),$$

$$-18\rho x=q_1q_2+q_2q_3+\ldots+q_7q_8-16x(1+y)$$

$$=4x(1+y)-4\frac{N_9}{y^3(y-x-y^2)-(y-x)^3}-16x(1+y) \qquad (596),$$

$$\tfrac{9}{2}\rho=3(1+y)+\frac{N_9}{y^3(y-x-y^2)-(y-x)^3} \qquad (597),$$

reducing ultimately to (*Proc. Lond. Math. Soc.*, Vol. XXV., p. 265)

$$\tfrac{9}{2}\rho=3+y-z\left(1-\frac{c}{t}\right)$$

$$=-q^3+3q^2+16q+13-3\sqrt{Q} \qquad (598).$$

Therefore ρ and P vanish, if

$$(q^3-3q^2-6q-13)^2-9Q=0 \qquad (599),$$

or $$q^6+3q^5+6q^4+10q^3-3q^2-15q-20=0 \qquad (600).$$

This sextic equation has only two real roots

$$+1{\cdot}21921268, \text{ and } -2{\cdot}301874537 \qquad (601),$$

which were calculated by Mr. Dewar, with Horner's method; he has also calculated the remaining quartic factor

$$q^4+1{\cdot}917338143q^3+6{\cdot}730647q^2+8{\cdot}09394q+7{\cdot}1264 \qquad (602),$$

and has resolved the sextic into its cubic factors in the two ways

$$(q^3+3{\cdot}914721q^2+5{\cdot}278q+3{\cdot}464)(q^3-0{\cdot}914721q^2+4{\cdot}3629q-5{\cdot}772) \qquad (603),$$

$$(q^3+2{\cdot}60635q^2+5{\cdot}43504q+10{\cdot}898)(q^3+0{\cdot}39365q^2-0{\cdot}46117q-1{\cdot}8352) \qquad (604).$$

If we had obtained a simple rational root, as in the case of $\mu = 10$, it would have been worth while to go on with the calculation of this algebraical catenary, which would possess the symmetry of the nonagon, the prime meridians being 10° apart.

$$\mu = 16.$$

50. The parameter of the associated pseudo-elliptic integral assumed by (29) is in this case

$$v = \tfrac{1}{4}\omega_3 \qquad (605),$$

and the catenary will be given by an equation of the form

$$(1-z^2)^2 e^{4\chi i} = (Hz^3+H_1z^2+H_2z+H_3)\sqrt{Z_1}+i(Lz^3+L_1z^2+L_2z+L_3)\sqrt{Z_2} \qquad (606),$$

leading to

$$4(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = Pz^2+Q \qquad (607).$$

According to p. 262 of "Pseudo-Elliptic Integrals," we must now take

$$x = -\frac{(a-1)(a^2+1)(a^4-a^3+a^2+3a+1)+(a^4-a^3+a^2-a-1)\sqrt{A'}}{2a^3(a^2-2a-1)} \qquad (608),$$

$$y = \frac{(a+1)(a^4-1)(a^2-2a-1)+(a^4-2a-1)\sqrt{A'}}{2a^5(a+1)(a^2-2a-1)} \qquad (609),$$

where

$$A' = (a^4-1)(a^2-2a-1) \qquad (610),$$

$$M^2 = \frac{a^7-a^6-3a^5+a^4-3a^3+3a^2+5a+1+(a^4+4a+1)\sqrt{A'}}{4a(a^4-1)} \qquad (611),$$

$$s_3 = \left\{\frac{a^3-a^2-3a-1+(2a+1)\sqrt{A'}}{2a^4(a+1)(a^2-2a-1)}\right\}^2 \qquad (612),$$

$$k^2 = \frac{(a+1)(a^4-1)(a^2-2a-1)+4a\sqrt{A'}}{-(a+1)(a^4-1)(a^2-2a-1)} \qquad (613),$$

$$I(\tfrac{1}{4}\omega_3) = \tfrac{1}{2}\int \frac{\rho_2 s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{4}\tan^{-1}\frac{(s+C)\sqrt{(4\,.\,s-s_1\,.\,s-s_2)}}{(4\rho_2 s+R)\sqrt{(s-s_3)}} \qquad (614),$$

where
$$4\rho_2 = -\frac{(a^2+1)(a^2-3)}{a}\sqrt{s_3}$$

$$= -(a^2+1)(a^2-3)\,\frac{a^3-a^2-3a-1+(2a+1)\sqrt{A'}}{2a^5(a+1)(a^2-2a-1)} \qquad (615),$$

$$R = -\frac{(a^2-1)(a^2+1)^2}{a^3}s_3^{\frac{3}{2}} \qquad (616),$$

$$C = -\frac{a^2+1}{a^2}s_3 \qquad (617),$$

$$4(s-s_1)(s-s_2) = 4s^2 - \frac{(a^4-1)(a^4-4a^2-1)}{a^2}ss_3 + \frac{(a^4-1)^2}{a^2}s_3^2 \qquad (618).$$

Better therefore take

$$\frac{s}{s_3} = s' \qquad (619)$$

as variable; and then

$$xy = \frac{A'}{a(a^2-2a-1)}s_3^{\frac{3}{2}} = \frac{a^4-1}{a}s_3^{\frac{3}{2}} \qquad (620),$$

$$\sqrt{(s_1 s_2)} = \frac{xy}{2s_3^{\frac{1}{2}}} = -\frac{a^4-1}{2a}s_3 \qquad (621).$$

The condition $p=0$ thus gives

$$-4\frac{(a^2+1)(a^2-3)}{a}\sqrt{s_3}+16(1+y)=0 \qquad (622),$$

leading to the sextic equation

$$15a^6+30a^5+15a^4+12a^3+9a^2-2a-7=0 \qquad (623),$$

which, according to Mr. Dewar, has two real roots

$$+0{\cdot}551764 \quad\text{and}\quad -1{\cdot}55711472 \qquad (624),$$

but this does not at present look promising enough to make it worth while to investigate the corresponding catenary.

51. Clebsch extends his investigations to the case where the spherical surface is made to spin about the vertical axis with constant angular velocity n; and now, changing to the absolute unit of force in the C.G.S. system, the tension

$$T = wg\,(h-z) + wn^2\,(b^2 - r^2) \qquad (625),$$

together with

$$Tr^2 \frac{d\psi}{ds} = H \qquad (626).$$

Equation (6) now becomes perfectly intractable, and of hyper-elliptic character, when gravity and centrifugal whirling are both taken into account; but when gravity is neglected in comparison with the whirling effect, by putting $g = 0$, equation (6) becomes

$$\psi = \int \frac{H\,dz}{r^2 \sqrt{\{w^2 n^4 (b^2 - r^2)^2 r^2 - H^2\}}} \qquad (627),$$

or, putting $H = wn^2 A$ (628),

and taking r^2 for independent variable,

$$\psi = \tfrac{1}{2} \int \frac{A\,dr^2}{r^2 \sqrt{(1-r^2)} \sqrt{\{(b^2-r^2)^2 r^2 - A^2\}}} \qquad (629),$$

an elliptic integral of the third kind.

Also

$$\frac{ds}{d\psi} = \frac{Tr^2}{H} = \frac{(b^2 - r^2)\,r^2}{A} \qquad (630),$$

so that the arc

$$s = \tfrac{1}{2} \int \frac{(b^2 - r^2)\,dr^2}{\sqrt{(1-r^2)} \sqrt{\{(b^2-r^2)^2 r^2 - A^2\}}} \qquad (631),$$

introducing elliptic integrals of the second kind.

52. Clebsch (*Crelle*, 57, p. 106) puts

$$R = (b^2 - r^2)^2 r^2 - A^2 = (r^2 - \rho^2)(r^2 - \sigma^2)(r^2 - \tau^2) \qquad (632),$$

where

$$2b^2 = \rho^2 + \sigma^2 + \tau^2 \qquad (633),$$

$$b^4 = \sigma^2\tau^2 + \tau^2\rho^2 + \rho^2\sigma^2 \qquad (634),$$

$$A^2 = \rho^2\sigma^2\tau^2 \qquad (635);$$

and therefore $$(\rho^2+\sigma^2+\tau^2)^2-4\,(\sigma^2\tau^2+\tau^2\rho^2+\rho^2\sigma_2)=0 \qquad (636),$$

or $$(\rho+\sigma+\tau)(\rho-\sigma-\tau)(-\rho+\sigma-\tau)(-\rho-\sigma+\tau)=0 \qquad (637),$$

so that, taking $$\rho=\sigma+\tau \qquad (638),$$

and therefore $$b^2=\sigma^2+\sigma\tau+\tau^2 \qquad (639),$$

$$R=r^2-(\sigma+\tau)^2\,.\,r^2-\sigma^2\,.\,r^2-\tau^2 \qquad (640),$$

$$T=wn^2\,(\sigma^2+\sigma\tau+\tau^2-r^2) \qquad (641),$$

and the integral is reduced to the standard form (29) by putting

$$r^2=\frac{\wp u-\wp v}{\wp u-\wp c} \qquad (642),$$

$$1-r^2=\frac{\wp v-\wp c}{\wp u-\wp c} \qquad (643),$$

$$R=\lambda\,\frac{\wp'^2 u}{(\wp u-\wp c)^3} \qquad (644).$$

But we shall find that these integrals are essentially the same as those required for the catenary assumed by a chain wrapped on a vertical paraboloid, whether spinning about its axis, or at rest; the investigation of this new problem had better be reserved for another paper.

[*Additional Note*, 7*th May*, 1896.

The algebraical case of the Spherical Catenary for $\mu=7$, indicated in (408), p. 159, has now been completed by Mr. Dewar, and the figure obtained is similar to that on p. 170, but having heptagonal instead of pentagonal symmetry.

The numerical data, exhibited in a tabular form, similar to that on p. 169, are given in the table on next page.

The disentanglement of the roots z_0, h_1, l_1, h_2, ... near the North Pole is facilitated by drawing the osculating plane of the catenary at its highest point, and calculating by spherical trigonometry the points where the osculating small circle cuts the prime meridians; these points will be indistinguishable from points on the catenary for some considerable distance.

z	$\sin\lambda$	λ	ψ
z_0	$-0{\cdot}9970939$	85° 37′·85 N.	0
h_1	9969428	31·12	$\frac{90°}{7}$
l_1	9964255	9·24	$\frac{180}{7}$
h_2	9952666	84 25·35	$\frac{270}{7}$
l_2	9926029	83 1·60	$\frac{360}{7}$
h_3	9850008	80 3·83	$\frac{450}{7}$
l_3	9479943	71 26·44	$\frac{540}{7}$
h_4	3250447	18 58·13	90°
0	0·	0	92° 20′·30
l_4	$+0{\cdot}6789442$	42 45·67 S.	$\frac{720°}{7}$
$\frac{1}{2}h$	6832409	43 5·84	
h_5	8414974	57 15·91	$\frac{810}{7}$
l_5	8915966	63 4·47	$\frac{900}{7}$
h	9110921	65 39·40	$\frac{990}{7}$
z_1	9164583	66 24·82	$\frac{1080}{7}$
h_7	1·5496647		

The angular radius θ of the osculating small circle at any point, or $\rho = \sin\theta$, the radius of absolute curvature, is given by the simple expressions

$$\frac{1}{\rho^2} = \frac{1}{\sin^2\theta} = 1 + \frac{A^2}{(h-z)^4},$$

or
$$\tan\theta = \frac{(h-z)^2}{A};$$

this follows from equations (16)–(22), for

$$\frac{d}{ds}(h-z)\frac{dx}{ds} = -x(h-2z),$$

$$\frac{d}{ds}(h-z)\frac{dy}{ds} = -y(h-2z),$$

$$\frac{d}{ds}(h-z)\frac{dz}{ds} = -z(h-2z)-1,$$

so that

$$(h-z)\frac{d^2x}{ds^2}=\frac{dx}{ds}\frac{dz}{ds}-x(h-2z),$$

$$(h-z)\frac{d^2y}{ds^2}=\frac{dy}{ds}\frac{dz}{ds}-y(h-2z),$$

$$(h-z)\frac{d^2z}{ds^2}=\frac{dz}{ds}\frac{dz}{ds}-z(h-2z)-1.$$

Squaring and adding,

$$(h-z)^2\frac{1}{\rho^2}=\frac{dz^2}{ds^2}+(h-2z)^2-2\frac{dz}{ds}\frac{dz}{ds}+2z(h-2z)+1$$

$$=(h-z)^2+1-z^2-\frac{dz^2}{ds^2},$$

or, from (11),

$$(h-z)^2\frac{1}{\rho^2}=(h-z)^2+1-z^2-\frac{Z}{(h-z)^2}$$

$$=(h-z)^2+\frac{A^2}{(h-z)^2}.$$

Knowing the angular radius θ of the osculating small circle, and also ϕ, the angle at which the catenary crosses a parallel of latitude, from § 3, it is possible to employ Mr. C. V. Boys's method of drawing the curve by means of a celluloid scale, bent to the radius of the sphere, and pivoted instantaneously at the centre of the osculating small circle.

The angles θ and ϕ are connected with λ, the latitude, by the relations

$$\tan\theta=\frac{(h-\sin\lambda)^2}{A},$$

$$\tan^2\phi=\frac{\cos^2\lambda\,(h-\sin\lambda)^2}{A^2}-1,$$

or

$$\sec\phi\sec\lambda=\frac{h-\sin\lambda}{A},$$

$$\tan\theta=(h-\sin\lambda)\sec\lambda\sec\phi$$

$$=A\sec^2\lambda\sec^2\phi.$$

Suppose we put, in § 8,

$$x = -\frac{m^2 a}{(a-m)^2}, \quad y = -\frac{(1-2m)\,a}{a-m};$$

we now find that a root of equation (31) is

$$s = \frac{m^2 a^2}{(a-m)^2},$$

so that, in § 24,

$$k^2 = 4a-1,$$

$$h^2 = \frac{(4a-1)\left\{2a^2-2\,(m+1)\,a+1\right\}}{2a\,(a-m)},$$

$$\frac{h^2}{k^2} = \frac{2a^2-2\,(m+1)\,a+1}{2a\,(a-m)},$$

$$\frac{\xi}{M} = -\frac{m}{a-m}.]$$

Thursday, January 9th, 1896.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

Miss Grace Chisholm, Ph.D. Göttingen, and Dr. Robert Bryant, were elected members.

Prof. Elliott, by a method used in connexion with seminvariants, showed how to obtain a criterion as to whether or not a rational integral homogeneous function of y, a function of x, and its derivatives, is an exact differential, and further showed that, if it is, its integral can be found by differential operations only.

The President announced the title of a paper by Prof. Lloyd Tanner, viz., "On a certain Ternary Cubic." The paper, in the absence of the author, was taken as read.

Mr. S. H. Burbury made a further communication "On Boltzmann's Minimum Function."

Mr. Love communicated "Some Examples illustrating Lord

Rayleigh's Theory of the Stability or Instability of certain Fluid Motions."

Messrs. Cunningham and Larmor spoke on the subject of the papers.

The following presents to the Library were received :—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 11; Leipzig, 1895.

"Nautical Almanac for the Year 1899," 8vo; London, 1896.

"Bulletin des Sciences Mathématiques," Tome XIX., Nov. et Déc. 1895; Paris.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 3; New York, 1895.

"Memorias y Revista de la Sociedad Científica Mexico," Tomo VIII. (1894–5), Nos. 1–2, 3–4; Mexico.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Geschäftliche Mittheilungen, 1895, Heft 2; Mathematisch-Physikalische Klasse, 1895, Heft 3, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. I., Fasc. 11; Napoli, November, 1895.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4e Serie, Tome V.; Paris, 1895.

Rayet, G.—"Observations Pluviométriques et Thermométriques faites dans la Gironde de Juin 1893 à Mai 1894," 8vo; Bordeaux, 1894.

Peter, B.—"Beobachtungen am Sechszölligen Repsoldschen Heliometer der Leipziger Sternwarte," roy. 8vo; Leipzig, 1895.

His, W.—"Anatomische Forschungen über Johann S. Bach's Gebeine und Antlitz," roy. 8vo; Leipzig, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. IV., Fasc. 10, 11; Roma, 1895.

"Educational Times," January, 1896.

"Indian Engineering," Vol. XVIII., Nos. 21, 22.

Notes on a Ternary Cubic. *By* H. W. Lloyd Tanner, M.A.

Received and read January 9th, 1896.

In the general ternary cubic

$$(a,\ b,\ c;\ f,\ g,\ h;\ i,\ j,\ k;\ l)(x,\ y,\ z)^3$$
$$= ax^3+by^3+cz^3+6lxyz$$
$$+3fy^2z+3gz^2x+3hx^2y+3iyz^2+3jzx^2+3kxy^2,$$

we assume $a=1$, $h=0$, and that the form is the product of three factors linear in x, y, z. Denoting this by $F(x, y, z)$, we have

$$F(x,\ y,\ z) = x^3+by^3+cz^3+6lxyz$$
$$+3fy^2z+3gz^2x+3iyz^2+3jzx^2+3kxy^2$$
$$= (x+\theta y+\phi z)(x+\theta_1 y+\phi_1 z)(x+\theta_2 y+\phi_2 z),$$

where the θ, ϕ are to be determined. For this purpose write $y=-1$, $z=0$, and it is seen that the three θ are roots of the equation

$$F(\theta,\ -1,\ 0) = (1,\ 0,\ k,\ b)(\theta,\ -1)^3 = 0.$$

From a comparison of the coefficients of x^2z, xyz, and y^2z in the two expressions for $F(x, y, z)$, we find

$$\phi = (j\theta^2-2l\theta+f)/(\theta^2+k) = A\theta^2+B\theta+C,$$

where $\quad A = 2kf-2bl-2k^2j\ (\div),\quad B = bf+4k^2l-kbj\ (\div),$

$$C = 4k^2f-4kbl+b^2j\ (\div),$$

the common denominator being $= b^2+4k^3 = \Delta$.

The θ, ϕ represent any one of the pairs θ, ϕ; θ_1, ϕ_1; θ_2, ϕ_2.

In this way the first factor of $F(x, y, z)$ becomes

$$x+y\theta+z\theta^2 = x+Cz+(y+Bz)\,\theta+Az\,.\,\theta^2$$
$$= x'+y'\theta+z'\theta^2, \text{ say},$$

which is in the standard form of a complex integer, if x', y', z' are integral. And then

$$F(x,\ y,\ z) = N(x'+y'\theta+z'\theta^2).$$

When the norm is developed and the symmetric functions of θ, θ_1, θ_2 are replaced by their values in terms of k, b, we get

$$N(x'+y'\theta+z'\theta^2) = (1, b, b^2;\ 0, 3k^2, 0;\ kb, -2k, k;\ -\tfrac{1}{2}b)(x', y', z')^3$$
$$= \Phi(x', y', z').$$

It is this form Φ which is considered in the present communication. Obviously it has the important property that the coordinates of a complex property $x'+y'\theta+z'\theta^2$ of any real integer m give a representation of m by the form Φ.

In general the forms F, Φ are not equivalent, since the transformations are not integral.

When $k=0$, the discriminant of $(1, 0, k, b)$, namely, b^2+4k^3, is positive, and two of the factors of F (or Φ) are imaginary. This case has been discussed in the *Proceedings** by Professor Mathews, who gives references to several memoirs. The case in which all the factors are real will be here considered, and it will be assumed that b, k are integers, such that b^2+4k^3 is negative, and that $(1, 0, k, b)(\theta, -1)^3$ is irreducible.

Units and Automorphs.

Let u, v, w be a representation of 1 by the form Φ, so that

$$\Phi(u, v, w) = 1 = N(u+v\theta+w\theta^2),$$

and let $x+y\theta+z\theta^2$, $\xi+\eta\theta+\zeta\theta^2$ be two integers, such that

$$x+y\theta+z\theta^2 = (u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2).$$

Hence
$$\begin{aligned}\Phi(x, y, z) &= N(x+y\theta+z\theta^2)\\ &= N(u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2)\\ &= N(\xi+\eta\theta+\zeta\theta^2)\\ &= \Phi(\xi, \eta, \zeta).\end{aligned}$$

Also
$$\begin{aligned}&x+y\theta+z\theta^2\\ &= (u+v\theta+w\theta^2)(\xi+\eta\theta+\zeta\theta^2)\\ &= u\xi+(v\xi+u\eta)\,\theta+(w\xi+v\eta+u\zeta)\,\theta^2+(w\eta+v\zeta)\,\theta^3+w\zeta\theta^4.\end{aligned}$$

Reducing this by means of the equation

$$\theta^3+3k\theta-b = 0,$$

* Vol. xxi., pp. 280–287.

we obtain
$$x+y\theta+z\theta^2$$
$$=u\xi+bw\eta+bv\zeta+\left[v\xi+u\eta+3k\,(w\eta+v\zeta)+bw\zeta\right]\theta$$
$$+(w\xi+v\eta+u\zeta-3kw\zeta)\,\theta^2,$$

and hence, on account of the irreducibility of the equation for θ,
$$x=u\xi+bw\eta+bv\zeta,$$
$$y=v\xi+(u-3kw)\,\eta+(bw-3kv)\,\zeta,$$
$$z=w\xi+v\eta+(u-3kw)\,\zeta.$$

These equations may be written
$$x,\,y,\,z=v\,(\xi,\,\eta,\,\zeta),$$
where
$$v=\begin{pmatrix} u, & bw, & bv \\ v, & u-3kw, & bw-3kv \\ w, & v, & u-3kw \end{pmatrix},$$
and, since
$$\Phi\,(x,\,y,\,z)=\Phi\,(\xi,\,\eta,\,\zeta),$$
it follows that v is an automorph of Φ, and, as is seen from its genesis, a proper automorph.

There is plainly a correspondence between the proper automorphs, v, of the form Φ, and the complex units $u+v\theta+w\theta^2$ ($=U$ say), the coordinates of the unit being the terms of the first column of the corresponding automorph. And it is easily proved that the product v_1v_2 of two proper automorphs corresponds to the product U_1U_2 of the two corresponding units.

For, if $(x,\,y,\,z)=v_1\,(x',\,y',\,z')$ and $(x',\,y',\,z')=v_2\,(x'',y'',z'')$, then, for the corresponding units,
$$(x+y\theta+z\theta^2)=U_1\,(x'+y'\theta+z'\theta^2),\quad (x'+y'\theta+z'\theta^2)=U_2\,(x''+y''\theta+z''\theta^2).$$
But, from the matrical equations,
$$(x,\,y,\,z)=v_1\,(x',\,y',\,z')=v_1v_2\,(x'',\,y'',\,z''),$$
and from the unit equations,
$$x+y\theta+z\theta^2=U_1U_2\,(x''+y''\theta+z''\theta^2),$$
which shows that v_1v_2 and U_1U_2 correspond; and further proves that the sequence of factors in a product of proper automorphs is indifferent, assuming—what will presently appear—that to every unit corresponds only one proper automorph. The results extend to any number of factors, which need not all be different.

The coordinates of a complex unit give the elements of the first column of the corresponding proper automorph; and this first column determines the whole of the automorph. For the second column is formed from the first, and the third from the second by the matrix

$$\begin{pmatrix} . & . & b \\ 1 & . & -3k \\ . & 1 & . \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta),$$

acting on the three elements of the preceding column. Thus the scheme v includes circulants as a particular case. And, again, recalling the properties of circulants, the development of a determinant formed by this rule from the first column x, y, z is the form $\Phi(x, y, z)$.

There are some curious relations (which may be useful) arising from the nature of the matrix β. For instance,

$$\Phi\,.\,\beta\,(x, y, z) = b\,.\,\Phi\,(x, y, z),$$

so that, if (x, y, z) is a representation of m, then we have a representation also of $b^r . m$, viz., $\beta^r(x, y, z)$. The inverse matrix β^{-1} may also be used to derive a representation of m from a known representation (bx, y, z) of mb, in which the first element is a multiple of b. We have, in fact,

$$\Phi\,.\,\beta^{-1}(bx, y, z) = \Phi\,(3kx+y, z, x),$$

and
$$\Phi\,.\,\beta^{-1}(bx, y, z) = \frac{1}{b}\,\Phi\,(bx, y, z).$$

Fundamental Improper Automorphs.

If θ, θ_1, θ_2 be the three roots of the equation

$$\theta^3+3k\theta-b=0,$$

we have
$$\theta_1 = 2kq+p\theta+q\theta^2,$$
$$\theta_2 = -2k\,(p+z)+kq\theta-(p+1)\,\theta^2,$$

where
$$q = \pm k\sqrt{-3/\Delta}, \quad 2p+1 = bq/k.^*$$

Assigning one of these values to q, the other value of q and the

---

* If $k=0$, $q=0$, and $p^2+p+1=0$.

corresponding value of p are $-q$, $-p-1$. Therefore

$$\theta_2 = -2kq-(p+1)\,\theta-q\theta^2,$$
$$\theta_2^2 = \quad 2k\,(p-1)-kq\theta+p\theta^2.$$

These relations are merely a particular form of the well-known homographic relation between the roots of a cubic, but it is more easy to obtain them by writing coefficients for θ^0, θ, θ^2, which are determined by the results of the next paragraph.

The selection of one value for q determines the sequence of θ, θ_1, θ_2, but it in no wise identifies θ, which may be any one of the three roots.

From the above formulæ,

$$\begin{aligned} x+y\theta_1+z\theta_1^2 &= x+(2kq+p\theta+q\theta^2)\,y+\left[-2k\,(p+z)+kq\theta-(p+1)\,\theta^2\right]z \\ &= x+2kqy-2k\,(p+z)\,z+(py+kqz)\,\theta+\left[qy-(p+1)\,z\right]\theta^2 \\ &= \xi+\eta\theta+\zeta\theta^2, \end{aligned}$$

if
$$\xi,\,\eta,\,\zeta = \left[\begin{array}{ccc} 1, & 2kq, & -2k\,(p+2) \\ \cdot & p, & kq \\ \cdot & q, & -p-1 \end{array}\right]\!\!\big(x,\,y,\,z\big).$$

Similarly, if
$$x+y\theta_2+z\theta_2^2 = \xi'+\eta'\theta+\zeta'\theta^2,$$

$$\xi',\,\eta',\,\zeta' = \left[\begin{array}{ccc} 1, & -2kq, & 2k\,(p-1) \\ \cdot & -p-1, & -kq \\ \cdot & -q, & p \end{array}\right]\!\!\big(x,\,y,\,z\big).$$

Now the equation $\quad x+y\theta_1+z\theta_1^2 = \xi+\eta\theta+\zeta\theta^2$

is tantamount to $\quad x+y\theta+z\theta^2 = x+\eta\theta_2+\zeta\theta_2^2.$

But we have $\quad \xi'+\eta'\theta+\zeta'\theta^2 = x+y\theta_2+z\theta_2^2.$

Hence the first matrix of this article, which changes (x, y, z) to (ξ, η, ζ), is the inverse of the second, which changes (x, y, z) to (ξ', η', ζ'). In a similar way it is found that each matrix is the square of the other, and that each is a cube root of unity. Accordingly they will be represented by the symbols γ, γ^2. The results just found enable us to write down the formulæ of the last article.

[redacted]

The matrices γ, γ^2 are automorphs of Φ, for, since

$$x+y\theta_1+z\theta^2 = \xi+\eta\theta+\zeta\theta^2,$$

their norms are equal, that is to say,

$$\Phi(x, y, z) = \Phi(\xi, \eta, \zeta) = \Phi\gamma(x, y, z).$$

Since they change the θ in the complex factor $x+y\theta_1+z\theta_1^2$, they are improper automorphs.

It is sometimes convenient to write the automorphs γ, γ^2 in another way, which displays the irrational elements (if any). Writing $\Delta = -3r^2$, the symbol r is real but in general irrational. In the numerical example (p. 195) it is rational, and this is probably the case in most of the cubics that occur in connexion with cyclotomy. All the elements of γ, γ^2 can now be expressed in terms of k, b, r, and the result is

$$\gamma, \gamma^2 = \tfrac{1}{2}\begin{pmatrix} 2, & \cdot & -6k \\ \cdot & -1, & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \pm \frac{1}{2r}\begin{pmatrix} \cdot & 2k^2, & -kb \\ \cdot & b, & 2k^2 \\ \cdot & 2k, & -b \end{pmatrix}.$$

Associated Automorphs.

Let v be a proper automorph whose first column consists of the coordinates of the complex unit $u+v\theta+w\theta^2$, and let $x+y\theta+z\theta^2$ be any complex integer. Then, as on p. 186, the coordinates of the product $(u+v\theta+w\theta^2)(x+y\theta+z\theta^2)$ are $v(x, y, z)$.

Again (p. 191), if

$$\gamma(x, y, z) = (\xi, \eta, \zeta),$$

we have
$$\xi+\eta\theta+\zeta\theta^2 = x+y\theta_1+z\theta_1^2.$$

Hence the complex integer in θ whose coordinates are $\gamma(x, y, z)$ may be written $x+y\theta_1+z\theta_1^2$.

Hence it follows that $v\gamma(x, y, z)$ are the coordinates of the complex number $(u+v\theta+w\theta^2)(x+y\theta_1+z\theta_1^2)$; while $\gamma u(x, y, z)$ are the coordinates of $(u+v\theta_1+w\theta_1^2)(x+y\theta_1+z\theta_1^2)$. In the first case, γ increases the subscript of θ, and the unit multiplication is then effected; in the second case, the action of v corresponds to multiplying $x+y\theta+z\theta^2$ by the complex unit, and the subsequent action of γ increases by 1 the subscript of every θ.

The automorphs most nearly related to v may be shown in a diagram :—

$$\begin{array}{ccc} v & v\gamma & v\gamma^2 \\ \gamma v\gamma^2 & \gamma v & \gamma v\gamma \\ \gamma^2 v\gamma & \gamma^2 v\gamma^2 & \gamma^2 v \end{array}$$

and the preceding explanations will make it easy to see that the first column contains the proper automorphs, the second contains improper automorphs of the γ-kind, and the third improper automorphs of the γ^2-kind. The automorphs of the first, second, and third rows correspond to the complex units $u+v\theta+w\theta^2$, $u+v\theta_1+w\theta_1^2$, and $u+v\theta_2+w\theta_2^2$, respectively. It follows that no two of the nine automorphs can be equivalent.

The automorphs v, γ satisfy the identity

$$u\gamma v\gamma v\gamma = u\gamma^2 v\gamma^2 v\gamma^2 = 1,$$

which differs from the similar identity for the elliptic modular function group only by containing a periodic automorph of the third instead of the second order. The proof of the identity comes from the observation that $v\gamma v\gamma v\gamma\,(x, y, z)$ are the coordinates of

$$(u+v\theta+w\theta^2)(u+v\theta_1+w\theta_1^2)(u+v\theta_2+w\theta_2^2)(x+y\theta+z\theta^2),$$

that is, of
$$x+y\theta+z\theta^2,$$
so that
$$v\gamma v\gamma v\gamma\,(x, y, z) = (x, y, z)\,;$$
and so for $v\gamma^2 v\gamma^2 v\gamma^2$.

The identity shows that the cube of every improper automorph is 1; this is visible on writing out the cube, say $\gamma v\gamma\,\gamma v\gamma\,\gamma v\gamma$. It proves that the product of two proper automorphs is independent of the sequence of the factors. This comes by multiplying the equation

$$\gamma u\gamma v\gamma = \gamma^2 v\gamma^2 v\gamma \quad (= v^{-1})$$

by the multipliers indicated below, on the left, and then changing the association of the factors as is allowable.

$$\begin{array}{ll} 1 \;\ldots & \gamma v\gamma^2 . \gamma^2 v\gamma = \gamma^2 v\gamma . \gamma v\gamma^2, \\ \gamma^2 \ldots \gamma & v . \gamma v\gamma^2 = \gamma v\gamma^2 . v, \\ \gamma \;\ldots \gamma^2 & \gamma^2 v\gamma^2 . v = v . \gamma^2 v\gamma^2. \end{array}$$

But any other pair of automorphs in the scheme are not commutative. If we denote by ω any one of the automorphs, then any pair of automorphs in the table (extended if need be) are consecutive automorphs (i.) in a horizontal line, as ω, $\omega\gamma$; or (ii.) in a dexter line, as ω, $\gamma\omega$; or (iii.) in a sinister line (ω, $\gamma^2\omega\gamma^2$); or (iv.) in a vertical line (ω, $\gamma^2\omega\gamma$). It will therefore suffice to prove that each of these four pairs gives different products when the sequence of the factors is altered.

Since $\gamma\omega$, $\omega\gamma$ are two automorphs in the scheme, they are different. Hence, if we multiply, as indicated on the left, and then reassociate the factors suitably, the inequation

$$\omega\gamma \neq \gamma\omega,$$

we obtain $\quad \omega \; \ldots \qquad \omega\,.\,\omega\gamma \neq \omega\gamma\,.\,\omega$ ....(i.),

$\qquad\qquad \ldots\; \omega \qquad \omega\,.\,\gamma\omega \neq \gamma\omega\,.\,\omega$ ....(ii.),

$\qquad\qquad \omega^{-1} \ldots \omega^{-1} \qquad \gamma\omega^{-1} \neq \omega^{-1}\gamma.$

But since $\qquad \omega^{-1} = \gamma^2\omega\gamma^2\omega\gamma^2,$

the last equation gives $\quad \omega\,.\,\gamma^2\omega\gamma^2 \neq \gamma^2\omega\gamma^2\,.\,\omega$ ....(iii.),

For automorphs in the same column it is necessary to take the column separately. For the second column the inequation

$$\gamma^2 v\, \gamma^2 v \gamma^2 \neq v^2$$

gives $\qquad \gamma^2 \ldots \qquad \gamma v\,.\,\gamma^2 v\gamma^2 \neq \gamma^2 v\gamma^2\,.\,\gamma v,$

$\qquad\qquad \gamma \ldots \gamma \qquad v\gamma\,.\,\gamma v \neq \gamma v\,.\,v\gamma,$

$\qquad\qquad \ldots\; \gamma^2 \qquad \gamma^2 v\gamma^2\,.\,v\gamma \neq v\gamma\,.\,\gamma^2 v\gamma^2,$

and similar results follow for the third column by using the inequation

$$\gamma v\gamma v\gamma \neq v^2.$$

There is an interesting speculation concerning the three proper automorphs v, $\gamma v\gamma^2$, $\gamma^2 v\gamma$ (or, say, for shortness v, ν, ω) which is connected with these results. We know that the powers of any proper automorph are commutative in a product; and the question is whether v, ν, ω can be represented as powers of one of them. Now, if $\nu = v^h$, we have also $\omega = \nu^h$ and $v = \omega^h$, so that $h^3 = 1$. But h obviously cannot be 1, and therefore the only values for this exponent are the imaginary cube roots of 1, viz., γ, γ^2. If we agree to define the symbol v^γ by the equation $v^\gamma = \nu$, then any product

$$v^a \nu^b \omega^c = v^{a+b\gamma+c\gamma^2},$$

and, supposing v to be a fundamental automorph, every proper automorph of the form, is a power of a fundamental automorph whose exponent is a cubic integer. There is thus a doubly infinite series. For the forms in which two factors are imaginary the exponent is a real integer and there is a singly infinite series of automorphs.

The units $u+v\theta+w\theta^2$, $u+v\theta_1+w\theta_1^2$, $u+v\theta_2+w\theta_2^2$ are connected in the same way. In Eisenstein's great memoir on cubic forms much use is made of the "regulator," viz., the expression $\log A-\gamma\log B$; where A, B are two conjugate factors of the form, and $\log A$, $\log B$ mean the logarithm of the absolute values of A, B. As the use of a complex exponent has been fruitful, it seemed worth while to notice the extension to matrices.

There is one point to which reference may be made. In the numerical example following, the automorphs include fractional elements. It will be found, however, that in no product of the automorphs does the common denominator exceed 9, and this fact leads to Eisenstein's proof of the existence of units (and therefore automorphs) with integral elements. There are two reasons for using the automorphs with fractional elements (with the property that the common denominator in all powers and products has a superior limit). In the first place, the integral elements are liable to be inconveniently large, or the first automorph with integral elements is a power of the fractional automorph, and the exponent may be any integer less than 3^n, where n is the common divisor. In the second place, the number m of which a representation is sought generally contains extraneous factors, especially in cyclotomic problems (for example, $x^2-5y^2=4p$, and the form that follows) which are also the factors of the common denominators of the fractional elements of the automorphs. So far from being a hindrance, the presence of fractions with such denominators is a real help to the calculator.

Numerical Example.

In the determination of a 7-ic complex prime, a factor of $p\ (=14\mu+1)$, the following equation is to be solved:—

$$F(x, y, z) = x^3+7y^3-189z^3+189y^2z-63z^2x-6yz^2-21xy^2 = 216p.$$

The equation for θ is $\qquad \theta^3-21\theta-7=0,$

$$k=-7,\quad b=7,\quad b^2+4k^3, =\Delta, =-27.49,\quad r=21.$$

The equation for ϕ is $\qquad 3\phi=-28-\theta+2\theta^2,$

giving $\qquad 3x'=3x-28z,\quad 3y'=3y-z,\quad 3z'=2z.$

[redacted]

These give integral values for x', y', z', only if z is a multiple of 3 (which is actually the case in the present problem). The transformed form is

$$\Phi(x, y, z) = x^3+7y^3+7^2z^3+21^2 . z^2x-147yz^2+42zx^2-21xy^2-21xyz.$$

Automorphs $\gamma = \frac{1}{3}\begin{pmatrix} 3, & \overline{14}, & 56 \\ . & \bar{2}, & \bar{7} \\ . & 1, & \bar{1} \end{pmatrix}$, $\gamma^2 = \frac{1}{3}\begin{pmatrix} 3, & 14, & 70 \\ . & \bar{1}, & 7 \\ . & \bar{1}, & \bar{2} \end{pmatrix}$

$$(p, q) = \tfrac{1}{3}(-1, -1), \quad \text{or} \quad \tfrac{1}{3}(-2, 1),$$

$$v = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & 0, & 7 \\ 1, & \bar{1}, & 21 \\ 0, & 1, & \bar{1} \end{pmatrix}, \quad v\gamma = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & 7, & \overline{21} \\ 1, & 3, & 14 \\ . & \bar{1}, & \bar{2} \end{pmatrix}, \quad v\gamma^2 = \tfrac{1}{3}\begin{pmatrix} \bar{1}, & \bar{7}, & \overline{28} \\ 1, & \bar{2}, & 7 \\ . & . & 3 \end{pmatrix},$$

$$\gamma v\gamma^2 = \tfrac{1}{9}\begin{pmatrix} \overline{17}, & 7, & \overline{14} \\ \bar{2}, & 4, & \overline{35} \\ 1, & \bar{2}, & 4 \end{pmatrix}, \quad \gamma v = \tfrac{1}{9}\begin{pmatrix} \overline{17}, & 70, & \overline{329} \\ \bar{2}, & 5, & \overline{35} \\ 1, & \bar{2}, & 22 \end{pmatrix}, \quad \gamma v\gamma = \tfrac{1}{9}\begin{pmatrix} \overline{17}, & 77, & \overline{371} \\ \bar{2}, & 1, & \overline{14} \\ 1, & 4, & 16 \end{pmatrix},$$

$$\gamma^2 v\gamma = \tfrac{1}{9}\begin{pmatrix} 11, & \bar{7}, & \bar{7} \\ \bar{1}, & \overline{10}, & \overline{28} \\ \bar{1}, & 1, & \overline{10} \end{pmatrix}, \quad \gamma^2 v\gamma^2 = \tfrac{1}{9}\begin{pmatrix} 11, & \overline{49}, & 224 \\ \bar{1}, & 2, & 14 \\ \bar{1}, & 2, & \overline{13} \end{pmatrix}, \quad \gamma^2 v = \tfrac{1}{9}\begin{pmatrix} 11, & 56, & 245 \\ \bar{1}, & 8, & \overline{28} \\ \bar{1}, & \bar{1}, & \overline{19} \end{pmatrix}.$$

Geometrical Theory.

A point P whose rectangular coordinates (x, y, z) satisfy the equation

$$\Phi(x, y, z) = 1$$

is upon a cubic surface; and every point on this surface with integral coordinates (x, y, z) corresponds to a unit $x+y\theta+z\theta^2$ and *vice versâ*. The surface has three real asymptotic planes

$$x+y\theta+z\theta^2 = 0, \quad x+y\theta_1+z\theta_1^2 = 0, \quad x+y\theta_2+z\theta_2^2 = 0,$$

where θ, θ_1, θ_2 are real irrational roots of

$$\theta^3+3k\theta-b = 0.$$

It is clear that, of the three units corresponding to any point P, all or only one must be positive. The surface consists of four sheets for which the three units have the signs $+++$; $+--$, $-+-$, $--+$, respectively. The first may be distinguished as the "positive" sheet. Of the eight compartments into which space is divided by the asymptote planes, that one which contains the axis of x also contains the positive sheet, which it meets at the "vertex" A, coordinates 1, 0, 0. This compartment is separated from each of the other occupied compartments by an edge. The tangent plane at the vertex (1, 0, 0) is given by the equation

$$x-2kz = 1.$$

Consider now the section of the surface by a plane

$$x-2kz = m.$$

This plane not being parallel to any of the asymptote planes, the section will always have three asymptotes that in general form a triangle. When m is greater than 1 the plane cuts the positive sheet, and the curve consists of an oval inside the triangle with three infinite branches (from the three negative sheets) in the outward angles of the triangle. The curve has three diameters, the medians of the asymptote triangle, which meet at the point where the axis of x pierces the plane. If the plane moves till $m = 1$, the oval in the curve shrinks up to an acnode, at the vertex A of the surface. These sections are two of the species added by Stirling to Newton's *Enumeratio*. When m is between 0 and 1, the triangle is empty and the infinite branches remain (Newton's 22nd species); when $m = 0$, the asymptotes are concurrent (Newton's 32nd species); and when m becomes negative, the triangle is empty and the infinite branches are separated from the triangle by its sides (Newton's 23rd species).

Consider now a point P, (x, y, z) on the unit surface, and its conjugate points whose coordinates are

$$\gamma(x, y, z) = x-3kz+(2ky-bz)/2r, \quad -\tfrac{1}{2}y+(by+2k^2z)/2r, \quad -\tfrac{1}{2}z+(2ky-bz)/2r,$$

$$\gamma^2(x, y, z) = \text{the same with } -r \text{ for } r.$$

It is then seen that $\qquad x-2kz$

has the same value for the three conjugate points P, γP, $\gamma^2 P$, so that these points lie upon a plane parallel to the tangent plane at the vertex, viz., the plane

$$x-2kz = \text{const.}$$

A second relation common to the three points can also be found by direct substitution or by determining the volume of tetrahedron $O, P, \gamma P, \gamma^2 P$, namely,

$$(x-2kz)(ky^2-byz-k^2z^2)/2r.$$

This volume is the same for each of the three points, and one of its factors is also constant. Hence the three points lie upon the cylinder whose axis is the axis of x,

$$-ky^2+byz+k^2z^2 = \text{const.},$$

which can be written $(2ky-bz)^2+3r^2z^2 = \text{const.}$,

and is seen to be elliptic.

The geometric construction of the two points γP, $\gamma^2 P$ conjugate with a given point P now becomes very simple. Let ABC be the asymptote triangle on the plane through P, parallel to the tangent at the vertex. Then P will be a point inside or outside the triangle according as it is on the positive sheet or one of the negative sheets. Through P draw lines parallel to the sides of the triangle, upon which mark points P', Q', R', respectively, so that the middle points of PP', QQ', RR' are the middle points of the segments intercepted by the triangle. A similar construction at P', or Q', or R', or all of them, will give two new points Q, R. Then P, Q, R are conjugate points, and so likewise are P', Q', R'.

The proof comes out thus. The six points are on the cubic curve because the intercepts between the cubic and two asymptotes on a line parallel to the third are equal. Hence, P being on the curve, P', Q', R', and therefore also Q, R, are on the curve.

Similarly for the conic, observing that the medians of the triangle ABC are diameters of the conic conjugate to the sides they bisect, it is seen that the six points are on the conic.

The medians AF, BG, CH divide the triangle into six parts with a common vertex at O', the centroid. The effect of γ is to transpose the asymptote planes cyclically. Hence A becomes B, B becomes C, C becomes A, while O' is unchanged. Hence the triangles $O'BF$, $O'CG$, $O'AH$ are conjugate, and any point in $O'BF$ has one conjugate in $O'CG$ and the other $O'AH$. But this is equivalent to the mode of distributing the six points into conjugate sets, as is seen at once from a figure.

As another application, consider a point U, coordinates (u, v, w); P, as before, the point (x, y, z), and examine the effect of $\boldsymbol{v}$, the proper automorph with u, v, w for its first column. This automorph determines a homogeneous strain in space; and, since it is proper, the three edges of the asymptote system remain unchanged. The point A (1, 0, 0) comes to U; the plane containing P, γP, $\gamma^2 P$, which was parallel to the tangent plane at A, when transformed, contains the points vP, $v\gamma P$, $v\gamma^2 P$, and is parallel to the tangent plane at U, the new position of A. In the same way, using the automorph γv, we find that the plane of the points γvP, $\gamma v\gamma P$, $\gamma v\gamma^2 P$ is parallel to the tangent plane at γU.

It may be noted that the fundamental units must be represented by points on the negative sheets, unless indeed there are no points with integral coordinates upon these sheets. For, if (u, v, w) is on the positive sheet, the three conjugate units are positive. Hence every power has three positive conjugates, and cannot lie on a negative sheet because units on a negative sheet have two of the conjugates negative.

Examples illustrating Lord Rayleigh's Theory of the Stability or Instability of certain Fluid Motions. By A. E. H. Love. Read January 9th, 1896.

In papers published in the *Proceedings*, Vols. XI. and XIX., Lord Rayleigh has discussed the oscillations possible in a stream of fluid flowing between two fixed planes, which arise from difference of spin (or molecular rotation) in different parts. He has especially attended to cases where the spin changes discontinuously at certain planes between the boundaries. A difficulty occurs in these solutions through the fact that the places where the stream-velocity is equal to the wave-velocity are singularities of the integrals of the differential equation on which the small varied motion depends. This difficulty Lord Rayleigh has sought to evade in a recent paper (*Proc.*, Nov., 1895). It was felt, however, that, in the case of continuously varying spin, the complete discussion of the problem for a particular law of velocity would be desirable. The present paper contains such a discussion as appears possible (without solving the differential equation) of a case where there are two separated singular places of the integral. The general conclusion seems to be that wave-motions of Lord Rayleigh's type can only occur in some very special cases, and that his method does not avail for the determination of a criterion of stability when the disturbance is of a general character. Some further examples are given in which the exact analytical form of the disturbance can be calculated for a definite wave-velocity.

1. A steady motion of fluid in the plane (x, y) is possible in which the stream-lines are parallel to a fixed line, the axis x, and the velocity of the fluid, everywhere parallel to this axis, is independent of x, and is any differentiable function of y. Suppose U is this velocity. The fluid may be bounded by two lines parallel to the axis of x, and between these lines $-\frac{1}{2}\frac{\partial U}{\partial y}$ is everywhere finite. This quantity is the spin (or molecular rotation) of the fluid at any point, and it is variable only with y, and we shall suppose it to be a differentiable function* of y. We shall suppose that U and $\frac{\partial U}{\partial y}$ are given for all values of y between the two limiting lines which contain the fluid.

We seek now to determine a motion of the fluid which shall, at all times, and for all points between the given boundaries, differ very little from the steady motion just described. Suppose $U+u$ and v are the component velocities parallel to x and y in the varied motion, and suppose that u and v are at any one time each less than some given arbitrarily small quantity ϵ for all values of x, and for all values of y between the given boundaries,† and further, suppose that u and v do not tend to increase indefinitely with the time; then the quantities u and v are always small, of the order ϵ at most, and the steady motion is stable. For the investigation of such varied motions, the equations determining u and v are reduced to linear equations by rejecting all terms that involve their squares or products, or higher powers, and then it may be assumed that u and v are proportional to simple harmonic functions of the time, with a period to be determined. If the equation determining this period has only real roots, the motion is stable.

* The case in which $\partial U/\partial y$ is discontinuous has been sufficiently considered by Lord Rayleigh in the papers already cited.

† This limitation excludes *a priori* the solutions discussed by Lord Rayleigh for which $\partial v/\partial y$, and consequently u, becomes infinite. For the limitation itself, *cf.* Routh's *Treatise on the Stability of Motion*, p. 1.

2. In the problem before us the quantities u and v are to be determined from the equation of continuity, and the condition that the spin of any element is independent of the time. The former gives

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

and the latter gives

$$\frac{\partial}{\partial t}(Z+\zeta)+(U+u)\frac{\partial}{\partial x}(Z+\zeta)+v\frac{\partial}{\partial y}(Z+\zeta)=0,$$

where $Z, =-\frac{1}{2}\,\partial U/\partial y$, is the spin in steady motion, and ζ, $=\frac{1}{2}\,(\partial v/\partial x-\partial u/\partial y)$, is the added spin at any point. By the process already sketched this reduces to

$$\frac{\partial \zeta}{\partial t}+U\frac{\partial \zeta}{\partial x}+v\frac{\partial Z}{\partial y}=0 \quad\text{.......................(1).}$$

We shall now, following Lord Rayleigh, suppose that the varied motion is a wave-motion with wave-length $2\pi/\kappa$ and velocity V, so that

$$u=u_1\cos\kappa\,(x-Vt+a),\quad v=v_1\sin\kappa\,(x-Vt+a)\text{.........(2),}$$

where u_1 and v_1 are functions of y only. The equation of continuity then becomes

$$-\kappa u_1+\frac{\partial v_1}{\partial y}=0 \quad\text{..........................(3),}$$

and the equation (1) for the varied spin becomes

$$(U-V)\left(\frac{\partial^2 v_1}{\partial y^2}-\kappa^2 v_1\right)-\frac{\partial^2 U}{\partial y^2}v_1=0,$$

since

$$\zeta=\tfrac{1}{2}\cos\kappa\,(x-Vt+a)\left(\kappa v_1-\frac{\partial u_1}{\partial y}\right).$$

We may suppress the suffix attached to v_1, and we see that v satisfies the equation

$$(U-V)\left(\frac{\partial^2 v}{\partial y^2}-\kappa^2 v\right)-\frac{\partial^2 U}{\partial y^2}v=0\text{....................(4).}$$

The further conditions which have to be satisfied are that $v=0$ at the two fixed boundaries, *i.e.*, for two particular values of y.

3. In order that the disturbance may be propagated by waves in the manner supposed it must be possible to assign a real quantity V so that a function v may exist which (i.) satisfies the differential equation (4) for all values of y in a certain real interval, (ii.) vanishes at the limits of this interval, (iii.) is finite, and has a finite and continuous differential coefficient in this interval. Further, in order that the method may apply to an arbitrary initial disturbance, it is necessary that there should be a series of such quantities V_r, and, associated with each, a function v_r of such a character that an arbitrary function of y can be expanded in a series of the form

$$\Sigma A_r v_r,$$

which converges in the given interval. The quantities V_r are required to exist for all real values of κ.

Lord Rayleigh has proved* that it is impossible to satisfy the differential equation and the boundary conditions with a complex value of V, if $\dfrac{\partial^2 U}{\partial y^2}$ is one-signed between the boundaries; and he concluded that, under this condition, the steady motion expressed by U must be stable. It appears however that this conclusion required additional justification, inasmuch as there is no evidence to show that every disturbance will be propagated by waves in the manner supposed. Lord Rayleigh has further remarked† that it is impossible to satisfy the differential equation and the boundary conditions with any value of V for which $U-V$ and $\dfrac{\partial^2 U}{\partial y^2}$ have the same sign everywhere between the boundaries.

We can construct an example for which $\dfrac{\partial^2 U}{\partial y^2}$ is one-signed, and for which $U-V$ has the opposite sign for values of y between the boundaries, by taking for U any quadratic function of y, and for V a real constant such that the roots of $U-V$ are two values of y between which the boundaries of the fluid lie.

4. Let
$$U = Ay^2+2By+C,$$
and
$$U-V = A(y-a)(y-b),$$
and let the boundaries of the fluid be $y=h_1$ and $y=h_2$; and to fix

* *Proc.*, Vol. XI., p. 69.
† *L.c.*, p. 70.

ideas take $b > h_2 > h_1 > a$; then the differential equation becomes

$$(y-a)(y-b)\left(\frac{\partial^2 v}{\partial y^2} - \kappa^2 v\right) = 2v \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

When κ is given we have only one parameter to dispose of, viz., V, and this is given by

$$V = \tfrac{1}{4}A\,(b-a)^2 + \frac{AC-B^2}{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

so that the parameter which is in our power is $b-a$, the sum of the roots $(b+a)$ being given.

We transform the equation to one with fixed singularities by changing the independent variable from y to y', where

$$\frac{y-a}{y-b} = \frac{y'}{y'-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

i.e., by putting $y = (b-a)\,y' + a$. We then have

$$\frac{\partial^2 v}{\partial y'^2} - \kappa^2 (b-a)^2 v = -\frac{2v}{y'(1-y')},$$

and the interval within which it is to be solved is such that

$$1 > y' > 0,$$

but the interval does not extend so far as $y' = 0$ or $y' = 1$.

Writing now λ^2 for $\kappa^2 (b-a)^2$, we see that we have to seek for real values of λ for which the differential equation

$$\frac{\partial^2 v}{\partial y'^2} = -\frac{2v}{y'(1-y')} + \lambda^2 v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

can be satisfied by a value of v which vanishes for two values of y' lying in the interval $1 > y' > 0$. The limiting values, h_1' and h_2', of y' are given by

$$h_1' = (h_1 - a)/(b-a) = \tfrac{1}{2}\left\{1 - (a+b-2h_1)/(b-a)\right\},$$

$$h_2' = (h_2 - a)/(b-a) = \tfrac{1}{2}\left\{1 - (a+b-2h_2)/(b-a)\right\},$$

so that they depend on h_1, h_2, and on the parameter $(b-a)$ in quite definite ways.

5. It is useless to attempt to solve the differential equation. Solutions in series can of course be obtained, but they afford no criterion for determining the roots of v. We can however use an intuitional method* by comparing the above equation (8) with

$$\frac{d^2x}{dt^2} = -\frac{2}{t(1-t)}x + \lambda^2 x.$$

This equation may be regarded as the equation of motion of a particle, which moves in a straight line under the action of a repulsive force $\lambda^2 x$ proportional to its displacement x, and an attractive force $\frac{2}{t(1-t)}x$ also proportional to x, but variable with the time t. At a certain time t_0 $(= h_1')$, the particle is passing through the origin with a finite velocity, and we wish to discover whether the forces can be so adjusted that before $t = 1$ it may have returned to the origin. We cannot determine the time that the particle would take to describe any distance x, or to return to the origin, but we may use the principle that any increase in the strength of the attractive force or decrease in the strength of the repulsive force would shorten the period that must elapse before it returns to the origin. We can now show that at least one of the limiting values must lie near to $t = 0$ or $t = 1$. Suppose, to take an example, that the instant of starting is $t = \frac{1}{5}$, we can see that the particle will not have returned before $t = \frac{4}{5}$; for during this interval the strength of the attractive force is constantly less than $\frac{25}{2}$, and the particle would not return in less time than $\frac{1}{5}\pi\sqrt{2}$ if the strength of the attractive force were constant and had its greatest value. More generally, if the particle starts at time $t_0 (= \epsilon)$, and returns at time t_1 $(\not> 1-\epsilon)$, the time taken to return to the origin is not greater than $1-2\epsilon$. The maximum strength of the attractive force is $2/\{\epsilon(1-\epsilon)\}$. The period is greater than

$$\frac{\pi}{\sqrt{\left\{\frac{2}{\epsilon(1-\epsilon)} - \lambda^2\right\}}},$$

so that this quantity is certainly $< 1-2\epsilon$. Further we have

$$\lambda = \kappa(b-a) = \kappa(a+b-2h_1)/(1-2\epsilon),$$

---

* Suggested by Klein's discussion of Lamé's equation in his lithographed *Vorlesungen über lineare Differentialgleichungen der zweiten Ordnung*, Göttingen, 1894.

and we hence find that

$$\epsilon < \tfrac{1}{3}\left[1-\sqrt{\left\{\frac{\pi^2+\kappa^2\,(a+b-2h_1)^2}{8+\pi^2+\kappa^2\,(a+b-2h_1)^2}\right\}}\right].$$

In the case where κ is small, *i.e.*, for very long waves in the hydrodynamical problem, $\epsilon < \frac{1}{8}$ and for greater values of κ still smaller values of ϵ are required.

Again, let us suppose the particle starts at time $t_0\,(\nless \epsilon)$ and returns at time $t_1\,(=1-\epsilon)$; then, just as before, we find

$$\epsilon < \tfrac{1}{3}\left[1-\sqrt{\left\{\frac{\pi^2+\kappa^2\,(a+b-2h_2)^2}{8+\pi^2+\kappa^2\,(a+b-2h_2)^2}\right\}}\right].$$

This discussion shows that equation (8) with the conditions $v=0$ when $y'=h_1'$ and when $y'=h_2'$ cannot be satisfied unless either h_1' is very near 0 or h_2' is very near 1. In general we can only satisfy these conditions, either by taking a very near to h_1, or by taking b very near to h_2.

The results here obtained are by themselves sufficient to show that in many cases the problem can have no solution. We have found that either

$$\frac{a+b-2h_1}{b-a} > \text{a quantity} > \tfrac{3}{4},$$

or

$$\frac{a+b-2h_2}{b-a} > \text{a quantity} > \tfrac{3}{4}.$$

The former of these gives

$$b-a < \tfrac{4}{3}\,(a+b-2h_1),$$

but, since

$$b-a > h_2-h_1,$$

we must certainly have

$$h_2-h_1 < \tfrac{4}{3}\,(a+b-2h_1),$$

or

$$3h_2+5h_1 < 4\,(a+b).$$

In like manner the inequality

$$\frac{a+b-2h_2}{b-a} > \tfrac{3}{4}$$

leads to

$$11h_2-3h_1 < 4\,(a+b).$$

We conclude that there are some restrictive conditions on h_1, h_2 in order that a solution may be possible, and the character of these would appear to be that the breadth of the stream must not exceed one or other of certain limits depending on the given quantity $(a+b)$.

6. But now suppose these conditions are satisfied, and consider the velocity-time curve of the motion of the particle. Let us suppose that λ is sufficiently small for the resultant force on the particle to be attractive throughout the interval $1 > t > 0$. Since the attractive force has minimum strength when $t = \frac{1}{2}$, this requires $\lambda^2 < 8$. Then the velocity constantly diminishes until the particle returns to the origin, and, as we shall see in § 8, the velocity becomes in general positively or negatively infinite at $t = 1$. The curve has therefore an asymptote $t = 1$. Further, we shall prove in § 8 that when the velocity is infinite the displacement remains finite and thus the area between the curve and the asymptote is finite. Now the area contained between the curve, the t axis, and two ordinates $t = t_0$ and $t = t_1$ is the space passed over between these times. If then the particle starts from the origin at time t_0 with a finite positive velocity, it may happen that there is a value $t_1 (< 1)$ of t for which the positive area between t_0 and the point where the velocity vanishes is equal to the negative area between this point and t_1. Since $\frac{d^2x}{dt^2} = 0$ when $x = 0$, the points t_0 and t_1 must be points on the velocity-time curve where the ordinate is a maximum or a minimum. Thus λ must be chosen so that the distance between a maximum and the next minimum ordinate may be $\kappa (h_2 - h_1)/\lambda$, and the area between the maximum ordinate and the point where the velocity vanishes must be equal to the area between this point and the minimum ordinate (see Fig. 1). It may be possible to secure the satisfaction of these conditions when h_1, h_2 are suitably chosen, but it may be impossible. We shall proceed on the supposition that one value of λ has been chosen which satisfies all the conditions.

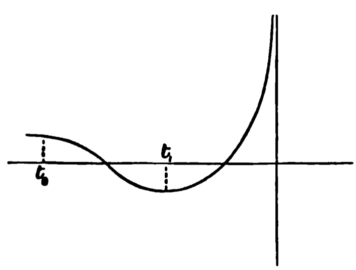


FIG. 1.

7. It is next important to inquire whether the conditions can be satisfied by more than one value of λ. In the first place we show

that if the particle can return once to the origin with any value of λ it cannot return for any greater value of λ. For the period in which it is required to return, being $\kappa(h_2 - h_1)/\lambda$, is diminished by increasing λ; but the period in which it does return is certainly increased by increasing λ, since λ^2 is the strength of the repulsive force. In the next place we observe that a diminution in the value of λ increases the time in which the particle is required to return to the origin, but diminishes the period in which the particle does return once to the origin. If then with one value of λ the particle can return once to the origin with any smaller λ for which the conditions can be satisfied, it must return more than once. Thus, if more than one value of λ can be chosen which satisfies all the conditions, with these different λ's the particle returns to the origin once, twice, three times, and so on. Hence these values of λ form a discrete series of diminishing quantities. But, since $\lambda \not< \kappa(h_2 - h_1)$, there is only a finite number of possible values for λ.

Hence, going back to the hydrodynamical problem, it is proved that, though there may be a finite number of values of V for which the differential equation

$$(U-V)\left(\frac{\partial^2 v}{\partial y^2} - \kappa^2 v\right) = \frac{d^2 U}{dy^2} v$$

has a solution v, which vanishes when $y = h_1$ and when $y = h_2$, there cannot be an indefinite series of such values. It follows that, though there may be particular types of disturbance which can be propagated by wave-motion in the manner supposed, this cannot be true for a general disturbance.

Further, since, as we have seen, possible values of λ cannot exist for arbitrary values of h_1 and h_2, there will not be such wave-motions unless some (unknown) relations subsist between the constants in U and the ordinates h_1, h_2 of the boundary lines.

We conclude that so long as the wave-velocity differs from the stream-velocity at all points between the boundaries there will be wave-motions of the kind supposed only for some special types of disturbance, and with the breadth of the stream subject to some restrictions.

We shall see presently that the application of the method is still more restricted when the wave-velocity is allowed to become equal to the stream-velocity.

8. Before passing on to examine how our results are modified if either of the roots of $U-V$ lies between the boundaries $y=h_1$ and $y=h_2$, it will be convenient to investigate the character of the solutions of the differential equation in the neighbourhood of the roots of $U-V$. Taking the equation in the form

$$\frac{\partial^2 v}{\partial y'^2} = \lambda^2 v - \frac{2v}{y'(1-y')},$$

we observe that in the neighbourhood of $y'=0$ there is one particular integral, say v_1, which can be expanded in powers of y', the first term being the term of degree unity in y'. We may write

$$v_1 = g_1 y' + g_2 y'^2 + g_3 y'^3 + \ldots + g_n y'^n + \ldots;$$

then the differential equation gives us

$$(y'-1)\left[\sum_2^\infty n(n-1)\, g_n y'^{n-1} - \lambda^2 \sum^\infty g_n y'^{n+1}\right] - 2\sum^\infty g_n y'^n = 0,$$

from which

$$g_2 = -g_1,$$

$$6g_3 = \lambda^2 g_1,$$

$$n\left[(n-3)\, g_{n-1} - (n-1)\, g_n\right] + \lambda^2 (g_{n-2} - g_{n-3}) = 0, \quad n = 4, 5, \ldots.$$

We may assign an arbitrary value to g_1 and then all the other coefficients are determinate.

There is also a particular integral, say v_2, which cannot be so expressed, but

$$v_2 = v_1 \log y' + v',$$

where v' is a series of the form

$$v' = h_0 + h_2 y'^2 + h_3 y'^3 + \ldots,$$

in which h_0 is different from zero. The differential equation gives us

$$\frac{\partial^2 v'}{\partial y'^2} + \left(\frac{2}{y'(1-y')} - \lambda^2\right) v' + \frac{2}{y'}\,\frac{dv_1}{dy'} - \frac{1}{y'^2} v_1 = 0,$$

or

$$(1-y')\left[\sum_2^\infty n(n-1)\, h_n y'^{n-1} - \lambda^2 \left(h_0 y' + \sum^\infty h_n y'^{n+1}\right)\right]$$

$$+ 2\left(h_0 + \sum_2^\infty h_n y'^n\right) + (1-y')\left[2\sum^\infty n g_n y'^{n-1} - \sum^\infty g_n y'^{n-1}\right] = 0,$$

from which
$$2h_0+g_1=0,$$
$$2h_2-\lambda^2h_0+3g_2-g_1=0,$$
$$6h_3+\lambda^2h_0+5g_3-3g_2=0,$$
$$12h_4-4h_3-\lambda^2h_2+7g_4-5g_3=0,$$
$$n\left[(n-1)h_n-(n-3)h_{n-1}\right]-\lambda^2(h_{n-2}-h_{n-3})+(2n-1)g_n-(2n-3)g_{n-1}=0,$$
$$n=5,\ 6,\ \ldots.$$

This work is the application to our particular equation of a known general theory.* It is further known that the series v_1 and v' converge when $1>y'>-1$, the other singularity of the differential equation being at $y'=1$, but they do not converge for the value $y'=1$. There will in like manner be two particular integrals which in the neighbourhood of $y'=1$ can be expressed in the forms

$$g_1(1-y')+g_2(1-y')^2+\ldots+g_n(1-y')^n+\ldots,$$

and $$\log(1-y')\left[g_1(1-y')+g_2(1-y')^2+\ldots+g_n(1-y')^n+\ldots\right]$$
$$+h_0+h_2(1-y')^2+h_3(1-y')^3+\ldots+h_n(1-y')^n+\ldots,$$

the coefficients being clearly the same as before, on account of the complete symmetry of the differential equation with respect to $y'=0$ and $y'=1$. The series here given converge when $1>1-y'>-1$, *i.e.*, when $2>y'>0$.

We notice that neither solution becomes infinite at the singularities, but v_2 has an infinite differential coefficient at $y'=0$. Thus, in general, $\partial v/\partial y$ is infinite at the singularities, but v is finite.

9. The results just found may be interpreted in accordance with the intuitional method in which y' is replaced by the time t, and v is regarded as the displacement of a particle which moves in a straight line under a repulsive force and an attractive force. The particle will generally have infinite velocity at the instants $t=0$ and $t=1$ (supposing the time of motion to include those instants), but it may have a finite velocity at one of these instants, and in that case it must be at the origin at that instant. Further, before the time $t=0$, and after the time $t=1$, the force is repulsive, so that if the particle

* See *e.g.* Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, Bd. I., Leipzig, 1895.

is moving through the origin at time $t = 0$ with a finite velocity it can never have been at the origin before, and if the particle is moving through the origin at time $t = 1$ with a finite velocity it can never return to the origin. Again, since the constant h_0 of the v_2 solution must be different from zero, we see that the particle can never have infinite velocity when at the origin.

Taking as before the supposition that the force never becomes repulsive, the velocity-time curves in the cases where the velocity is finite at $t = 0$ and $t = 1$ respectively are of the forms shown in Fig. 2 and Fig. 3, so that it appears probable that λ may be so chosen

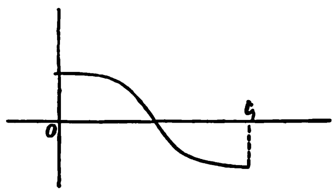


Fig. 2.

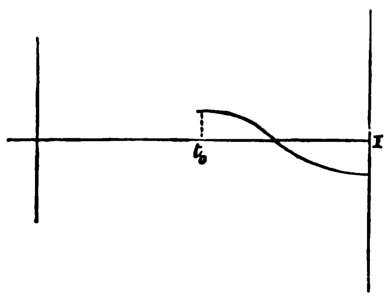


Fig. 3.

that in the first the area between $t = 0$ and $t = t_1$ (<1) shall vanish, and in the second that the area between $t = t_0$ (>0) and $t = 1$ shall vanish.

10. Going back now to the hydrodynamical problem, if we take $a = h_1$, b will be fixed since $a+b$ is given. We shall suppose that $b > h_2$. Then the solution corresponding to v_1 with its differential coefficients remains finite when $y = h_1$ and throughout the range $h_2 > y > h_1$, and it appears probable that the value of V which makes $a = h_1$ will make v vanish for some definite values of $h_2 < b$. But it is not certain that this will be the case; and, if it does so happen,

the values of h_2 will be thereby fixed, and may not include the given value of h_2. Thus there will be, with given $U (= Ay^2+2By+C)$, at most a finite number of definite breadths of stream for which a disturbance of wave-length κ can be propagated by waves in the manner supposed with a wave-velocity equal to the value of U on one of the boundary lines. Further, as in the preceding case, the disturbances which can be so propagated will not be arbitrary disturbances of wave-length κ but quite definite disturbances. Like results follow if we take $b = h_2$.

11. Any other supposition as to the relative situation of a, b, h_1, h_2 may be similarly disposed of. Without going through all the work we shall tabulate the results :—

(i.) $\left\{\begin{matrix} b > a > h_2 > h_1 \\ h_2 > h_1 > b > a \end{matrix}\right\}$. No disturbance possible. (This is the supposition disposed of by Lord Rayleigh in *Proc.*, Vol. XI., p. 70.)

(ii.) $b > h_2 > h_1 > a$. At most a finite number of values of V for any κ, so that special types of disturbance only are capable of being propagated by waves in the manner supposed.

(iii.) $b > h_2 > h_1$, $h_1 = a$. No value of V unless h_2 has one of certain definite values, and possibly no value of V for any h_2. So that at most this would give a finite number of values of V when h_2 has one of certain definite values; *i.e.*, at most there are a finite number of special types of disturbance, in streams of special breadths, which can be propagated by waves in the manner supposed, with $V = U(h_1)$.

(iv.) $b = h_2$, $h_2 > h_1 > a$. This case is exactly like the preceding.

(v.) $b < h_2$, $h_2 > h_1 > a$. In this case $\partial v/\partial y$ can only be finite, when $y = b$, if v vanishes there, and then v cannot vanish for any greater value of y, so that in particular it cannot vanish for h_2. Hence there is no solution.

(vi.) $a > h_1$, $b > h_2 > h_1$. This case is like the preceding, and there is no solution.

(vii.) $a = h_1$, $b = h_2$. Since the value of v which makes $\partial v/\partial y$ finite at a vanishes at a, and is finite and different from zero at b, there is no solution.

12. The case where U is a linear function of y is very simple. Suppose as before that the boundaries are $y = h_1$ and $y = h_2$. The differential equation becomes

$$\frac{\partial^2 v}{\partial y^2} - \kappa^2 v = 0 \quad \text{..............................(9)},$$

and a solution vanishing when $y = h_1$ is

$$v = A \sinh \kappa (y - h_1),$$

but we cannot make it vanish also when $y = h_2$. In this case it has been suggested that a possible wave-motion might be found by taking V equal to the value of U at one line, $y = a$ say, between h_1 and h_2. Then the differential equation (9) need not be satisfied when $y = a$. We should then have to take

$$v = A \sinh \kappa (y - h_1), \quad a > y > h_1,$$

$$v = B \sinh \kappa (h_2 - y), \quad h_2 > y > a.$$

To make v and $\partial v / \partial y$ continuous at $y = a$, we should require

$$A \sinh \kappa (a - h_1) = B \sinh \kappa (h_2 - a),$$

$$A \cosh \kappa (a - h_1) = -B \cosh \kappa (h_2 - a),$$

and these cannot be satisfied when h_1 is different from h_2. Thus there would be in this case no disturbance which could be propagated by waves in the manner supposed. Yet the example afforded by initial disturbances

$$u = C (2y - h_1 - h_2) \cos \kappa x,$$

$$v = \kappa C (y - h_1)(y - h_2) \sin \kappa x,$$

shows that some varied motion is possible which initially is periodic in x with given wave-length. Lord Rayleigh's method does not avail for the discovery of this motion, nor for determining whether the original steady motion is stable for this type of disturbance.

13. In the examples hitherto discussed there has either been no solution, or we have been able to show that the solution, if it exists, corresponds to a special type of disturbance, but we have not, in any case where there could be a solution, found the explicit form of the solution. In the examples now to be given the exact analytical form of v will be obtained when U has a definite form, and V has a

particular value. We observe that the differential equation depends only on $U-V$. Writing W for this, the equation is

$$\frac{\partial^2 v}{\partial y^2}-\kappa^2 v = \frac{\partial^2 W}{\partial y^2}\frac{v}{W} \qquad \text{......................(10).}$$

If the form of W is assigned so that this equation can be solved, it is only necessary to determine whether the boundary conditions can be satisfied.

(*a*) Let $$W = A \sin \mu\,(y-a);$$

then $$v = C \sin \lambda\,(y-h_1), \quad \text{where} \quad \lambda^2 = \mu^2-\kappa^2,$$

and, if $\lambda\,(h_2-h_1)/\pi =$ an integer, we have a solution. Thus there are in this case solutions corresponding to every integral value of s which is such that $\mu^2-s^2\pi^2/(h_2-h_1)^2$ is positive.

(*b*) Let $$W = Ay\frac{d}{dy}\left(\frac{\sin \mu y}{y}\right),$$

then $$\frac{\partial^2 W}{\partial y^2} = W\left(-\mu^2-\frac{2}{y^2}\right),$$

and if $\mu^2-\kappa^2$ is positive, and $= \lambda^2$ say, we have

$$v = y\frac{d}{dy}\left(\frac{C_1 \sin \lambda y + C_2 \cos \lambda y}{y}\right).$$

If λ is suitably chosen, v can vanish when $y = h_1$, and when $y = h_2$. In fact this requires

$$\tan \lambda\,(h_1-h_2) = \lambda\,(h_1-h_2)/(1+\lambda^2 h_1 h_2).$$

This equation has an infinite number of real roots, but only such of them may be taken as make $\mu^2-\lambda^2$ positive.

In this case the singular point $y = 0$ may be included in the interval between the boundaries if the latter are suitably chosen. We must then take $C_2 = 0$, and the boundaries h_1, h_2 must be roots of the same equation of the form $\tan \lambda h = \lambda h$. Thus if $h_2 = -h_1 = h$ any value of λ ($\ngtr \mu$) which satisfies this equation gives a solution. It is to be noted here that $W = 0$ at $y = 0$, and v also $= 0$ at $y = 0$; but the limits for y are not necessarily $y = 0$ and $y = h$, in contrast with the case where U was a quadratic function, for which we found that, if a zero of $U-V$ is included in the region occupied by fluid, it must lie on the boundary.

On Boltzmann's Law of the Equality of Mean Kinetic Energy for each Degree of Freedom. By S. H. Burbury.

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1. The following investigation is based on the assumption that if a great number of molecules are moving in any large space, and if the velocities of translation of any group be denoted by $u_1 \dots u_n$, $v_1 \dots v_n$, $w_1 \dots w_n$ (and they may have also other velocities, *e.g.*, of rotation $\dot{q}_1 \dots \dot{q}_n$), then the chance that at any instant these velocities shall lie within assigned limits $u_1 \dots u_1 + du_1$, &c., is proportional to $e^{-hQ} du_1 \dots dw_n d\dot{q}_1 \dots$, in which Q is a quadratic function, containing squares of the velocities multiplied by certain coefficients, and also containing the products $u_1 u_2$, &c., $v_1 v_2$, &c., $w_1 w_2$, &c., of the translation velocities. In other words, I assume that the translation velocities are generally—*i.e.*, always except in the limiting case of an infinitely rare medium—*correlated*. In order completely to prove the permanence of such a distribution of velocities, it would be necessary to show (1) that it would not be destroyed by diffusion, which I hope to do hereafter; and (2) that it will, if certain conditions be satisfied among the coefficients in Q, not be disturbed by encounters between the molecules. The object of the present investigation is to find what relations must exist among the coefficients in Q in order that the distribution may not be disturbed by encounters. If we denote by u_1, u_2, &c., the velocities before encounter, and by u_1', u_2', &c., those after encounter, we have to find what the relations among the coefficients must be in order that when u_1, u_2, &c., are expressed in terms of u_1', u_2', &c., according to the known laws of an encounter, Q shall be the same function of u_1', u_2', &c., as it was of u_1, u_2, &c. That is, it must be unchanged in form. I confine myself now to the case in which the molecules are, or behave as, rigid elastic bodies, and I shall use the term "collision" for "encounter." In that case it is as convenient to use "velocities" as "momenta." The characteristic of collisions between elastic bodies is that, if two colliding bodies have between them n degrees of freedom, there are $n-1$ linear functions of their n velocities which remain unchanged—let them be S_1, S_2, ... S_{n-1}—and one, R, which remains unchanged in absolute magnitude, but changes sign. It is

then possible to express the n velocities concerned in the collision in terms of $S_1, \dots S_{n-1}, R$. If, when they are so expressed, Q contains R only in the second degree, then evidently Q is the same function of the post-collision velocities u_1', &c., as of the pre-collision velocities u_1, &c., because both $S_1, \dots S_{n-1}$ and R^2 are the same functions of u_1', &c., as of u_1, &c. That then is the condition of which we are in search. We now deal with four cases.

2. *Case I. The molecules are equal elastic spheres.*—Two colliding spheres having between them six degrees of freedom, there are five linear functions of the velocities $S_1, \dots S_5$ which remain unchanged. Four of these are the component velocities of the two spheres in the common tangent plane at collision, which need not concern us. Let the normal velocities be u_1, u_2 before collision, and u_1', u_2' after. Then we have, by conservation of momentum,

$$\left.\begin{aligned} u_1+u_2 &= u_1'+u_2' = S \\ \text{or} \qquad u_1-u_1' &= u_2'-u_2 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots (1).$$

By conservation of energy,

$$u_1^2-u_1'^2 = u_2'^2-u_2^2 \dots\dots\dots\dots\dots\dots\dots\dots\dots (2),$$

and, from (1) and (2), $\qquad u_1+u_1' = u_2'+u_2,$

and $\qquad u_1-u_2 = R = u_2'-u_1';$

whence $\qquad u_1 = \dfrac{S+R}{2},$

$$u_2 = \frac{S-R}{2}.$$

In this case let

$$\begin{aligned} Q = a_1(u_1+v_1^2+w_1^2)+a_2(u_2^2+v_2^2+w_2^2)+\&c. \\ +b_{12}(u_1u_2+v_1v_2+w_1w_2)+\&c. \\ +b_{13}(u_1u_3+v_1v_3+w_1w_3)+\&c. \\ +b_{23}(u_2u_3+v_2v_3+w_2w_3)+\&c. \end{aligned}$$

Substitute for u_1, u_2 their values $\dfrac{S+R}{2}$ and $\dfrac{S-R}{2}$. That makes

$$Q = a_1\left(\frac{S+R}{2}\right)^2 + a_2\left(\frac{S-R}{2}\right)^2 + \&c.$$

$$+ b_{12}\frac{(S+R)(S-R)}{4} + \&c.$$

$$+ b_{13}\frac{S+R}{2}u_3 + \&c.$$

$$+ b_{23}\frac{S-R}{2}u_3 + \&c.$$

In order that Q, so expressed, may contain R only in the form R^2, we have the necessary condition $a_1 = a_2$, and similarly $a_1 = a_3$, &c.

A further necessary condition is $b_{13} = b_{23}$. That is that, if the two spheres to which the suffixes 1 and 2 relate collide with each other, then $b_{13} = b_{23}$, $b_{14} = b_{24}$, &c. And evidently a sufficient condition is that all the b's are equal. But, if b_{12}, b_{13}, &c., be functions of the distance between the spheres to which the suffixes relate, it may conceivably be the case that the difference $b_{13} - b_{23}$ becomes evanescent when spheres (1) and (2) are close together, as in collision they must be, without making them generally equal. The absolute equality of all the b's, though sufficient, may not be necessary.

3. *Case II. Two sets of unequal elastic spheres.*—Let their masses be m, M respectively, and their velocities u, v, w, U, V, W, respectively. The treatment of this case is the same as that of Case I., except that, in lieu of $u+U=S$ for conservation of momentum, we have

$$mu + MU = S,$$

and, as before,

$$u - U = R;$$

whence

$$u = \frac{S+MR}{M+m}, \quad U = \frac{S-mR}{M+m};$$

and, in order that Q, when u, U are expressed in terms of S and R, may contain R only in the second degree, we require that $\dfrac{dQ}{dR}$ shall vary as R.

That is

$$\frac{dQ}{du}\frac{du}{dR} + \frac{dQ}{dU}\frac{dU}{dR} \propto R.$$

Now, if u, U relate to two colliding spheres, and u_1, v_1, w_1, U_1, V_1, W_1 to other type spheres of either class respectively, we have

$$\begin{aligned} Q = {} & a\,(u^2+v^2+w^2)+A\,(U^2+V^2+W^2) \\ & +a\,(u_1^2+v_1^2+w_1^2)+A\,(U_1^2+V_1^2+W_1^2) \\ & +\beta\,(uU+vV+wW) \\ & +b\,(uu_1+vv_1+ww_1) \\ & +B\,(UU_1+VV_1+WW_1)+\&c. \\ & +\beta\,(uU_1+vV_1+wW_1)+\beta\,(Uu_1+Vv_1+Ww_1)+\dots; \end{aligned}$$

whence
$$\frac{dQ}{du} = 2au+\beta U+bu_1+\beta U_1,$$

$$\frac{du}{dR} = \frac{M}{M+m},$$

also
$$\frac{dQ}{dU} = 2AU+\beta u+BU_1+\beta u_1,$$

$$\frac{dU}{dR} = -\frac{m}{M+m}.$$

Therefore

$$\frac{dQ}{dR} = \frac{1}{M+m}\left(\overline{2aM-m\beta}\,u-\overline{2Am-M\beta}\,U\right)$$

$$+\frac{1}{M+m}\,(bM-\beta m)\,u_1-\frac{1}{M+m}\,(Bm-\beta M)\,U_1,$$

also
$$R = u-U,$$

and in order that for all values of the variables $\dfrac{dQ}{dR}$ may vary as R, we must have

$$2aM-m\beta = 2Am-M\beta, \quad \text{or} \quad 2aM-2Am+\beta\,\overline{M-m} = 0,$$

$$b = \frac{m}{M}\beta, \quad B = \frac{M}{m}\beta.$$

I have here assumed all the β's to be equal, and similarly for the B's and b's. This, however, may be more than is necessary, as in the case of equal spheres. I have also assumed, as proved in Case I., that all the m's have the same a, and all the M's the same A.

4. If β be negative, it follows that, if $M > m$, $A < a$, or the greater mass has in this distribution the greater mean kinetic energy. This is contrary to Boltzmann's law, but it is a necessary consequence of the existence of the coefficient β. If Boltzmann's law be assumed as an axiom, we must then draw the conclusion that in the permanent state β must be zero. But Boltzmann's law has never been proved, except upon assumptions which are tantamount to assuming $\beta = 0$. And I hope to show later that under certain circumstances β cannot $= 0$. It has, of course, always been known that if *the entire mass* of molecules has any visible motion, *e.g.*, the earth's motion in space, the molecules of greater mass have, taking this motion into account, the greater mean energy. I have only extended this to the case where, not the entire mass, but only portions of it too small to be isolated, have some common motion, which is implied by the existence of β.

5. *Case III. Any number, say n, sets of unequal spheres.*—Let their masses be $m_1, \dots m_n$.

Let
$$Q = a_1 u_1^2 + a_2 u_2^2 + \dots + a_n u_n^2 + a_1 v_1^2 + \&c.$$
$$+ \beta_{12}(u_1 u_2) + \beta_{13}(u_1 u_3) + \&c.$$
$$+ b_{11} u_1 u_1' + b_{22} u_2 u_2' + \&c.$$

We have then to satisfy $\frac{n\,.\,\overline{n-1}}{2}$ equations (A) of the form

$$2a_p m_q - 2a_q m_p + \beta_{pq}(m_q - m_p) = 0,$$

and $n\,.\,\overline{n-1}$ equations (B) of the form

$$b_{pp} = \frac{m_p}{m_q}\beta_{pq}.$$

If any one β, as β_{12}, be given, we can, by the equations B, find every other β in terms of it in the form

$$\beta_{pq} = \frac{m_p m_q}{m_1 m_2}\beta_{12}.$$

For
$$\beta_{pp} = \frac{m_p}{m_q}\beta_{pq},$$

also
$$\beta_{pp} = \frac{m_p}{m_1}\beta_{p1};$$

whence
$$\beta_{pq} = \frac{m_p}{m_1}\beta_{1q} = \frac{m_p}{m_1}\,\frac{m_q}{m_2}\beta_{12},$$

and, substituting these in equations (A), we can, if any a, as a_1, be given, find every other a; for instance,

$$2a_1m_2 - 2a_2m_1 + \beta_{12}(m_2 - m_1) = 0$$

determines a_2, and so on. So that, as in every other case, there are among the coefficients in Q two arbitrary ones, one a and one β, and the rest are determinate.

6. Let us now revert to the case of two sets of spheres of masses m and M respectively. Let there be in any space n spheres m, and N spheres M. The x velocity of their common centre of gravity is

$$\frac{\Sigma(MU+mu)}{NM+nm}.$$

The energy of this motion shall be $\frac{T_s}{3}$. Then

$$\frac{T_s}{3} = \frac{1}{2}\frac{\left[\Sigma(MU+mu)\right]^2}{NM+nm}.$$

The whole energy of the motion in x is

$$\tfrac{1}{2}(M\Sigma U^2 + m\Sigma u^2).$$

Similar expressions hold for y and z, v, V, &c. If therefore T_r be the kinetic energy of relative motion,

$$\begin{aligned}
T_r = T - T_s = {} & \tfrac{1}{2}\left[M\Sigma(U^2+V^2+W^2) + m\Sigma(u^2+v^2+w^2)\right] \\
& -\tfrac{1}{2}\frac{\left[\Sigma(MU+mu)^2 + (\Sigma\overline{MV+mv})^2 + (\Sigma\overline{MW+mw})^2\right]}{NM+nm} \\
= {} & \tfrac{1}{2}\Sigma\left(M - \frac{M^2}{NM+nm}\right)(U^2+V^2+W^2) \\
& + \tfrac{1}{2}\Sigma\left(m - \frac{m^2}{NM+nm}\right)(u^2+v^2+w^2) \\
& - \frac{M^2}{NM+nm}\Sigma\Sigma(U_pU_q + V_pV_q + W_pW_q) \\
& - \frac{m^2}{NM+nm}\Sigma\Sigma(u_pu_q + v_pv_q + w_pw_q) \\
& - \frac{Mm}{NM+nm}\Sigma\Sigma(Uu + Vv + Ww),
\end{aligned}$$

the last three terms including all products of the form indicated that exist in T_s.

If now we make $Q = T + \chi T_r$, where χ is any constant whatever, we shall find that this form of Q satisfies the condition of Case II. (p. 217).

For in this case let

$$\begin{aligned} T+\chi T_r = \; & A\Sigma\,(U^2+V^2+W^2)+a\Sigma\,(u^2+v^2+w^2) \\ & +B\Sigma\Sigma\,(UU'+VV'+WW') \\ & +b\Sigma\Sigma\,(uu'+vv'+ww') \\ & +\beta\Sigma\Sigma\,(Uu+Vv+Ww). \end{aligned}$$

Then we shall have

$$2A = M\,\overline{1+\chi}-\frac{M^2\chi}{NM+nm},$$

$$2a = m\,\overline{1+\chi}-\frac{m^2\chi}{NM+nm},$$

$$B = -\frac{M^2\chi}{NM+nm},$$

$$b = -\frac{m^2\chi}{NM+nm},$$

$$\beta = -\frac{Mm\chi}{NM+nm},$$

and these satisfy the conditions (p. 217).

7. The index Q in the form now presented is analogous to the $T+\kappa T_r$ of my first paper. We saw reason in that paper to attribute to κ a value $\frac{2}{3}\pi c^3\rho$, where ρ is the number of molecules (equal spheres) in unit volume, and $\frac{2}{3}\kappa T_r$ represents the increase of pressure arising when the molecules, from being material points, become spheres of finite diameter. It is not difficult to attribute to χ an analogous value for two sets of spheres, and, if that be done, I think that, at least for small values of χ—*e.g.*, if χ^2 but not χ be negligible—$Q = T+\chi T_r$ represents the true distribution of velocities in the permanent state.

8. *Case IV. Two sets of rigid elastic bodies of any kind.*—Let m be the mass, A, B, C the principal moments of inertia for a body of one set, m', A', B', C' for the other set; and suppose a collision to take place between an m and an m'. If we suppose for a moment m to be at rest, and to move from rest under the impulses given by the impact of m', the velocities with which m moves off would be as follows. Refer the system to axes instantaneously coinciding with

the principal axes of m. Let x, y, z be the coordinates of P, the point of contact at collision; λ, μ, ν the direction cosines of the normal at P. Let u be the velocity assumed by the centre of inertia of m (which is parallel to the normal at P), and ω_x, ω_y, ω_z the angular velocities assumed by m round its principal axes; also let

$$\nu y - \mu z = p,$$
$$\lambda z - \nu x = q,$$
$$\mu x - \lambda y = r.$$

Then, by known methods, we obtain the equations

$$A\omega_x - mpu = 0,$$
$$B\omega_y - mqu = 0,$$
$$C\omega_z - mru = 0.$$

If the motion of m be not from rest, let U, Ω_x, Ω_y, Ω_z be velocities before collision corresponding to u, ω_x, ω_y, ω_z after collision. Evidently the velocities of the centre of inertia of m in the common tangent plane are unchanged. Then we shall have for m

$$A(\omega_x - \Omega_x) - mp(u - U) = 0,$$
$$B(\omega_y - \Omega_y) - mq(u - U) = 0,$$
$$C(\omega_z - \Omega_z) - mr(u - U) = 0.$$

Similarly, for m',

$$A'(\omega'_x - \Omega'_x) - m'p'(u' - U') = 0,$$
$$B'(\omega'_y - \Omega'_y) - m'q'(u' - U') = 0,$$
$$C'(\omega'_z - \Omega'_z) - m'r'(u' - U') = 0.$$

We have then, in addition to four components of translation velocity in the tangent plane, the following seven linear functions of the velocities unchanged, namely,

and

$$\left.\begin{aligned}
mu + m'u' &= S_1 = mU + m'U' \\
mpu - A\omega_x &= S_2 = mpU - A\Omega_x \\
mqu - B\omega_y &= S_3 = mqU - B\Omega_y \\
mru - C\omega_z &= S_4 = mrU - C\Omega_z \\
m'p'u' - A'\omega'_x &= S_5 = m'p'U' - A'\Omega'_x \\
m'q'u' - B'\omega'_x &= S_6 = m'q'U' - B'\Omega'_y \\
m'r'u' - C'\omega'_z &= S_7 = m'r'U' - C'\Omega'_z
\end{aligned}\right\} \quad \text{..............(I.)},$$

the first of which expresses conservation of momentum.

[redacted]

Hence we have the system of seven linear equations

$$\left.\begin{array}{llll} m(u-U)+m'(u'-U') & & & =0 \\ mp(u-U) & -A(\omega_x-\Omega_x) & & =0 \\ mq(u-U) & & -B(\omega_y-\Omega_y) & =0 \\ \quad \&c. & & & \end{array}\right\} \ldots \text{(II.)}$$

among the eight quantities $u-U \ldots \omega_z'-\Omega_z'$. And to these comes, by conservation of energy, the eighth equation

$$m(u+U)(u-U)+m'(u'+U')(u'-U')+A(\omega_x+\Omega_x)(\omega_x-\Omega_x)+\&c.=0,$$

and we can now equate the determinant to zero in the form

$$\begin{vmatrix} m & m & & & & & & \\ mp & & -A & & & & & \\ mq & & & -B & & & & \\ mr & & & & -C & & & \\ & m'p' & & & & -A' & & \\ & m'q' & & & & & -B' & \\ & m'r' & & & & & & -C' \\ m\,\overline{u+U} & m'(u'+U') & A(\omega_x+\Omega_x) & \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} & C'(\omega_z'+\Omega_z') \end{vmatrix}=0,$$

and therefore, separating the accented from the unaccented terms in the last row,

$$R=D=\begin{vmatrix} m & m' & & & & & & \\ mp & & A & & & & & \\ mq & & & B & & & & \\ mr & & & & C & & & \\ & m'p' & & & & A' & & \\ & m'q' & & & & & B' & \\ & m'r' & & & & & & C' \\ mu & m'u' & A\omega_x & B\omega_y & C\omega_z & A'\omega_x' & B'\omega_y' & C'\omega_z' \end{vmatrix}.$$

If D_1 be the minor formed by omitting the first column and last row, D_2 that formed by omitting the second column and last row, and so on, then

$$R=muD_1+m'u'D_2+A\omega_xD_3+\&c.$$

Also, from the system

$$\begin{array}{llll} mu+m'u' & & & = S_1, \\ mpu & -A\omega_x & & = S_2, \\ mqu & & -B\omega_y & = S_3, \\ \quad \&c., & & & \\ muD_1+m'u'D_2+\&c. & & & = R, \end{array}$$

expressing u, &c., in terms of $S_1 \dots S_7$, R, we find

$$\frac{du}{dR}=\frac{D_1}{D},\quad \frac{du'}{dR}=\frac{D_2}{D},\quad \frac{d\omega^x}{dR}=\frac{D_3}{D},$$

and so on.

Now let $\quad Q = a\,(u^2+v^2+w^2)+a'\,(u'^2+v'^2+w'^2)+\beta\,(uu'+vv'+ww')$
$\qquad + a_\omega\,(A\omega_x^2+B\omega_y^2+C\omega_z^2)+a'_\omega\,(A'\omega_x'^2+B'\omega_y'^2+C'\omega_z'^2),$

and let it be required to find the relations which must subsist among the coefficients a, a', β, a_ω, a'_ω, in order that Q, when expressed as a function of $S_1 \dots S_7$, R, may contain R only in the second degree; that is, that $\dfrac{dQ}{dR}$ may vary as R for all values of u, u', &c. We have

$$\frac{dQ}{dR}=\frac{dQ}{du}\,\frac{du}{dR}+\frac{dQ}{du'}\,\frac{du'}{dR}+\&c.;$$

that is, $\qquad \dfrac{dQ}{dR}=(2au+\beta u')\,\dfrac{du}{dR}+(2a'u'+\beta u)\,\dfrac{du'}{dR},$

with similar terms for the v's and w's,

$$+2a_\omega\left(A\omega_x\frac{d\omega_x}{dR}+B\omega_y\frac{d\omega_y}{dR}+C\omega_z\frac{d\omega_z}{dR}\right)$$
$$+2a'_\omega\left(A'\omega'_x\frac{d\omega'_x}{dR}+B'\omega'_y\frac{d\omega'_y}{dR}+C'\omega'_z\frac{d\omega'_z}{dR}\right),$$

and $\dfrac{du}{dR}$ is found from the linear equation expressing u in terms of R and $S_1 \dots S_7$, and so on, that is $\dfrac{du}{dR}=\dfrac{D_1}{D}$, &c.,

whence $\qquad D\dfrac{dQ}{dR}=(2au+\beta u')\,D_1+(2a'u'+\beta u)\,D_2+\&c.$

$$+2a_\omega\,(A\omega_xD_3+B\omega_yD_4+C\omega_zD_5)$$
$$+2a'_\omega\,(A'\omega'_xD_6+B'\omega_yD_7+C'\omega'_zD_8).$$

But evidently, by inspection of the determinant,

$$mD_1=-m'D_2,\quad \text{or}\quad D_2=-\frac{m}{m'}D_1,$$

so $\qquad D\dfrac{dQ}{dR}=\left(2a-\dfrac{m}{m'}\beta\right)uD_1+\left(2a'-\dfrac{m'}{m}\beta\right)u'D_2+\&c.$

$$+2a_\omega\,(A\omega_xD_3+B\omega_yD_4+C\omega_zD_5)$$
$$+2a'_\omega\,(A'\omega'_xD_6+B'\omega'_yD_7+C'\omega'_zD_8).$$

Also, as we have seen,

$$DR = muD_1 + m'u'D_2 + A\omega_x D_3 + \&c.$$

Comparing coefficients of uD_1, $u'D_2$, &c., in $\frac{dQ}{dR}$ and R, our condition requires that

$$2a - \frac{m}{m'}\beta : m :: 2a' - \frac{m'}{m}\beta : m',$$

or $$2am' - m\beta = 2a'm - m'\beta.$$

Also that $$2a - \frac{m}{m'}\beta : m :: 2a_\omega : 1,$$

or $$2am' - m\beta = 2a_\omega mm',$$

and similarly $$2am' - m\beta = 2a'_\omega mm'.$$

Hence our result is

$$2am' - m\beta = 2a'm - m'\beta = 2a_\omega mm' = 2a'_\omega mm'$$

or $$a_\omega = a'_\omega \quad\text{..............(1)},$$

$$2am' - 2a'm + \beta(m' - m) = 0 \quad\text{..............(2)},$$

and $$\left.\begin{aligned} 2a_\omega mm' - 2a'm + m'\beta &= 0 \\ 2a_\omega mm' - 2am' + m\beta &= 0 \end{aligned}\right\} \quad\text{..............(3).}$$
or

So that among the five coefficients a, a', β, a_ω, a'_ω, two only, namely, β and one other, are arbitrary.

9. I have applied the proof only to rigid elastic bodies; but it would apply equally well to any material systems, provided that the momenta before encounter are connected with the momenta after encounter by linear equations of the form I. and II. of Article 8.

10. If β be negative and $m = m'$, we see that $a_\omega > a$ or a', and therefore the mean energy of rotation is less than that of translation. This is an unavoidable result of the existence of β, that is, of the translation velocities being *correlated.* If it can be shown, as I think it can, or if it be true, that in a dense medium they must be correlated, then the supposed law of equality of kinetic energy for each degree of freedom is nothing more than a special property of a very special system—that, namely, of the infinitely rare gas. If, on the other hand, β has no existence however dense the medium, then writers on the kinetic theory have, as it seems to me, unnecessarily restricted the generality of their own theory.

Thursday, February 13th, 1896.

Prof. M. J. M. HILL, F.R.S., Vice-President, in the Chair.

Miss Chisholm was admitted into the Society.

The Chairman read the opening paragraphs of a paper entitled "Geodesics on Quadrics, not of Revolution," by Prof. Forsyth.

Prof. Elliott gave an account of a paper, by Mr. A. L. Dixon, on "The Potential of Cyclides."

Mr. Love communicated a paper on "Solid Ellipsoidal Vortex," by Mr. R. Hargreaves.

Dr. J. Larmor and Lt.-Col. Cunningham took part in a discussion on the last paper.

Prof. Hill (Mr. Jenkins, Vice-President, *pro tem.* in the Chair) and Mr. Tucker made short impromptu communications.

The following presents to the Library were received:—

"Proceedings of the Royal Society," Vol. LIX., No. 353.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 12; Bd. XX., St. 1; Leipzig, 1895-96.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg," Bd. III., Heft 6; 1896.

"Nyt Tidsskrift for Matematik," A., Aarg. VII., Nos. 5-7; Copenhagen, 1895.

"Proceedings of the Physical Society of London," Vol. XIII., Pt. 13, No. 63, December, 1895; Vol. XIV., Pt. 1, No. 64, January, 1896.

"Revue Semestrielle des Publications Mathématiques," Tome IV., Pt. 1; Amsterdam, 1896.

"Wiskundige Opgaven met de Oplossingen," N. Reeks, Deel VI.; Amsterdam, 1894-6.

"Monatshefte für Mathematik und Physik," Jahrgang XI., 1895, Hefte 1-12; Wien, 1895.

"Bulletin de la Société Mathématique de France," Tome XXIII., Nos. 9, 10; Paris.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 4; New York, 1896.

"Tokyo Mathematical-Physical Society," Maki No. 7, Dai 3.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. I., Fasc. 12; Napoli, 1895.

"Annali di Matematica," Serie II., Tome XXIV., Fasc. 1; Milano.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. IV., 1895, Fasc. 12, Sem. 2; Vol. V., 1896, Fasc. 1, 2; Roma.

"Educational Times," February, 1896.

"Transactions of the Royal Irish Academy," Vol. xxx., Pts. 15–17; Dublin, 1895.

"Proceedings of the Royal Irish Academy," Vol. III., No. 4; December, 1895, Dublin.

"Acta Mathematica," xx., 1; Stockholm, 1896.

"Journal für die reine und angewandte Mathematik," Bd. cxvi., Heft 1; Berlin, 1896.

"Annals of Mathematics," Vol. IX., Nos. 4, 5, May–July, 1895; University of Virginia.

"Indian Engineering," Vol. XVIII., Nos. 25, 26; Vol. XIX., Nos. 1–3; December 21, 1895, to January 18, 1896.

"Euclid's Elements of Geometry," by H. M. Taylor, Books I.–VI., XI., XII., 8vo; Cambridge, 1895.

"Catalogue of Scientific Papers (1874–1883) compiled by the Royal Society of London," Pet-Zyb, Vol. XI., 4to; London, 1896.

Cayley, A.—"Collected Mathematical Papers," Vol. IX., 4to; Cambridge, 1896.

"On the Numerical Factors of a^n-1," by C. E. Bickmore. From the author. Offprint from "Messenger of Mathematics," Vol. xxv., pp. 1–44.

"American Journal of Mathematics," Vol. XVIII., No. 1; Baltimore, 1896.

The Potential of Cyclides. *By* A. L. Dixon.

Received and Read February 13th, 1896.

In this paper I propose to find an expression for the potential of certain classes of cyclides, both as solids and shells, and in particular to consider the case of the anchor ring.

The first part contains the necessary analysis, and the rest the application to cyclides, which are throughout considered as being given by equations in pentaspherical coordinates, in which form they have been investigated by Darboux and Casey.

The expressions found have a marked analogy with the usual form of the potential of an ellipsoid, viz.,

$$\int_{\lambda}^{\infty} \frac{d\lambda}{\sqrt{(a+\lambda)(b+\lambda)(c+\lambda)}} \left(\frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} - 1 \right),$$

of which, indeed, they may be regarded as a generalization.

The method of proof for the case of a solid is to find two functions V_0 and V_ι, such that

$$\nabla^2 V_0 = 0, \quad \text{and} \quad \nabla^2 V_\iota = -4\pi\rho,$$

and such that at the surface of the solid $V_0 - V_\iota$ and its first differential coefficients vanish; with corresponding modifications in the case of a shell.

I have added as an appendix a direct proof of a theorem analogous to the well known one "that, two confocal homœoids of equal mass being given, the attraction of one at any point of the other is equal to the attraction of the other at the corresponding point of the one."

Part I.

1. Take any number of variables x_1, x_2, x_3, &c., and write ∇^2 for the operator

$$\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} + \frac{d^2}{dx_3^2} + \ldots,$$

S for the function

$$C + \frac{x_1^2}{a_1+\lambda} + \frac{x_2^2}{a_2+\lambda} + \frac{x_3^2}{a_3+\lambda} + \ldots,$$

where C is any constant; P for the product $(a_1+\lambda)(a_2+\lambda)\ldots$, F for

$$F\left(\frac{x_1}{a_1+\lambda},\ \frac{x_2}{a_2+\lambda},\ \frac{x_3}{a_3+\lambda},\ \ldots\right).$$

We have obviously
$$\frac{dS}{d\lambda} = -\Sigma\,\frac{x^2}{(a+\lambda)^2},$$

$$\frac{dF}{d\lambda} = \Sigma\,\frac{dF}{dx}(a+\lambda)\,\frac{-x}{(a+\lambda)^2}$$

$$= -\Sigma\,\frac{x}{a+\lambda}\,\frac{dF}{dx}.$$

Then
$$\nabla^2\,\frac{S^r}{\sqrt{P}} = \Sigma\,\frac{d^2}{dx^2}\,\frac{S^r}{\sqrt{P}}$$

$$= \Sigma\,\frac{4r}{\sqrt{P}}\left\{(r-1)\,S^{r-2}\,\frac{x^2}{(a+\lambda)^2} + \tfrac{1}{2}S^{r-1}\,\frac{1}{a+\lambda}\right\}$$

$$= -4r\,\frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}\right),$$

$$\nabla^2\left(\frac{S^r}{\sqrt{P}}\,F\right) = \frac{S^r}{\sqrt{P}}\,\nabla^2 F + 2\,.\,\Sigma\,\frac{d}{dx}\left(\frac{S^r}{\sqrt{P}}\right)\frac{dF}{dx} + F\,\nabla^2\,\frac{S^r}{\sqrt{P}}.$$

Now
$$\Sigma \frac{d}{dx}\left(\frac{S^r}{\sqrt{P}}\right)\frac{dF}{dx} = \Sigma\, 2r \frac{S^{r-1}}{\sqrt{P}}\, \frac{x}{a+\lambda}\, \frac{dF}{dx}$$
$$= -2r \frac{S^{r-1}}{\sqrt{P}}\, \frac{dF}{d\lambda};$$

and therefore
$$\nabla^2\left(\frac{S^r}{\sqrt{P}}F\right) = \frac{S^r}{\sqrt{P}}\nabla^2 F - 4r\frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}F\right),$$

and multiplying by any function of λ, $f(\lambda)$, and integrating,
$$\nabla^2 \int \frac{S^r}{\sqrt{P}} F . f(\lambda)\, d\lambda = \int \frac{S^r}{\sqrt{P}} \nabla^2 F . f(\lambda)\, d\lambda - 4r \int f(\lambda) \frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}F\right) d\lambda.$$

Integrating the last term by parts, we get
$$\nabla^2 \int \frac{S^r}{\sqrt{P}} F . f(\lambda)\, d\lambda$$
$$= \int \frac{S^r}{\sqrt{P}} \nabla^2 F . f(\lambda)\, d\lambda + 4r \int f'(\lambda) \frac{S^{r-1}}{\sqrt{P}} F .\, d\lambda - 4r . f(\lambda) \frac{S^{r-1}}{\sqrt{P}} F.$$

Also putting $(a_1+\lambda)\dfrac{d^2F}{dx_1^2}$ for F—which may be done since the only condition for F is that it be a function of $\dfrac{x_1}{a_1+\lambda}$, &c.—and putting
$$f'(\lambda) = \frac{1}{(a_1+\lambda)^2}$$
—and therefore
$$f(\lambda) = \frac{\lambda - c}{(a_1+c)(a_1+\lambda)},$$
where c is any constant—we get
$$\nabla^2 \int \frac{S^r}{\sqrt{P}}\, \frac{(\lambda-c)(a_1+\lambda)}{a_1+c}\, \frac{d^2F}{dx_1^2} d\lambda$$
$$= \int \frac{S^r}{\sqrt{P}} \nabla^2 \frac{(\lambda-c)(a_1+\lambda)}{a_1+c}\, \frac{d^2F}{dx_1^2} d\lambda + 4r \int \frac{S^{r-1}}{\sqrt{P}}\, \frac{d^2F}{dx_1^2} d\lambda$$
$$-4r \frac{S^{r-1}}{\sqrt{P}}\, \frac{(\lambda-c)(a_1+\lambda)}{a_1+c}\, \frac{d^2F}{dx_1^2}.$$

Similar equations with the other variables written for x_1 can be obtained, and by addition we shall get, putting
$$\delta \equiv \Sigma \frac{(\lambda-c)(a+\lambda)}{a+c}\, \frac{d^2}{dx^2},$$
$$\nabla^2 \int \frac{S^r}{\sqrt{P}} \delta F .\, d\lambda = \int \frac{S^r}{\sqrt{P}} \nabla^2 \delta F .\, d\lambda + 4r \int \frac{S^{r-1}}{\sqrt{P}} \nabla^2 F .\, d\lambda - 4r \frac{S^{r-1}}{\sqrt{P}} \delta F.$$

2. Now let us see what would be the effect of writing $\delta^n F$ instead of F in our original expression, that is, let us find $\nabla^2\left(\frac{S^r}{\sqrt{P}}\delta^n F\right)$. To do this we must evaluate $\Sigma \frac{d}{dx}\frac{S^r}{\sqrt{P}}\frac{d\delta^n F}{dx}$, that is

$$2r\frac{S^{r-1}}{\sqrt{P}}\Sigma\frac{x}{a+\lambda}\frac{d\delta^n F}{dx}.$$

Now $\left(\text{putting } D_1 \text{ for } \frac{d}{dx_1}\right)$

$$\delta^n(x_1 D_1 F) = \left\{\frac{(\lambda-c)(a_1+\lambda)}{a_1+c}D_1^2+\dots\right\}^n(x_1 D_1 F)$$

$$= x_1\delta^n D_1 F + \frac{d\delta^n}{dD_1}D_1 F$$

$$= x_1\frac{d}{dx_1}(\delta^n F) + 2\frac{(\lambda-c)(a_1+\lambda)}{a_1+c}n\,.\,\delta^{n-1}\frac{d^2F}{dx_1^2};$$

and therefore

$$\Sigma\frac{x}{a+\lambda}\frac{d}{dx}(\delta^n F) = \delta^n\left(\Sigma\frac{x}{a+\lambda}\frac{dF}{dx}\right) - n\delta^{n-1}\Sigma\left\{\frac{2(\lambda-c)}{a_1+c}\frac{d^2F}{dx^2}\right\}.$$

Now $$\frac{d\delta}{d\lambda} = \Sigma\frac{2(\lambda-c)}{a+c}\frac{d^2}{dx^2}+\Sigma\frac{d^2}{dx^2},$$

and we get therefore

$$\Sigma\frac{x}{a+\lambda}\frac{d}{dx}(\delta^n F) = -\delta^n\left(\frac{dF}{d\lambda}\right) - n\delta^{n-1}\left(\frac{d\delta}{d\lambda}-\nabla^2\right)F$$

$$= -\frac{d}{d\lambda}(\delta^n F) + n\,.\,\nabla^2\delta^{n-1}F.$$

We have then

$$\nabla^2\left(\frac{S^r}{\sqrt{P}}\delta^n F\right)$$

$$= \frac{S^r}{\sqrt{P}}\nabla^2\delta^n F + 4r\frac{S^{r-1}}{\sqrt{P}}\left(-\frac{d}{d\lambda}\delta^n F + n\nabla^2\delta^{n-1}F\right) - 4r\,.\,\delta^n F\frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}\right)$$

$$= \frac{S^r}{\sqrt{P}}\nabla^2\delta^n F + 4rn\frac{S^{r-1}}{\sqrt{P}}\nabla^2\delta^{n-1}F - 4r\frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}\delta^n F\right),$$

[redacted]

and we may write down therefore the following series of equations

$$\nabla^2\left(\frac{S^r}{\sqrt{P}}F\right) = \frac{S^r}{\sqrt{P}}\nabla^2 F \qquad -4r\frac{d}{d\lambda}\left(\frac{S^{r-1}}{\sqrt{P}}F\right),$$

$$\nabla^2\left(\frac{S^{r+1}}{\sqrt{P}}\delta F\right) = \frac{S^{r+1}}{\sqrt{P}}\nabla^2\delta F + 4(r+1)\frac{S^r}{\sqrt{P}}\nabla^2 F - 4(r+1)\frac{d}{d\lambda}\left(\frac{S^r}{\sqrt{P}}\delta F\right),$$

$$\nabla^2\left(\frac{S^{r+2}}{\sqrt{P}}\delta^2 F\right) = \frac{S^{r+2}}{\sqrt{P}}\nabla^2\delta^2 F + 4(r+2)\,2\frac{S^{r+1}}{\sqrt{P}}\nabla^2\delta F - 4(r+2)\frac{d}{d\lambda}\left(\frac{S^{r+1}}{\sqrt{P}}\delta^2 F\right),$$

$$\nabla^2\left(\frac{S^{r+3}}{\sqrt{P}}\delta^3 F\right) = \frac{S^{r+3}}{\sqrt{P}}\nabla^2\delta^3 F + 4(r+3)\,3\frac{S^{r+2}}{\sqrt{P}}\nabla^2\delta^2 F - 4(r+3)\frac{d}{d\lambda}\left(\frac{S^{r+2}}{\sqrt{P}}\delta^3 F\right),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\nabla^2\left(\frac{S^{r+n}}{\sqrt{P}}\delta^n F\right) = \frac{S^{r+n}}{\sqrt{P}}\nabla^2\delta^n F + 4(r+n)\,n\frac{S^{r+n-1}}{\sqrt{P}}\nabla^2\delta^{n-1} F - 4(r+n)\frac{d}{d\lambda}\left(\frac{S^{r+n-1}}{\sqrt{P}}\delta^n F\right);$$

and therefore

$$\nabla^2\frac{1}{\sqrt{P}}\left\{S^r - \frac{S^{r+1}}{1(r+1)}\frac{\delta}{4} + \frac{S^{r+2}}{1.2(r+1)(r+2)}\frac{\delta^2}{4^2} - \dots \right.$$

$$\left. \dots \cdots \frac{(-)^n S^{r+n}}{n!\,(r+1)(r+2)\dots(r+n)}\frac{\delta^n}{4^n}\dots\right\} F^*$$

$$= -4r\frac{d}{d\lambda}\left\{\frac{1}{\sqrt{P}}\left(S^{r-1} - \frac{S^r}{1.r}\frac{\delta}{4} + \frac{S^{r+1}}{1.2.r(r+1)}\frac{\delta^2}{4^2}\dots\right)F\right\}.$$

3. Now let us take F as some power of a linear function of $\frac{x_1}{a_1+\lambda}$, $\frac{x_2}{a_2+\lambda}$, ..., *i.e.*, put $F = T^{-m}$, where

$$T = \frac{p_1 x_1}{a_1+\lambda} + \frac{p_2 x_2}{a_2+\lambda} + \dots,$$

also put
$$\theta = \frac{p_1^2}{a_1(a_1+\lambda)} + \frac{p_2^2}{a_2(a_2+\lambda)} + \dots,$$

and take $c = 0$, so that

$$\delta \equiv \Sigma\frac{\lambda(a+\lambda)}{a}\frac{d^2}{dx^2}.$$

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* The series on the left-hand side of this equation has been already given by Mr. Dyson in his paper on "Potentials of Ellipsoids," *Quarterly Journal*, xxv.

Then
$$\delta T^n = n\,(n-1)\,T^{n-2}\,.\,\Sigma\,\frac{\lambda\,.\,p^2}{a\,(a+\lambda)}$$
$$= n\,(n-1)\,T^{n-2}\,.\,\lambda\theta,$$

and so the preceding formula becomes

$$\nabla^2\,\frac{S^r}{T^m\sqrt{P}}\left\{1-\frac{m\,(m+1)}{1\,(r+1)}\,\frac{\lambda\theta S}{4T^2}+\frac{m\,(m+1)(m+2)(m+3)}{1\,.\,2\,(r+1)(r+2)}\left(\frac{\lambda\theta S}{4T^2}\right)^2\right.$$
$$\left.-\frac{m\,(m+1)(m+2)(m+3)(m+4)(m+5)}{1\,.\,2\,.\,3\,(r+1)(r+2)(r+3)}\left(\frac{\lambda\theta S}{4T^2}\right)^3+\dots\right\}$$
$$=-4r\,\frac{d}{d\lambda}\left\{\frac{S^{r-1}}{T^m\sqrt{P}}\left[1-\frac{m\,(m+1)}{1\,.\,r}\,\frac{\lambda\theta S}{4T^2}\right.\right.$$
$$\left.\left.+\frac{m\,(m+1)(m+2)(m+3)}{1\,.\,2\,.\,r\,(r+1)}\left(\frac{\lambda\theta S}{4T^2}\right)^2-\dots\right]\right\}.$$

4. Now $\dfrac{1}{\pi}\displaystyle\int_0^\pi \frac{\sin^{2r}\phi\,d\phi}{\{1-x\cos\phi\}^m}$ (r being integral), when expanded and integrated term by term, becomes the series

$$\frac{1\,.\,3\,.\,5\,\dots\,(2r-1)}{2\,.\,4\,.\,6\,\dots\,2r}\left\{1+\frac{m\,(m+1)}{2!}\,\frac{x^2}{2r+2}\right.$$
$$+\frac{m\,(m+1)(m+2)(m+3)}{4!}\,\frac{1\,.\,3\,x^4}{(2r+2)(2r+4)}$$
$$\left.+\frac{m(m+1)(m+2)(m+3)(m+5)(m+6)}{6!}\,\frac{1\,.\,3\,.\,5\,x^6}{(2r+2)(2r+4)(2r+6)}\dots\right\},$$

i.e.,

$$\frac{1.3.5\dots(2r-1)}{2.4.6\dots2r}\left\{1+\frac{m\,(m+1)}{1\,(r+1)}\,\frac{x^2}{4}+\frac{m\,(m+1)(m+2)(m+3)}{1.2\,(r+1)(r+2)}\,\frac{x^4}{4^2}+\dots\right\},$$

and so the preceding formula becomes

$$\nabla^2\int_0^\pi\frac{1}{\sqrt{P}}\,\frac{S^r\sin^{2r}\phi\,d\phi}{\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^m}$$
$$=-(4r-2)\,\frac{d}{d\lambda}\int_0^\pi\frac{1}{\sqrt{P}}\,\frac{S^{r-1}\sin^{2r-2}\phi\,d\phi}{\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^m}\quad\dots\dots\dots\dots\text{(A)}.$$

The verification of this result by the actual calculation of each expression is, of course, necessary, and can be effected without much difficulty.

5. Write this equation, for shortness,

$$\nabla^2 Q = -\frac{dR}{d\lambda},$$

and let us consider the value of $\nabla^2 \int_\lambda^\beta Q\,d\lambda$, where $R_\beta = 0$ and λ is a root of the equation $S = 0$. Assuming $r \geqq 1$, we have $Q_\lambda = 0$, and, therefore, we get

$$\nabla^2 \int_\lambda^\beta Q\,d\lambda = R_\lambda - \Sigma \frac{dQ_\lambda}{dx}\,\frac{d\lambda}{dx}$$

$$= 0, \quad \text{when} \quad r > 1;$$

when $r = 1$,

$$R_\lambda = \frac{2\pi}{\sqrt{P_\lambda T_\lambda^m}},$$

$$\Sigma \frac{dQ_\lambda}{dx}\,\frac{d\lambda}{dx} = \int_0^\pi \frac{\Sigma \dfrac{dS}{dx}\,\dfrac{\dfrac{2x}{a+\lambda}}{\Sigma \dfrac{x^2}{(a+\lambda)^2}}}{\sqrt{P_\lambda T_\lambda^m}} \sin^2\phi\,d\phi$$

$$= \frac{2\pi}{\sqrt{P_\lambda T_\lambda^m}};$$

and therefore $$\nabla^2 \int_\lambda^\beta Q\,d\lambda = 0.$$

Also, putting 0 for the lower limit,

$$\nabla^2 \int_0^\beta Q\,d\lambda = -R_0 = (4r-2)\,\frac{S_0^{r-1}}{\sqrt{P_0 T_0^m}} \int_0^\pi \sin^{2r-2}\phi\,d\phi$$

$$= -\frac{S_0^{r-1}}{\sqrt{P_0 T_0^m}}\,\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots(2r-2)}\,2\pi.$$

Further, the first differential coefficients of $\int_\lambda^\beta Q\,d\lambda$ with regard to any x are the same as those of $\int_0^\beta Q\,d\lambda$, when $\lambda = 0$.

6. The case when $r = 0$ must be investigated separately. We find in the same way that

$$\nabla^2 \int_0^\pi \frac{d\phi}{\sqrt{P}\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^m}$$

$$= 2m(m+1)\frac{d}{d\lambda}\int_0^\pi \frac{\lambda\theta\sin^2\phi\, d\phi}{\sqrt{P}\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^{m+2}} \quad \text{...(B)}$$

Writing this as before,

$$\nabla^2 Q = 2m(m+1)\frac{dR}{d\lambda},$$

we have
$$\nabla^2 \int_0^\rho Q\, d\lambda = 0,$$

and we may prove that
$$\nabla^2 \int_\lambda^\rho Q\, d\lambda = 0.$$

Further, when $\lambda = 0$, we have

$$\int_\lambda^\rho Q\, d\lambda = \int_0^\rho Q\, d\lambda;$$

but
$$\Sigma\left(\frac{d}{dx}\int_\lambda^\rho Q\, d\lambda - \frac{d}{dx}\int_0^\rho Q\, d\lambda\right)^2 = \Sigma\left(Q_\lambda \frac{d\lambda}{dx}\right)^2_{\lambda=0} = \left(\frac{2\pi\rho}{\sqrt{P_0}\,T_0^m}\right)^2,$$

where
$$\frac{1}{\rho^2} = \Sigma\left(\frac{x}{a}\right)^2.$$

Part II.

7. The equation to a system of confocal cyclides can be written

$$\frac{X^2}{A+\lambda}+\frac{Y^2}{B+\lambda}+\frac{Z^2}{C+\lambda}+\frac{U^2}{D+\lambda}+\frac{V^2}{E+\lambda} = 0 \quad \text{.........(I.)}$$

where X, Y, Z, U, V, are Darboux's pentaspherical coordinates, viz. the powers of a point with regard to a set of five mutually orthogonal spheres, divided by the corresponding radius, and the two following identical relations hold:—

$$X^2+Y^2+Z^2+U^2+V^2 = 0 \quad \text{......................(1)}$$

and $$\frac{X}{r_1}+\frac{Y}{r_2}+\frac{Z}{r_3}+\frac{U}{r_4}+\frac{V}{r_5}=-2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, in addition, $$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

By means of the relation (1) we can transform

$$\frac{X^2}{A+\lambda}+\frac{Y^2}{B+\lambda}+\frac{Z^2}{C+\lambda}+\frac{U^2}{D+\lambda}+\frac{V^2}{E+\lambda},$$

which we will call S, as follows:—

$$S(E+\lambda)=X^2\left(\frac{E+\lambda}{A+\lambda}-1\right)+\ldots+U^2\left(\frac{E+\lambda}{A+\lambda}-1\right)$$

$$=X^2\frac{E-A}{A+\lambda}+\ldots,$$

$$S(E+\lambda)^2=X^2\frac{(E-A)(E+\lambda)}{A+\lambda}+\ldots$$

$$=\frac{X^2}{\dfrac{1}{E-A}-\dfrac{1}{E+\lambda}}$$

$$=\frac{X^2}{a+\lambda'}+\frac{Y^2}{b+\lambda'}+\frac{Z^2}{c+\lambda'}+\frac{U^2}{d+\lambda'}.$$

That is to say, we can, if convenient, take

$$\frac{X^2}{a+\lambda}+\frac{Y^2}{b+\lambda}+\frac{Z^2}{c+\lambda}+\frac{U^2}{d+\lambda}=0 \quad\ldots\ldots\ldots\ldots\ldots(\text{II.})$$

as our standard form instead of (I.).

In general, when one or more of the variables are absent from T, it will be simpler to use (II.), and so reduce the function to one involving not more than four variables.

In cases where a coordinate V is present in T but absent in S, we take the corresponding parameter e to be infinite, and put

$$T=\frac{p_1X}{a+\lambda}+\frac{p_2Y}{b+\lambda}+\ldots+p_5V,$$

$$\theta=\frac{p_1^2}{a(a+\lambda)}+\ldots+p_5^2.$$

8. It is easy to express Laplace's equation in terms of this system of coordinates. If ϕ be a homogeneous function of degree n in penta-spherical coordinates, we get, by direct transformation from rectangular Cartesian coordinates (x, y, z),

$$\tfrac{1}{4}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right) = \Sigma\,\frac{d^2\phi}{dX^2} + (n+\tfrac{1}{2})\,\Sigma\,\frac{1}{r}\,\frac{d\phi}{dX},$$

and in particular, if $n = -\frac{1}{2}$, Laplace's equation becomes

$$\frac{d^2\phi}{dX^2}+\frac{d^2\phi}{dY^2}+\frac{d^2\phi}{dZ^2}+\frac{d^2\phi}{dU^2}+\frac{d^2\phi}{dV^2} = 0.*$$

We also get, without difficulty,

$$\tfrac{1}{4}\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right\} = \Sigma\left(\frac{d\phi}{dX}\right)^2 + n\phi\,\Sigma\,\frac{1}{r}\,\frac{d\phi}{dX}\ \ldots\ldots(\text{C}),$$

and $$dx^2+dy^2+dz^2 = \tfrac{1}{4}\,\Sigma\,dX^2.$$

9. We see from this that we must put $m = 2r+\frac{1}{2}$ in the expression $\int_\lambda^\beta Q\,d\lambda$ found in § 4, and that, if β can be determined, and $\int_\lambda^\beta Q\,d\lambda$ vanish for points at an infinite distance and be continuous, then $\int_\lambda^\beta Q\,d\lambda$ will be the potential for any point outside, and $\int_0^\beta Q\,d\lambda$ for any point inside, a solid cyclide whose density at any point is

$$\frac{S_0^{r-1}}{\sqrt{P_0\,T_0^m}}\;\frac{1\,.\,3\,.\,5\,\ldots\,(2r-1)}{2\,.\,4\,.\,6\,\ldots\,(2r-2)}\cdot\frac{1}{2}.$$

10. For a thin cyclidal shell we take $\int_\lambda^\beta Q\,d\lambda$ from § 6, and now $\int_\lambda^\beta Q\,d\lambda$ will be the potential for a point outside, and $\int_0^\beta Q\,d\lambda$ for a point inside, a cyclidal shell of surface density $\dfrac{\rho}{\sqrt{P_0\,T_0^m}}$, where

$$\rho^{-2} = \Sigma\left(\frac{X}{a}\right)^2;$$

for the density will be equal to $\dfrac{1}{4\pi}\dfrac{d}{dn}(V_i - V_0)$, where V_0 is the outside

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* *Cf.* Darboux, *Comptes Rendus*, 1876, II., p. 1037.

potential, and V_i the inside potential, for a point on the surface, and dn is an element of length in a direction normal to the surface at that point, and therefore (putting ϕ for $V_i - V_o$)

$$\left(\frac{d\phi}{dn}\right)^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2$$

$$= 4\Sigma\left(\frac{d\phi}{dX}\right)^2, \text{ since } \phi = 0, \text{ by (C), §8.}$$

11. Let us next consider the determination of β from the equation $R_\beta = 0$, that is,

$$\int_0^\pi \frac{S^{r-1}\sin^{2r-2}\phi\, d\phi}{\sqrt{P}\left\{T-(-\lambda\theta S)^{\frac12}\cos\phi\right\}^{2r-\frac32}} = 0 \quad (\lambda = \beta).$$

If we take $\beta = \infty$, then T is of the same order as $\dfrac{1}{\beta}$, if T does not contain V, $\lambda\theta$ is the same as 1, and S as $\dfrac{1}{\beta}$ (S being in the standard form (II.) with only four terms), and therefore the whole expression is of the same order as $\dfrac{\beta^{\frac12}}{\sqrt{P}}$, which vanishes, since $\sqrt{P}$ is of order $\beta^{\frac32}$ at least.

Similar considerations will show that we may take $\beta = \infty$, when T contains all five coordinates, whether S is in form I. or II.

12. Further, the expression we have found is of degree $-\frac12$ in our coordinates, and in general, therefore, will reduce for points at a very great distance to the form $\dfrac{C}{r}$, where C is some constant, and we see that, when C has been found, our expression, when multiplied by $\dfrac{M}{C}$, where M is the mass of the attracting body, will give the actual potential due to the body of mass M.

13. It is very often convenient to take the special system of coordinates in which three of the spheres of reference become the three coordinate planes of the ordinary rectangular Cartesian system, so that we put $X = 2x$, $Y = 2y$, $Z = 2z$, and

$$\iota r U = x^2+y^2+z^2+r^2, \quad rV = x^2+y^2+z^2+r^2,$$

and relation (2) of § 7 becomes

$$\frac{U}{\iota r}+\frac{V}{r}=-2.$$

Any general system can be inverted into this special system, and, since the potential of a body is known when that of the inverse body is known, there would be no loss of generality in taking this as the standard system.

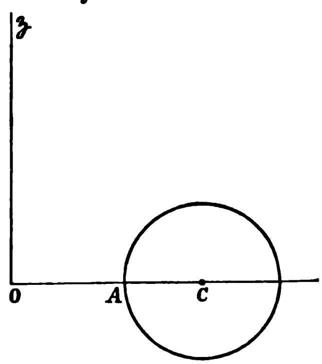
U is a pure imaginary quantity in all cases when X, Y, Z, V are taken as real.

14. As an interesting and important example let us consider the case of an anchor ring or tore of uniform density.

The Cartesian equation of an anchor ring is

$$(\sqrt{x^2+y^2}-c)^2+z^2=a^2,$$

where a is the radius of the circular plane section made by any plane through the axis of z (CA in the figure), and c is the radius of the circular axis of the ring in the plane of xy. This becomes



$$x^2+y^2+z^2+c^2-a^2=2c\sqrt{x^2+y^2},$$

or
$$(r^2+c^2-a^2)^2=4c^2(x^2+y^2),$$

or putting
$$U=\frac{r^2+c^2-a^2}{\iota\sqrt{c^2-a^2}},\quad V=\frac{r^2-c^2+a^2}{\sqrt{c^2-a^2}},$$

$$X=2x,\quad Y=2y,\quad Z=2z,$$

as in the last paragraph,

$$U^2(c^2-a^2)+c^2(X^2+Y^2)=0;$$

using the identical relation, this becomes

$$U^2a^2+c^2(V^2+Z^2)=0,$$

and the system of confocals is

$$\frac{U^2}{c^2+\lambda}+\frac{V^2+Z^2}{a^2+\lambda}=0.$$

15. Now, taking

$$-T = \frac{c^2}{c^2+\lambda}\,\frac{U}{\iota\sqrt{c^2-a^2}} + \frac{a^2}{a^2+\lambda}\,\frac{V}{\sqrt{c^2-a^2}}$$

$$= \frac{-2c^2(a^2+\lambda)-(r^2-c^2+a^2)\,\lambda}{(c^2+\lambda)(a^2+\lambda)}.$$

$$T = \frac{2a^2c^2+\lambda\,(r^2+a^2+c^2)}{(c^2+\lambda)(a^2+\lambda)},$$

so that $T_0 = 2$, and the density is $\dfrac{1}{8\sqrt{2}\,ca^2}$, and

$$\theta = \frac{-\lambda}{(c^2+\lambda)(a^2+\lambda)},$$

and also putting $\quad S = \dfrac{U^2}{c^2+\lambda} + \dfrac{V^2+Z^2}{a^2+\lambda}$

$$= -\frac{(X^2+Y^2)(\lambda-\lambda')}{(c^2+\lambda)(a^2+\lambda)}$$

$$= -\frac{a^2U^2+c^2\,(V^2+Z^2)}{(c^2+\lambda)(a^2+\lambda)}\,\frac{\lambda-\lambda'}{\lambda'},$$

we have the following expression for the potential at an external point:—

$$\int_{\lambda'}^{\infty}\int_0^{\pi} \frac{-(c^2+\lambda)(a^2+\lambda)^{\frac{1}{2}}\,(x^2+y^2)(\lambda-\lambda')\sin^2\phi\,d\lambda\,d\phi}{\left[\lambda\,(r^2+a^2+c^2)+2a^2c^2-2\iota\lambda\left\{(x^2+y^2)(\lambda-\lambda')\right\}^{\frac{1}{2}}\cos\phi\right]^{\frac{3}{2}}},$$

where $\quad \lambda' = \dfrac{a^2U^2+c^2\,(V^2+Z^2)}{X^2+Y^2}$

$$= \frac{(r^2+c^2-a^2)^2-4c^2\,(x^2+y^2)}{4\,(x^2+y^2)}.$$

For an internal point we have

$$\int_0^{\infty}\int_0^{\pi} \frac{(c^2+\lambda)(a^2+\lambda)^{\frac{1}{2}}\left\{(r^2+c^2-a^2)^2-4\,(c^2+\lambda)(x^2+y^2)\right\}\sin^2\phi\,d\lambda\,d\phi}{\left[\lambda(r^2+a^2+c^2)+2a^2c^2-\lambda\iota\left\{4(c^2+\lambda)(x^2+y^2)-(r^2+c^2-a^2)^2\right\}^{\frac{1}{2}}\cos\phi\right]^{\frac{3}{2}}}.$$

16. As an interesting verification let us find the value of this for a point on the axis of z, the potential at any point of which is known* to be

$$2\frac{M}{\pi}\int_0^\pi \frac{\sin^2\psi\, d\psi}{(x^2+y^2+c^2+a^2-2a\sqrt{x^2+y^2+c^2}\cos\psi)^{\frac{3}{2}}}$$

$$=\frac{M}{R}\left\{1-\tfrac{1}{8}\alpha^2-\tfrac{1}{64}\alpha^4-\ldots-2\,\frac{1^2.3^2\ldots(2n-3)^2(2n-1)}{2^2.4^2\ldots(2n-2)^2(2n)^2}\,\frac{\alpha^{2n}}{2n+2}-\ldots\right\},$$

where $R^2 = x^2+y^2+c^2$ and $\alpha = \dfrac{a}{R}$.

17. On the axis $X^2+Y^2=0$, and therefore λ' is infinitely great, but

$$S=-\frac{a^2U^2+c^2(V^2+Z^2)}{(c^2+\lambda)(a^2+\lambda)}\,\frac{\lambda-\lambda'}{\lambda'}=\frac{(c^2-a^2)\,U^2}{(c^2+\lambda)(a^2+\lambda)}\,\frac{\lambda-\lambda'}{\lambda'};$$

and we have therefore

$$\int_{\lambda'}^{\infty}\int_0^{\pi}\frac{-(c^2+\lambda)(a^2+\lambda)^{\frac{1}{2}}B^2(\lambda-\lambda')\sin^2\phi\, d\lambda\, d\phi}{\lambda'\left(A\lambda+2a^2c^2-2\lambda B\sqrt{\dfrac{\lambda-\lambda'}{\lambda'}}\cos\phi\right)^{\frac{3}{2}}},$$

where $A = R^2+a^2 = R^2(1+\alpha^2)$ and $B = R^2-a^2 = R^2(1-\alpha^2)$.

Now in this integral put $\lambda = u\lambda'$, and then put λ' infinite, and we get

$$B^2\int_1^{\infty}\int_0^{\pi}\frac{(u-1)\sin^2\phi\, du\, d\phi}{u\,(A+B\iota\sqrt{u-1}\cos\phi)^{\frac{3}{2}}}$$

$$=2B^2\int_0^{\infty}\int_0^{\pi}\frac{u^3\sin^2\phi\, du\, d\phi}{(1+u^2)(A+B\iota u\cos\phi)^{\frac{3}{2}}}\qquad\text{(putting } 1+u^2 \text{ for } u\text{)}$$

$$=-\tfrac{4}{3}B\iota\int_0^{\infty}\int_0^{\pi}\frac{u^2\cos\phi\, du\, d\phi}{(1+u^2)(A+B\iota u\cos\phi)^{\frac{1}{2}}}$$

(integrating by parts with respect to ϕ)

$$=-\tfrac{4}{3}B\iota\int_{-\infty}^{+\infty}dx\int_0^{\infty}\frac{x\,dy}{(1+x^2+y^2)(A+B\iota x)^{\frac{1}{2}}}$$

(putting $u\cos\phi = x$ and $u\sin\phi = y$)

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* *Cf. Phil. Trans.*, 1893, "The Potential of an Anchor Ring," by Mr. F. W. Dyson.

$$= -\tfrac{4}{3} B\iota \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{1+x^2}\,(A+B\iota x)^{\frac{3}{2}}}$$

$$= \frac{4\pi}{3} \int_{-\infty}^{\infty} \frac{dx}{(A+B\iota x)^{\frac{1}{2}}\,(1+x^2)^{\frac{3}{2}}}$$

$$= \text{real part of } \frac{8\pi}{3} \int_0^{\infty} \frac{dx}{(A+B\iota x)^{\frac{1}{2}}\,(1+x^2)^{\frac{3}{2}}}$$

$$= \quad ,, \qquad ,, \qquad \frac{8\pi}{3} \int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{(A+B\iota\tan\theta)^{\frac{1}{2}}}$$

$$= \quad ,, \qquad ,, \qquad \frac{8\pi}{3R} \int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{\{1+a^2+(1-a^2)\,\iota\tan\theta\}^{\frac{1}{2}}}.$$

18. Now $$\int_0^{\frac{1}{2}\pi} \frac{\cos\theta\,d\theta}{\{1+a^2+(1-a^2)\,\iota\tan\theta\}^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{\cos^{\frac{3}{2}}\theta\,d\theta}{(e^{\iota\theta}+a^2e^{-\iota\theta})^{\frac{1}{2}}}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{e^{-\frac{1}{2}(\iota\theta)}\cos^{\frac{3}{2}}\theta}{(1+a^2e^{-2\iota\theta})^{\frac{1}{2}}};$$

and therefore the real part of this series is

$$\Sigma_0 (-)^n \frac{1\,.\,3\,.\,5\,\ldots\,2n-1}{2\,.\,4\,.\,6\,\ldots\,2n} a^{2n} u_n,$$

where $$u_n = \int_0^{\frac{1}{2}\pi} \cos\left(2n+\tfrac{1}{2}\right)\theta\,\cos^{\frac{3}{2}}\theta\,d\theta.$$

But $$\frac{d}{d\theta} \sin\left(2n-\tfrac{1}{2}\right)\theta\,(\cos\theta)^{\frac{5}{2}}$$

$$= \left(2n-\tfrac{1}{2}\right)\cos\left(2n-\tfrac{1}{2}\right)\theta\,(\cos\theta)^{\frac{5}{2}} - \tfrac{5}{2}\sin\left(2n-\tfrac{1}{2}\right)\theta\,\sin\theta\,(\cos\theta)^{\frac{3}{2}}$$

$$= (\cos\theta)^{\frac{3}{2}}\left\{(n+1)\cos\left(2n+\tfrac{1}{2}\right)\theta + \left(n-\tfrac{3}{2}\right)\cos\left(2n-\tfrac{3}{2}\right)\theta\right\};$$

and therefore, integrating between 0 and $\frac{\pi}{2}$,

$$u_n = -\frac{2n-3}{2n+2} u_{n-1}$$

$$= (-)^{n-1} \frac{(2n-3)}{2n+2} \frac{2n-5}{2n} \ldots \tfrac{1}{4} u_1,$$

and $$u_1 = \tfrac{3}{2}u_0 = \tfrac{3}{2}\int_0^{\frac{1}{2}\pi} \cos\frac{\theta}{2}\cdot(\cos\theta)^{\frac{3}{2}}\,d\theta$$

$$= 3\int_0^{1/\sqrt{2}} (1-2s^2)^{\frac{3}{2}}\,ds = \frac{3\sqrt{2}}{4}\int_0^1 (1-y)^{\frac{3}{2}}\,y^{-\frac{1}{2}}\,dy$$

$$= \frac{3\sqrt{2}}{4}\,\frac{\Gamma(\frac{5}{2})\,\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{9\sqrt{2}}{32}\,\pi,$$

and therefore the real part of $\dfrac{8\pi}{3R}\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\cos\theta\,d\theta}{\{1+a^2+(1-a^2)\,\iota\tan\theta\}^{\frac{1}{2}}}$ is

$$\frac{3\sqrt{2}}{4}\,\frac{\pi^2}{R}\left\{1-\tfrac{1}{8}a^2-\tfrac{1}{64}a^4\,\ldots-2\,\frac{1^2.\,3^2\ldots(2n-3)^2\,(2n-1)}{2^2.\,4^2\ldots(2n-2)^2\,(2n)^2}\,\frac{a^{2n}}{2n+2}\ldots\right\},$$

which identifies this form of the potential for a point on the axis with that already quoted in § 16, and shows that the actual expression for the potential is

$$\frac{32}{3}\,\frac{Ma^2c}{\pi^2}\int_\lambda^\infty\int_0^\pi\frac{S\sin^2\phi\,d\lambda\,d\phi}{\sqrt{P}\,\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^{\frac{3}{2}}},$$

when the density is unity.

19. The potential for the surface of an anchor ring covered with matter of uniform density will be obtained by differentiating the expression found in § 18 with regard to a.

The expression found in § 6, viz.,

$$\int_\lambda^\infty\int_0^\pi\frac{d\phi}{\sqrt{P}\,\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^{\frac{1}{2}}},$$

is the potential when the surface density is proportional to U^{-1}, *i.e.*, to $(x^2+y^2+z^2+c^2-a^2)^{-1}$.

The working out of the potential of a solid whose surface equation is

$$\frac{U^2}{a}+\frac{V^2}{b}+\frac{Z^2}{c}=0$$

will be almost identical with that just given for an anchor ring.

20. Returning to the consideration of the general case, we see at once that the formulæ of §§4 and 6 are immediately applicable whenever $\lambda\theta S$ remains positive between the limits of the integration, or when T^2 is greater than $-\lambda\theta S$, if $\lambda\theta S$ is negative. In the next place, we notice that the equation $S = 0$ in its standard form (II.) can always be so chosen that to every point exterior to S shall correspond one and only one value of λ lying between 0 and ∞. For the system of confocal surfaces will be made up of three distinct series,* separated from one another and ended by the limiting cases when one of the system becomes a portion of the surface of one of the real spheres X, Y, Z, V, and such that through every point passes one of each series; and, if S_0, the surface whose potential is required, lie between Z and V (for example) in the series, we must take S in the form without Z or V (*i.e.*, so that $\lambda = \infty$ shall correspond to Z or V) according as the limiting focal curve on Z is found to lie without or within S_0.†

Now, if S be supposed in this form (without V), S is negative for all values of λ between λ', for which $S = 0$, and ∞, for which it has the same sign as $-V^2$, and so the first condition is satisfied if θ is negative.

Also θ will be negative if S_0 and V do not intersect, and if T_0 be the square of the distance of any point from some fixed point on V. For, if

$$S_0 \equiv \frac{U^2}{d} + \frac{X^2}{a} + \frac{Y^2}{b} + \frac{Z^2}{c} = 0$$

do not intersect $V = 0$, a, b, c must all be less than d; and, if

$$-T = \frac{a}{a+\lambda} XX' + \frac{b}{b+\lambda} YY' + \frac{c}{c+\lambda} ZZ' + \frac{d}{d+\lambda} UU',$$

where

$$X'^2 + Y'^2 + Z'^2 + U'^2 = 0,$$

$$\theta = \frac{a}{a+\lambda} X'^2 + \frac{b}{b+\lambda} Y'^2 + \frac{c}{c+\lambda} Z'^2 + \frac{d}{d+\lambda} U'^2$$

$$= \frac{\lambda(a-d)}{(a+\lambda)(d+\lambda)} X'^2 + \frac{\lambda(b-d)}{(b+\lambda)(d+\lambda)} Y'^2 + \frac{\lambda(c-d)}{(c+\lambda)(d+\lambda)} Z'^2,$$

and therefore θ is always negative. A particular case of this is that, if a cyclide has a plane of symmetry, its potential when filled with matter of uniform density can be found.

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* Just as in the case of confocal conicoids.

† There is a very useful table of the forms of confocal cyclides (and the surfaces into which they degenerate) on p. 65 of *Die Reihenentwickelungen der Potentialtheorie*," by Herr Maxime Bôcher (Teubner).

21. Further there will always be at least one pair (Z, V, say) of the spheres X, Y, Z, V whose curve of intersection does not cut S_0, and, just as above, the formulæ are immediately applicable if we take

$$-T = \frac{a}{a+\lambda}XX' + \frac{b}{b+\lambda}YY' + \frac{d}{d+\lambda}UU',$$

where $$X'^2 + Y'^2 + U'^2 = 0.$$

The corresponding particular case is that, if a cyclide have two planes of symmetry, its potential when filled with matter of uniform density can be found.

22. As an example of another kind let us suppose the density proportional to $V^{-\frac{3}{2}}$, and let us take

$$S \equiv \frac{X^2}{a+\lambda} + \frac{Y^2}{b+\lambda} + \frac{Z^2}{c+\lambda} + \frac{U^2}{d+\lambda},$$

$$T = V;$$

then $$\theta = 1.$$

If we take d greater than a, b, or c, S_0 will not meet V, and we have

$$S = -\frac{V^2}{d+\lambda} + \frac{1}{d+\lambda}\left\{\frac{(d-a)X^2}{a+\lambda} + \frac{(d-b)Y^2}{b+\lambda} + \frac{(d-c)Z^2}{c+\lambda}\right\},$$

and, as S is always negative, it is numerically less than $\frac{V^2}{d+\lambda}$; and therefore when λ is positive $(-\lambda\theta S)^{\frac{1}{2}}$ is real and less than V.

Therefore we have for the potential

$$\int_{\lambda'}^{\infty}\int_0^{\pi} \frac{S\sin^2\phi\, d\phi\, d\lambda}{\sqrt{P}\left\{V - (-\lambda\theta S)^{\frac{1}{2}}\cos\phi\right\}^{\frac{3}{2}}}.$$

If, however, we take V very small and λ' very large in this expression, it vanishes. Also we know that S_0 consists of two detached surfaces, one inside and one outside V, S_1 and S_2, say. Thus the above expression must be the potential of S_1 and S_2 together, the density of S_1 being $(-V)^{\frac{3}{2}}$,* and that of S_2 $(V)^{\frac{3}{2}}$, and we must change the sign of V on passing through the surface $V = 0$, and S_1 and S_2 are inverse masses in the sense required for Thomson's theorem† on the determination of potentials by inversion, V being the sphere with regard to which the inversion is made.

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* V is negative inside the sphere $V = 0$.

† *v.* Routh, *Analytical Statics*, Vol. II., p. 80.

[redacted]

I find similarly that in any case in which S is taken not to contain V, while T does contain it, the potential function found vanishes for points on $V=0$, and the sign of V must be changed on passing through the surface.

23. As a particular case included in the last paragraph, we have that the potential of the hyperboloid of two sheets

$$\frac{x^2}{a}-\frac{y^2}{b}-\frac{z^2}{c}-1=0$$

filled with matter whose density at any point is proportional to x^{-m} is

$$\int_\lambda^{-a}\int_0^\pi \frac{\left(\dfrac{x^2}{a+\lambda}-\dfrac{y^2}{b+\lambda}-\dfrac{z^2}{c+\lambda}-1\right)\sin^2\phi\,d\phi\,d\lambda}{\sqrt{p}\left[\dfrac{ax}{a+\lambda}-\left\{\dfrac{-a\lambda}{a+\lambda}\left(\dfrac{x^2}{a+\lambda}-\dfrac{y^2}{b+\lambda}-\dfrac{z^2}{c+\lambda}-1\right)\right\}^{\frac{1}{2}}\cos\phi\right]^m},$$

where $$p=(a+\lambda)(b+\lambda)(c+\lambda),$$

and in which m must be greater than $\frac{1}{2}$.

24. The potential of a cyclide the density of which is an algebraical function of U, V, X, Y, Z may be derived from the foregoing in the manner developed by Dr. N. M. Ferrers in Vol. XIV. of the *Quarterly Journal*, in a paper entitled "On the Potentials of Ellipsoids, &c." in which he shows that, if V be the potential of a body filled with matter of density ρ, at any point, the potential of the same body filled with matter of density $\dfrac{d\rho}{dx}$ will be $\dfrac{dV}{dx}$, if ρ vanishes at the surface of the solid.

In the case of the functions considered in this paper it is evident that, if we take

$$I=\int_\lambda^\infty\int_0^\pi \frac{S^r\sin^{2r}\phi\,d\lambda\,d\phi}{\sqrt{P}\left\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\right\}^{2r-1}},$$

and I_0 the corresponding function when the lower limit is zero, we shall have, if $r \not< 2$,

$$\nabla^2\frac{dI}{dX}=0,\quad \nabla^2\frac{dI_0}{dX}=\frac{d\rho}{dX},$$

and, further, $$\frac{d}{dX}(I-I_0)$$

and all its first differential coefficients equal to zero, when λ is put equal to zero, and so a potential for a density $\frac{d\rho}{dX}$ is derived. In order that $\frac{dI}{dX}$ should be of degree $-\frac{1}{2}$, I must be of degree $+\frac{1}{2}$. This process, for example, would give for a density proportional to U

$$\int_\lambda^\infty \int_0^\pi \frac{\sin^4\phi \, d\phi \, d\lambda}{\sqrt{P}} \, \frac{d}{dU} \, \frac{S^2}{\{T-(-\lambda\theta S)^{\frac{1}{2}}\cos\phi\}^{\frac{1}{2}}},$$

where $$-T = \Sigma \frac{aX}{(a+\lambda)\, r_1};$$

and any algebraical function could be built up term by term, in the manner given by Dr. Ferrers.

25. The potential of an attracting plane lamina bounded by a bicircular quartic curve may be deduced from that of a solid cyclide by a method also taken from the same paper.

If we take the equation of a cyclide to be

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}+\frac{U^2}{d^2}+\frac{V^2}{e^2}=0,$$

where $Z=0$ represents a plane, and $Z\equiv 2z$, and suppose c^2 to become very small, then Z becomes very small also, and we may put $Z=0$ in all terms except $\frac{Z^2}{c^2}$, and we get $\frac{Z^2}{c^2}=\sigma$, where $\sigma=0$ is the equation to the plane section of the cyclide by $z=0$.

Now, if the density at any point of the cyclide be $S^{r-1}f(UVXY)$, the mass of the solid prism which stands on a small element of area α of the plane base $z=0$ is

$$\int \alpha f S^{r-1} dz = \tfrac{1}{2}\alpha f \int_{-c\sqrt{\sigma}}^{c\sqrt{\sigma}} \left(\sigma - \frac{Z^2}{c^2}\right)^{r-1} dZ$$

$$= \alpha f \sigma^{r-\frac{1}{2}} c \frac{\Gamma(r)\,\Gamma(\frac{1}{2})}{\Gamma(r+\frac{1}{2})},$$

and so the potential of a lamina of density proportional to $\sigma^{r-\frac{1}{2}} f$ is found.

26. If $r = \frac{1}{2}$ and T_0 is constant, we have the case of a uniform disc, and it is curious that the potential function for this case reduces to a much simpler form.

Referring back to § 4, we get

$$\int_0^\pi \frac{1}{\sqrt{P}} \frac{S^{\frac{1}{2}} \sin\phi \, d\phi}{\{T-(-\lambda\theta S)^{\frac{1}{2}} \cos\phi\}^{\frac{3}{2}}},$$

which is the real part of

$$\frac{1}{2} \frac{\iota}{(\lambda\theta)^{\frac{1}{2}} \sqrt{P} \{T-(-\lambda\theta S)^{\frac{1}{2}}\}^{\frac{1}{2}}},$$

and in fact we may verify at once that

$$\nabla^2 \frac{\iota}{(\lambda\theta)^{\frac{1}{2}} \sqrt{P} \{T-(-\lambda\theta S)^{\frac{1}{2}}\}^{\frac{1}{2}}} = -\frac{d}{d\lambda} \left[\frac{1}{S^{\frac{1}{2}} \{T-(-\lambda\theta S)^{\frac{1}{2}}\}^{\frac{1}{2}}} \right],$$

and so, for a uniform disc bounded by the cyclidal curve

$$\frac{X^2}{A} + \frac{Y^2}{B} + \frac{U^2}{D} + \frac{V^2}{E} = 0,$$

we have the potential function

$$\int_\lambda^\infty \frac{d\lambda}{(\lambda\theta)^{\frac{1}{2}} \sqrt{P} \{T-(-\lambda\theta S)^{\frac{1}{2}}\}^{\frac{1}{2}}},$$

where $$T = \frac{AX}{(A+\lambda) r_1} + \frac{BY}{(B+\lambda) r_2} + \frac{DU}{(D+\lambda) r_4} + \frac{EV}{(E+\lambda) r_5},$$

and $$S = \frac{X^2}{A+\lambda} + \frac{Y^2}{B+\lambda} + \frac{Z^2}{\lambda} + \frac{U^2}{D+\lambda} + \frac{V^2}{E+\lambda}.$$

We may notice further that in every case when $r = n + \frac{1}{2}$, where n is an integer, the potential function becomes a single integral, as the integration with respect to ϕ can always be effected.

27. The potential of a spherical area, cut off by a cyclide from one of its focal spheres, and loaded with attracting matter, may be obtained in the same way.

For, if, in § 25, we suppose $Z = 0$ to be a sphere of radius a, and take r as the distance from the centre of any point, we have

$$dZ = \frac{r\,dr}{a}.$$

and for all points for which Z is very small $r = a$, and we have

$$dZ = dr,$$

and the working out will be exactly the same as for a plane area.

It may be noticed that the curve of intersection of a sphere and a cyclide may always be considered as the curve of intersection of a sphere and a conicoid, and, further, that a cyclide can always be described through the intersection of a sphere and conicoid so that the sphere shall be a focal sphere of the cyclide.

For, if $Z = 0$ be a given sphere, we have nine arbitrary constants left in the equation

$$\frac{X^2}{a}+\frac{Y^2}{b}+\frac{Z^2}{c}+\frac{U^2}{d}+\frac{V^2}{e}=0,$$

and, by substituting $x^2+y^2+z^2$, from $Z = 0$, we can reduce this equation to one of the second degree in x, y, z.

Cyclidal Cylinders.

28. The formulæ of §§ 4 and 6 can also be used for infinite cylinders on plane bicircular quartics as bases. For, if we choose as coordinates the power of a point with regard to four mutually orthogonal circles, divided by the corresponding radius, the only difference will be that we shall have four coordinates instead of five, and that, instead of

$$\nabla^2 \equiv \Sigma \frac{d^2}{dX^2}+(n+\tfrac{1}{2})\,\Sigma\frac{1}{r_1}\,\frac{d}{dX},$$

we shall get
$$\nabla^2 \equiv \Sigma \frac{d^2}{dX^2}+n\,\Sigma\frac{1}{r_1}\,\frac{d}{dX},$$

and that therefore, instead of putting $m = 2r+\frac{1}{2}$ as in § 9, we must have $m = 2r$.

Part III.

If P and P' be two corresponding points on two confocal cyclidal shells S and S', whose equations are

$$\frac{X^2}{a+\lambda}+\frac{Y^2}{b+\lambda}+\frac{Z^2}{c+\lambda}+\frac{U^2}{d+\lambda}=0$$

and
$$\frac{X^2}{a+\lambda'}+\frac{Y^2}{b+\lambda'}+\frac{Z^2}{c+\lambda'}+\frac{U^2}{d+\lambda'}=0,$$

and which are loaded with matter the density of which at any point is

$$V^{-\frac{1}{2}}\left\{\Sigma\frac{X^2}{(a+\lambda)^2}\right\}^{-\frac{1}{2}},$$

then the ratio of the potential of S at P' to the potential of S' at P is a constant multiple of the ratio of $V_P^{-\frac{1}{2}}$ to $V_{P'}^{-\frac{1}{2}}$.

Take any point Q on S, and let λ_1, λ_2, λ_3 be the parameters of the three confocals through Q, and ds_2, ds_3 elements of length on the surface at Q in the directions of λ_3 constant and λ_2 constant respectively, so that the element of area $ds_2\,ds_3$ is a rectangular element. Then we have $ds^2 = \frac{1}{4}\Sigma dX^2$ in general

$$= \tfrac{1}{4}V^2\,(d\xi^2+d\eta^2+d\zeta^2+dv^2),$$

where $$\xi = X/V,\quad \eta = Y/V,\ \&c.$$

Also, since

$$\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}+\frac{\zeta^2}{c+\lambda}+\frac{v^2}{d+\lambda} \equiv -\frac{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}{(a+\lambda)(b+\lambda)(c+\lambda)(d+\lambda)},$$

we have $$\xi^2 = \frac{(a+\lambda_1)(a+\lambda_2)(a+\lambda_3)}{(b-a)(c-a)(d-a)},\ \&c.,\ \&c.;$$

and therefore $$2\frac{d\xi}{d\lambda_2} = \frac{\xi}{a+\lambda_2},\ \&c.,\ \&c.;$$

and therefore $$ds_2 = VP_2\,d\lambda_2,$$

where $$P_2^2 = \Sigma\frac{\xi^2}{(a+\lambda_2)^2},$$

and $$ds_3 = VP_3\,d\lambda_3;$$

and therefore the potential of the element of mass at Q on a unit mass at P' is

$$V^{\frac{1}{2}}P_1^{-1}P_2P_3\,d\lambda_2\,d\lambda_3/PQ.$$

But $$P_1^2 = -\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a+\lambda_1)(b+\lambda_1)(c+\lambda_1)(d+\lambda)},$$

$$P_2^2 = -\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(a+\lambda_2)(b+\lambda_2)(c+\lambda_2)(d+\lambda_2)},$$

&c.;

and therefore $$\frac{P_2 P_3}{P_1} = \frac{\pi_1}{\pi_2 \pi_3}(\lambda_2 - \lambda_3),$$

where $$\pi = \sqrt{(a+\lambda)(b+\lambda)(c+\lambda)(d+\lambda)}\,;$$

and therefore the potential of mass at Q at P' is

$$\frac{\pi_1}{\pi_2 \pi_3}(\lambda_2 - \lambda_3)\, d\lambda_2 d\lambda_3 \frac{V_Q^{\frac{1}{2}}}{P'Q}.$$

Also, if we take the point Q' on S' corresponding to Q or S, the parameters of the confocals through Q' are λ_1', λ_2, λ_3; and therefore the potential of the element of mass at Q' at P is

$$\frac{\pi_1'}{\pi_2 \pi_3}(\lambda_2 - \lambda_3)\, d\lambda_2 d\lambda_3 \frac{V_{Q'}^{\frac{1}{2}}}{PQ'}.$$

But we know that $$\frac{V_{P'}^{\frac{1}{2}} V_Q^{\frac{1}{2}}}{P'Q} = \frac{V_P^{\frac{1}{2}} V_{Q'}^{\frac{1}{2}}}{PQ'};^*$$

and therefore the theorem is true for every element of mass at corresponding points on the two confocal cyclides S and S', and therefore for the whole shells.

To make the analogy with conicoids more complete it is easy to show that the thickness of the shell between

$$\frac{X^2}{a} + \frac{Y^2}{b} + \frac{Z^2}{c} + V^2 = 0$$

and $$\frac{X^2}{a(1+\kappa)} + \frac{Y^2}{b(1+\kappa)} + \frac{Z^2}{c(1+\kappa)} + V^2 = 0,$$

where κ is small, is proportional to

$$V^2 \left(\Sigma \frac{X^2}{a^2}\right)^{-\frac{1}{2}}.$$

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* *Cf.* "A Theorem for Confocal Cyclides corresponding to Ivory's Theorem," *Proc. Lond. Math. Soc.*, Vol. XXIV.

Geodesics on Quadrics, not of Revolution. By A. R. Forsyth.

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Central Quadrics. §§ 1–15.

1. One of the best known properties (Joachimstahl's) of any geodesic drawn upon an ellipsoid (or upon any central quadric) is represented by the equation

$$pD = \text{constant} = k^2,$$

where p is the perpendicular from the centre on the tangent plane at the point, and D is the length of a central semi-diameter parallel to the direction of the geodesic through the point; the quantity k is constant along the geodesic.

But an equation of precisely the same form characterizes lines of curvature upon central quadrics, the difference between the two arising in the value of the constant k for the particular curve. Yet even this difference disappears when the equation is used in a form

$$\frac{d}{ds}(pD) = 0,$$

current along the curve. The property, thus stated, does not distinguish between a geodesic and a line of curvature; it might, indeed, belong to curves of other classes passing through the point. A question is thus suggested as to the curves which are determined by either of the equivalent equations

$$pD = \text{constant}, \quad \frac{d}{ds}(pD) = 0.$$

2. Taking the quadric in the form

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} + \frac{z^2}{\gamma} = 1,$$

and denoting the tangential direction of the curve through x, y, z, by l, m, n, so that

$$l,\ m,\ n = x',\ y',\ z',$$

where dashes imply differentiation with regard to the arc s, we have

$$\frac{x^2}{a^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{1}{p^2},$$

$$\frac{l^2}{a}+\frac{m^2}{\beta}+\frac{n^2}{\gamma}=\frac{1}{D^2},$$

$$\frac{lx}{a}+\frac{my}{\beta}+\frac{nz}{\gamma}=0,$$

$$l^2+m^2+n^2=1.$$

From these we have

$$\frac{\dfrac{lx}{a^2}+\dfrac{my}{\beta^2}+\dfrac{nz}{\gamma^2}}{\dfrac{x^2}{a^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}}=-\frac{1}{p}\frac{dp}{ds},$$

$$\frac{\dfrac{ll'}{a}+\dfrac{mm'}{\beta}+\dfrac{nn'}{\gamma}}{\dfrac{l^2}{a}+\dfrac{m^2}{\beta}+\dfrac{n^2}{\gamma}}=-\frac{1}{D}\frac{dD}{ds};$$

whence, if we use the characteristic equation in the form

$$\frac{1}{p}\frac{dp}{ds}+\frac{1}{D}\frac{dD}{ds}=0,$$

we have $$l'\frac{D^2l}{a}+m'\frac{D^2m}{\beta}+n'\frac{D^2n}{\gamma}=-\frac{p^2l}{a^2}x-\frac{p^2m}{\beta^2}y-\frac{p^2n}{\gamma^2}z.$$

Again, we have

$$\frac{lx'}{a}+\frac{my'}{\beta}+\frac{nz'}{\gamma}+\frac{l'x}{a}+\frac{m'y}{\beta}+\frac{n'z}{\gamma}=0,$$

that is, $$l'\frac{D^2x}{a}+m'\frac{D^2y}{\beta}+n'\frac{D^2z}{\gamma}=-1;$$

and $$l'l+m'm+n'n=0.$$

There are thus three equations to determine l', m', n', and they will determine these quantities uniquely unless they are not independent of one another.

When we solve them, we have, as the coefficient of l', the quantity

$$D^4 \begin{vmatrix} \frac{l}{\alpha}, & \frac{m}{\beta}, & \frac{n}{\gamma} \\ \frac{x}{\alpha}, & \frac{y}{\beta}, & \frac{z}{\gamma} \\ l, & m, & n \end{vmatrix},$$

which is equal to

$$\frac{D^4}{\alpha\beta\gamma}\left\{xmn(\beta-\gamma)+ynl(\gamma-\alpha)+zlm(\alpha-\beta)\right\} = \frac{D^4}{\alpha\beta\gamma}\Theta, \text{ say.}$$

The value of $\frac{D^4}{\alpha\beta\gamma}\Theta l'$ is

$$= -p^2\left(\frac{lx}{\alpha^2}+\frac{my}{\beta^2}+\frac{nz}{\gamma^2}\right)D^2\left(\frac{ny}{\beta}-\frac{mz}{\gamma}\right)-D^2mn\left(\frac{1}{\gamma}-\frac{1}{\beta}\right).$$

Now $$\frac{lx}{\alpha^2}+\frac{my}{\beta^2}+\frac{nz}{\gamma^2} = -\frac{1}{\alpha}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)+\frac{my}{\beta^2}+\frac{nz}{\gamma^2}$$

$$= -\frac{my}{\beta}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)-\frac{nz}{\gamma}\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right);$$

and therefore the coefficient of $-p^2D^2$ is

$$-\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{mn}{\beta^2}y^2-\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{n^2}{\beta\gamma}yz+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{m^2}{\beta\gamma}yz+\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{mn}{\gamma^2}z^2.$$

Also the quantity $-D^2mn\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)$ is equal to $-p^2D^2$ multiplied by

$$\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\beta^2}y^2+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\alpha^2}x^2+\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\gamma^2}z^2.$$

Hence the whole expression for $\frac{D^4}{\alpha\beta\gamma}\Theta l'$ is

$$p^2D^2\left\{-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mn}{\alpha^2}x^2\right.$$

$$+\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\left(\frac{mn}{\beta^2}y^2+\frac{n^2}{\beta\gamma}yz\right)$$

$$\left.+\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\left(\frac{m^2}{\beta\gamma}yz+\frac{mn}{\gamma^2}z^2\right)\right\}.$$

But $$\frac{mn}{\beta^2}y^2+\frac{n^2}{\beta\gamma}yz=\frac{ny}{\beta}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)=-\frac{lnxy}{\alpha\beta},$$

and $$\frac{m^2}{\beta\gamma}yz+\frac{mn}{\gamma^2}z^2=\frac{mz}{\gamma}\left(\frac{my}{\beta}+\frac{nz}{\gamma}\right)=-\frac{lmxz}{\alpha\gamma};$$

and therefore

$$\frac{D^4}{\alpha\beta\gamma}\Theta l'=p^2D^2\frac{x}{\alpha}\left\{-\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)\frac{mnx}{\alpha}-\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\frac{lny}{\beta}-\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)\frac{lmz}{\gamma}\right\}$$

$$=-p^2D^2\frac{x}{\alpha^2\beta\gamma}\Theta,$$

so that, as D does not vanish, we have*

and similarly $$\left.\begin{aligned}\Theta l'&=-\Theta\frac{p^2}{D^2}\frac{x}{\alpha}\\ \Theta m'&=-\Theta\frac{p^2}{D^2}\frac{y}{\beta}\\ \Theta n'&=-\Theta\frac{p^2}{D^2}\frac{z}{\gamma}\end{aligned}\right\}.$$

3. If Θ does not vanish, we have

$$\frac{l'}{\frac{x}{\alpha}}=\frac{m'}{\frac{y}{\beta}}=\frac{n'}{\frac{z}{\gamma}}=-\frac{p^2}{D^2},$$

which are the equations† of a geodesic through x, y, z.

But, if Θ vanishes, the equations do not determine l', m', n'. In that case, we have

$$xmn(\beta-\gamma)+ynl(\gamma-\alpha)+zlm(\alpha-\beta)=0,$$

or, what is the equivalent,

$$\frac{x}{l}(\beta-\gamma)+\frac{y}{m}(\gamma-\alpha)+\frac{z}{n}(\alpha-\beta)=0.$$

This, together with

$$\left.\begin{aligned}\frac{lx}{\alpha}+\frac{my}{\beta}+\frac{nz}{\gamma}&=0\\ l^2+m^2+n^2&=1\end{aligned}\right\},$$

* See Salmon's *Solid Geometry*, 3rd edition, p. 353, note.
† Frost's *Solid Geometry*, 3rd ed., p. 314.

[redacted]

suffices to determine the (two) sets of values at x, y, z for l, m, n. That these two sets correspond to the lines of curvature can be seen easily as follows. The direction of either of the lines of curvature is normal to a confocal; so that, if ϕ be a root (other than zero) of the equation

$$\frac{x^2}{\alpha-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1,$$

the direction cosines λ, μ, ν of the line of curvature, that is normal to the ϕ confocal, are proportional to

$$\frac{x}{\alpha-\phi},\ \frac{y}{\beta-\phi},\ \frac{z}{\gamma-\phi}.$$

Hence
$$\frac{x}{\lambda}(\beta-\gamma)+\frac{y}{\mu}(\gamma-\alpha)+\frac{z}{\nu}(\alpha-\beta)$$

is proportional to

$$(\beta-\gamma)(\alpha-\phi)+(\gamma-\alpha)(\beta-\phi)+(\alpha-\beta)(\gamma-\phi),$$

that is, it vanishes; and

$$\frac{\lambda x}{\alpha}+\frac{\mu y}{\beta}+\frac{\nu z}{\gamma}$$

is proportional to $\dfrac{x^2}{\alpha(\alpha-\phi)}+\dfrac{y^2}{\beta(\beta-\phi)}+\dfrac{z^2}{\gamma(\gamma-\phi)}$,

that is, to

$$\frac{1}{\phi}\left\{\frac{x^2}{\alpha-\phi}+\frac{y^2}{\beta-\phi}+\frac{z^2}{\gamma-\phi}-\left(\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}\right)\right\},$$

so that it vanishes. Hence the two sets of values, determined for l, m, n, correspond to the lines of curvature.

Consequently, *the equation*

$$\frac{d}{ds}(pD)=0$$

determines either a geodesic or a line of curvature. When taken in the form

$$pD=k^2,$$

it determines either a geodesic or one of the lines of curvature according to the value of k.

The only exception is when both sets of equations, viz.,

$$\frac{l'}{\frac{x}{\alpha}} = \frac{m'}{\frac{y}{\beta}} = \frac{n'}{\frac{z}{\gamma}} = -\frac{p^2}{D^2},$$

and $$\Theta = 0,$$

are satisfied. This circumstance occurs when the geodesic, determined by the former, touches a line of curvature, determined by the latter; at the point, l, m, n have the same values. And, in fact, the quantity k, which is the parameter of a geodesic, can be equal to the parameter of some line of curvature, which accordingly is touched by the geodesic.

4. But, though the discrimination between the geodesic and the line of curvature cannot be made by the explicit form

$$\frac{d}{ds}(pD) = 0,$$

it can be secured by introducing into the differential equation the ellipsoidal surface-parameters. Denoting these by λ_1 and λ_2, the roots (other than zero) of the equation

$$\frac{x^2}{\alpha-\theta} + \frac{y^2}{\beta-\theta} + \frac{z^2}{\gamma-\theta} = 1,$$

we have, as usual, $$\frac{1}{p^2} = \frac{\lambda_1\lambda_2}{\alpha\beta\gamma},$$

$$x^2 = \text{A}\,(\alpha-\lambda_1)(\alpha-\lambda_2),$$

$$y^2 = \text{B}\,(\beta-\lambda_1)(\beta-\lambda_2),$$

$$z^2 = \Gamma\,(\gamma-\lambda_1)(\gamma-\lambda_2),$$

where, if $\square$ denote $$(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma),$$

then $$\square\text{A} = \alpha\,(\beta-\gamma),$$

$$\square\text{B} = \beta\,(\gamma-\alpha),$$

$$\square\Gamma = \gamma\,(\alpha-\beta)$$

they satisfy the equations

$$\mathrm{A}+\mathrm{B}+\Gamma=0, \qquad \frac{\mathrm{A}}{\alpha}+\frac{\mathrm{B}}{\beta}+\frac{\Gamma}{\gamma}=0.$$

Now
$$-2\frac{dx}{ds}=\mathrm{A}^{\frac{1}{2}}\left\{\left(\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\},$$
$$-2\frac{dy}{ds}=\mathrm{B}^{\frac{1}{2}}\left\{\left(\frac{\beta-\lambda_2}{\beta-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\beta-\lambda_1}{\beta-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\},$$
$$-2\frac{dz}{ds}=\Gamma^{\frac{1}{2}}\left\{\left(\frac{\gamma-\lambda_2}{\gamma-\lambda_1}\right)^{\frac{1}{2}}\frac{d\lambda_1}{ds}+\left(\frac{\gamma-\lambda_1}{\gamma-\lambda_2}\right)^{\frac{1}{2}}\frac{d\lambda_2}{ds}\right\}.$$

Hence
$$4=\left(2\frac{dx}{ds}\right)^2+\left(2\frac{dy}{ds}\right)^2+\left(2\frac{dz}{ds}\right)^2$$
$$=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\mathrm{A}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\mathrm{A}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\},$$

and
$$4\left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma}\right)$$
$$=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\}.$$

But, taking
$$pD=k^2,$$
we have
$$\frac{1}{D^2}=\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma},$$
$$\frac{1}{p^2}=\frac{\lambda_1\lambda_2}{\alpha\beta\gamma},$$
so that
$$\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}+\frac{z'^2}{\gamma}=\frac{\alpha\beta\gamma}{\lambda_1\lambda_2k^4}=\frac{\delta}{\lambda_1\lambda_2},$$
say, where
$$\delta=\frac{\alpha\beta\gamma}{k^4};$$

and, in the case of an ellipsoid for which $\alpha>\beta>\gamma$,
$$\alpha>\delta>\gamma.$$

Thus the second equation is
$$\frac{4\delta}{\lambda_1\lambda_2}=\left(\frac{d\lambda_1}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_2}{\alpha-\lambda_1}\right\}+\left(\frac{d\lambda_2}{ds}\right)^2\Sigma\left\{\frac{\mathrm{A}}{\alpha}\,\frac{\alpha-\lambda_1}{\alpha-\lambda_2}\right\}.$$

Now

$$\Sigma A \frac{\alpha-\lambda_2}{\alpha-\lambda_1} = \frac{\alpha(\beta-\gamma)}{\square}\frac{\alpha-\lambda_2}{\alpha-\lambda_1} + \frac{\beta(\gamma-\alpha)}{\square}\frac{\beta-\lambda_2}{\beta-\lambda_1} + \frac{\gamma(\alpha-\beta)}{\square}\frac{\gamma-\lambda_2}{\gamma-\lambda_1}$$

$$= \frac{\lambda_1(\lambda_1-\lambda_2)}{(\alpha-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma A \frac{\alpha-\lambda_1}{\alpha-\lambda_2} = \frac{\lambda_2(\lambda_2-\lambda_1)}{(\alpha-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma \frac{A}{\alpha}\frac{\alpha-\lambda_2}{\alpha-\lambda_1} = \frac{\lambda_1-\lambda_2}{(\alpha-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)},$$

$$\Sigma \frac{A}{\alpha}\frac{\alpha-\lambda_1}{\alpha-\lambda_2} = \frac{\lambda_2-\lambda_1}{(\alpha-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}.$$

Hence, writing $2d\Lambda_1 = \left\{\frac{\lambda_1-\lambda_2}{(\alpha-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)}\right\}^{\frac{1}{2}} d\lambda_1,$

$$2d\Lambda_2 = \left\{\frac{\lambda_2-\lambda_1}{(\alpha-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}\right\}^{\frac{1}{2}} d\lambda_2,$$

the equations are

$$\left.\begin{aligned} \lambda_1\left(\frac{d\Lambda_1}{ds}\right)^2 + \lambda_2\left(\frac{d\Lambda_2}{ds}\right)^2 &= 1 \\ \left(\frac{d\Lambda_1}{ds}\right)^2 + \left(\frac{d\Lambda_2}{ds}\right)^2 &= \frac{\delta}{\lambda_1\lambda_2} \end{aligned}\right\}.$$

Introducing a quantity $R\lambda$, defined for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ by the equation

$$R\lambda = -\lambda(\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda),$$

we have, on solving these equations

$$1 - \frac{\delta}{\lambda_2} = (\lambda_2-\lambda_1)\left(\frac{d\Lambda_2}{ds}\right)^2;$$

and therefore $\frac{1}{4}(\lambda_2-\lambda_1)^2\left(\frac{d\lambda_2}{ds}\right)^2 = \frac{1}{\lambda_2^2} R\lambda_2,$

so that $\frac{1}{2}(\lambda_2-\lambda_1)\frac{d\lambda_2}{ds} = \frac{1}{\lambda_2}\sqrt{R\lambda_2}.$

Similarly, $\frac{1}{2}(\lambda_1-\lambda_2)\frac{d\lambda_1}{ds} = \frac{1}{\lambda_1}\sqrt{R\lambda_1}.$

Consequently,

$$\left.\begin{aligned} \frac{\lambda_1 d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R\lambda_2}} &= 0 \\ \frac{d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{d\lambda_2}{2\sqrt{R\lambda_2}} &= du \\ -\frac{ds}{\lambda_1\lambda_2} &= du \end{aligned}\right\},$$

the final form of the differential equations; it agrees with the form given by Weierstrass* in 1861, obtained by other considerations.

5. These have been deduced on the supposition that the two equations involving $\frac{d\Lambda_1}{ds}$, $\frac{d\Lambda_2}{ds}$ could be solved properly. If, however, the curve under consideration be a line of curvature, we have either

$$\lambda_1 = \text{constant} \quad \text{or} \quad \lambda_2 = \text{constant}.$$

When λ_1 is constant, $d\Lambda_1$ vanishes; and so $\delta = \lambda_1$. The length of the arc is given by

$$ds = \tfrac{1}{2}\left\{\frac{\lambda_2(\lambda_2-\delta)}{(a-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)}\right\}^{\frac{1}{2}} d\lambda_2.$$

Similarly, when λ_2 is constant, $d\Lambda_2$ vanishes; and so $\delta = \lambda_2$. The length of the arc is given by

$$ds = \tfrac{1}{2}\left\{\frac{\lambda_1(\lambda_1-\delta)}{(a-\lambda_1)(\beta-\lambda_1)(\gamma-\lambda_1)}\right\}^{\frac{1}{2}} d\lambda_1.$$

From the earlier investigation it appeared that the equation pD = constant represents either a geodesic or a line of curvature; it consequently follows that *the proper equations of a geodesic are*

$$\left.\begin{aligned} \frac{\lambda_1 d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{\lambda_2 d\lambda_2}{2\sqrt{R\lambda_2}} &= 0 \\ \frac{d\lambda_1}{2\sqrt{R\lambda_1}} + \frac{d\lambda_2}{2\sqrt{R\lambda_2}} &= du \\ -\frac{ds}{\lambda_1\lambda_2} &= du \end{aligned}\right\},$$

where $$R\lambda = -\lambda(a-\lambda)(\beta-\lambda)(\gamma-\lambda)(\delta-\lambda),$$

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* *Ges. Werke*, t. I., p. 262.

and λ_1, λ_2 are the (non-zero) roots of

$$\frac{x^2}{a-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1.$$

6. When the given quadric is an *ellipsoid*, a, β, γ are all positive; take

$$a>\beta>\gamma>0.$$

Let λ_1 determine the confocal hyperboloid of two sheets, and λ_2 the confocal hyperboloid of one sheet; then we have

$$a>\lambda_1>\beta, \quad \beta>\lambda_2>\gamma.$$

Further, du must be real, and therefore both $R\lambda_1$ and $R\lambda_2$ must be positive. Taking account of the limits between which λ_1 and λ_2 must lie, we find that $R\lambda_1$ is positive if $\lambda_1>\delta$, and that $R\lambda_2$ is positive if $\lambda_2<\delta$; so that

$$\lambda_1>\delta>\lambda_2.$$

The only conditions other than these to which δ is subject are

$$a>\delta>\gamma.$$

They are covered by what precedes; hence the whole set of conditions is

$$a>\lambda_1>\begin{Bmatrix}\beta\\ \delta\end{Bmatrix}>\lambda_2>\gamma>0.$$

Three cases occur, according as

$$\text{(I.)}\quad \delta=\beta,$$

$$\text{(II.)}\quad \delta<\beta,$$

$$\text{(III.)}\quad \delta>\beta.$$

As regards the form of the curve, we have

$$p^2D^2=\frac{a\beta\gamma}{\delta}.$$

In the first case, when $\delta=\beta$, we have

$$p^2D^2=a\gamma;$$

the geodesic passes through an umbilicus, and therefore also through the centrally opposite umbilicus.

In each of the other two cases, the geodesic touches a line of curvature. At any point on its course, we have

$$p^2 = \frac{\alpha\beta\gamma}{\lambda_1\lambda_2},$$

so that

$$\delta = \frac{\lambda_1\lambda_2}{D^2}.$$

When a geodesic touches a line of curvature on a hyperboloid of one sheet, D is the same at the point of contact as for the line of curvature, that is, $D^2 = \lambda_1$; and hence at that point

$$\delta = \lambda_2$$

$$< \beta.$$

Hence, in the second case, when $\delta < \beta$, the geodesic touches a line of curvature lying on the confocal one-sheeted hyperboloid; and it undulates between the two lines of curvature that constitute the complete intersection of the ellipsoid and the confocal quadric.

When a geodesic touches a line of curvature on a hyperboloid of two sheets, D is the same at the point of contact as for the line of curvature, that is, $D^2 = \lambda_2$; and hence at that point

$$\delta = \lambda_1$$

$$> \beta.$$

Hence, in the third case, when $\delta > \beta$, the geodesic touches a line of curvature lying on a confocal two-sheeted hyperboloid; and it undulates between the two lines of curvature that constitute the complete intersection of the ellipsoid and the confocal quadric.*

In the case of the oblate spheroid, for which $\alpha = \beta$, the first of the above classes gives rise to the meridians; the second of them gives rise to the non-meridianal geodesics, the course of which is well known; the third of them gives rise also to the meridians, as a limiting form.

Likewise for a prolate spheroid.

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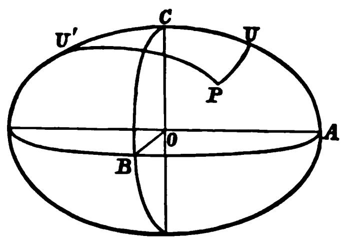
* *Cf.* Cayley, *Coll. Math. Papers*, Vol. VI., No. 425.

7. The differential relations of the geodesics can be replaced by expressions in terms of periodic functions.

(I.) In the first case, when $\delta = \beta$, the geodesics pass through the umbilici. As we take the lines of curvature from AB to UC, which lie on hyperboloids of one sheet, the quantity λ_2 increases; and as we



take the lines of curvature from OB to UA, which lie on hyperboloids of two sheets, the quantity λ_1 decreases. Hence at P, for the geodesic UP in the direction UP, we have

$$d\lambda_2 \text{ is negative,} \quad d\lambda_1 \text{ is positive.}$$

Also we take

$$\sqrt{R\lambda_1} = (\lambda_1-\beta)\left\{\lambda_1(\alpha-\lambda_1)(\lambda_1-\gamma)\right\}^{\frac{1}{2}} = (\lambda_1-\beta)\sqrt{\Lambda_1},$$

$$\sqrt{R\lambda_2} = (\beta-\lambda_2)\left\{\lambda_2(\alpha-\lambda_2)(\lambda_2-\gamma)\right\}^{\frac{1}{2}} = (\beta-\lambda_2)\sqrt{\Lambda_2}.$$

Moreover at U we have $\lambda_1 = \beta$, $\lambda_2 = \beta$. Hence at P the equations of the geodesic UP in the direction UP are

$$\left.\begin{aligned} \int_{\beta}^{\lambda_1} \frac{\theta}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} - \int_{\lambda_2}^{\beta} \frac{\theta}{\beta-\theta}\frac{d\theta}{\sqrt{\Theta}} &= 0 \\ \int_{\beta}^{\lambda_1} \frac{1}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} - \int_{\lambda_2}^{\beta} \frac{1}{\beta-\theta}\frac{d\theta}{\sqrt{\Theta}} &= 2u \\ s &= \int_{u}^{0} \frac{du}{\lambda_1\lambda_2} \end{aligned}\right\},$$

where u is chosen so as to vanish at U, and the arc s is measured from U.

The first two equations can be replaced by

$$\int_{\lambda_2}^{\lambda_1} \frac{\theta}{\theta-\beta}\frac{d\theta}{\sqrt{\Theta}} = 0,$$

$$2\beta u = -\int_{\lambda_2}^{\lambda_1} \frac{d\theta}{\sqrt{\Theta}},$$

where θ has continuous real values from λ_1 to λ_2, and in the former the principal value of the integral is to be taken. The first expresses the relation between λ_1 and λ_2 along the geodesic; for the explicit form of the relation, elliptic integrals of the third kind are necessary. In the second equation, the integral is elliptic of the first kind.

(II.) In the case when $\delta < \beta$ and the geodesic undulates between the two lines of curvature that are the complete intersection of the ellipsoid and a confocal hyperboloid of one sheet, the equations can be replaced by expressions involving hyperelliptic functions. We have

$$\alpha > \lambda_1 > \beta > \delta > \lambda_2 > \gamma > 0;$$

and we take

$$\left.\begin{aligned} a &= \int_\beta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta\, d\theta}{2\sqrt{R\theta}} \\ u &= \int_\beta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{d\theta}{2\sqrt{R\theta}} \end{aligned}\right\},$$

where a is an arbitrary constant; it is unnecessary to associate an arbitrary constant with u. Now introduce two new quantities, viz.,

$$\left.\begin{aligned} a-\gamma u = u_1 &= \int_\beta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\gamma}{2\sqrt{R\theta}}\, d\theta \\ a-\beta u = u_2 &= \int_\beta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\beta}{2\sqrt{R\theta}}\, d\theta \end{aligned}\right\},$$

these quantities u_1 and u_2 being the arguments of the hyperelliptic functions in Weierstrass's theory.* We take

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4 = \alpha,\ \beta,\ \delta,\ \gamma,\ 0;$$

and then we have

$$\frac{x^2}{\alpha} = \frac{(\alpha-\lambda_1)(\alpha-\lambda_2)}{(\beta-\alpha)(\gamma-\alpha)} = \frac{(a_0-\lambda_1)(a_0-\lambda_2)}{(a_1-a_0)(a_3-a_0)} = al_0^2\,(u_1,\ u_2),$$

$$\frac{y^2}{\beta} = \frac{(\beta-\lambda_1)(\beta-\lambda_2)}{(\alpha-\beta)(\gamma-\beta)} = \frac{(a_1-\lambda_1)(a_1-\lambda_2)}{(a_0-a_1)(a_3-a_1)} = \beta\,\frac{\beta-\delta}{\beta-\gamma}\,al_1^2\,(u_1,\ u_2),$$

$$\frac{z^2}{\gamma} = \frac{(\gamma-\lambda_1)(\gamma-\lambda_2)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{(a_3-\lambda_1)(a_3-\lambda_2)}{(a_3-a_0)(a_3-a_1)} = \gamma\,\frac{\delta-\gamma}{\beta-\gamma}\,al_3^2\,(u_1,\ u_2).$$

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* *Ges. Werke*, t. I., pp. 133–152, pp. 297–355; the special case required is given by $n = 2$.

Thus the equations of a geodesic are given by

$$\left.\begin{aligned} x &= \sqrt{a} \qquad\qquad al_0\,(a-\gamma u,\ a-\beta u) \\ y &= \beta\left(\frac{\beta-\delta}{\beta-\gamma}\right)^{\frac{1}{2}} al_1\,(a-\gamma u,\ a-\beta u) \\ z &= \gamma\left(\frac{\delta-\gamma}{\beta-\gamma}\right)^{\frac{1}{2}} al_2\,(a-\gamma u,\ a-\beta u) \end{aligned}\right\},$$

where a and δ are the arbitrary constants which can be determined by assigning any two points on the ellipsoid as points through which a geodesic is to be drawn and u is the parameter of the curve so drawn.

Again,
$$\begin{aligned} \frac{\lambda_1\lambda_2}{\beta\gamma} &= \frac{(a_4-\lambda_1)(a_4-\lambda_2)}{(a_4-a_1)(a_4-a_2)} \\ &= al_4^2\,(u_1,\ u_2) \\ &= 1+\frac{1}{a_1-a_4}\frac{\partial U}{\partial u_1}+\frac{1}{a_2-a_4}\frac{\partial U}{\partial u_2}, \end{aligned}$$

where U is the integral-function defined* by the equation

$$U=\int_\beta^{\lambda_1}+\int_\gamma^{\lambda_2}\frac{(\theta-\beta)(\theta-\gamma)}{2\sqrt{R\theta}}\,d\theta.$$

Thus
$$\lambda_1\lambda_2=\beta\gamma+\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}.$$

But
$$\begin{aligned} dU &= \frac{\partial U}{\partial u_1}du_1+\frac{\partial U}{\partial u_2}du_2, \text{ in general,} \\ &= -\left(\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}\right)du, \text{ in the present case;} \end{aligned}$$

and therefore

$$\begin{aligned} ds &= -\lambda_1\lambda_2\,du \\ &= -\beta\gamma\,du-\left(\gamma\frac{\partial U}{\partial u_1}+\beta\frac{\partial U}{\partial u_2}\right)du \\ &= dU-\beta\gamma\,du. \end{aligned}$$

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* Weierstrass, *l.c.*, pp. 337–346.

Consequently

$$s = \left[U - \beta\gamma u\right],$$

the right-hand side being taken between the values of u at two points on the geodesic, expresses the length of the arc between those points.

[*Added March* 16*th*, 1896.—The result can be obtained also as follows:—By the equations in § 4, we have

$$\frac{\dfrac{d\lambda_1}{2\sqrt{R\lambda_1}}}{\lambda_2} = \frac{\dfrac{d\lambda_2}{2\sqrt{R\lambda_2}}}{-\lambda_1} = \theta, \text{ say,}$$

so that

$$du = (\lambda_2 - \lambda_1)\,\theta.$$

Now

$$\begin{aligned} dU &= \frac{(\lambda_1-\beta)(\lambda_1-\gamma)}{2\sqrt{R\lambda_1}}\,d\lambda_1 + \frac{(\lambda_2-\beta)(\lambda_2-\gamma)}{2\sqrt{R\lambda_2}}\,d\lambda_2 \\ &= \theta\left\{\lambda_2(\lambda_1-\beta)(\lambda_1-\gamma) - \lambda_1(\lambda_2-\beta)(\lambda_2-\gamma)\right\} \\ &= \lambda_1\lambda_2(\lambda_1-\lambda_2)\,\theta + \beta\gamma(\lambda_2-\lambda_1)\,\theta \\ &= -\lambda_1\lambda_2\,du + \beta\gamma\,du \\ &= ds + \beta\gamma\,du, \end{aligned}$$

as before.]

(III.) In the case when $\delta > \beta$ and the geodesic undulates between the two lines of curvature that are the complete intersection of the ellipsoid and a confocal hyperboloid of two sheets, the result can similarly be expressed in terms of hyperelliptic functions. We now have

$$a > \lambda_1 > \delta > \beta > \lambda_2 > \gamma > 0,$$

and we take

$$a, \delta, \beta, \gamma, 0 = a_0, a_1, a_2, a_3, a_4.$$

Then introducing

$$\left.\begin{aligned} u_1 &= \int_\delta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\gamma}{2\sqrt{R\theta}}\,d\theta \\ u_2 &= \int_\delta^{\lambda_1} + \int_\gamma^{\lambda_2} \frac{\theta-\delta}{2\sqrt{R\theta}}\,d\theta \end{aligned}\right\},$$

so that

$$u_1 = a - \gamma u, \quad u_2 = a - \delta u,$$

we easily find

$$\left.\begin{aligned} x &= \sqrt{\alpha}\left(\frac{\alpha-\delta}{\alpha-\beta}\right)^{\frac{1}{2}} al_0(a-\gamma u,\ a-\delta u) \\ y &= \sqrt{\beta}\left(\frac{\delta-\beta}{\alpha-\beta}\right)^{\frac{1}{2}} al_2(a-\gamma u,\ a-\delta u) \\ z &= \gamma\, al_3(a-\gamma u,\ a-\delta u) \end{aligned}\right\},$$

In these expressions a and δ are the two arbitrary constants; they can be determined by any two points through which the geodesic passes. And u is the current parameter of the geodesic.

To find the arc, we introduce the integral-function U, where

$$U = \int_{\delta}^{\lambda_1} + \int_{\gamma}^{\lambda_2} \frac{(\theta-\gamma)(\theta-\delta)}{2\sqrt{R\theta}}\, d\theta\,;$$

and then the arc between any two points is equal to

$$\left[U-\beta\gamma u\right],$$

between the limiting values of u that determine the two points.*

It has been assumed throughout that $\alpha > \beta > \gamma$. Special cases arise when $\alpha = \beta$, viz., an oblate spheroid, and when $\beta = \gamma$, viz., a prolate spheroid. The corresponding formulæ then belong to elliptic functions.†

8. If numerical approximations are desired, they can be obtained, as pointed out by Weierstrass in his paper already quoted, by using the double theta-functions. The Abelian functions, that occur in the preceding solution, are expressible as quotients of these theta-functions in forms substantially agreeing with results first given by Rosenhain;‡ and when once the parameters, being small quantities for a surface nearly spherical, are determined, expansions can be obtained to any degree of accuracy required.

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* For the umbilical geodesics, see a paper by Cayley, "On the Geodesics on an Ellipsoid," *Coll. Math. Papers*, Vol. VII., 478. For the general geodesics on an ellipsoid, the paper by Weierstrass, referred to in § 4, should be consulted; also two papers by Cayley, *Coll. Math. Papers*, Vol. VIII., 508, 511.

† For the case of an oblate spheroid, see a paper by the author, *Messenger of Mathematics*, Vol. XXV. (1896), pp. 81–124.

‡ "Mémoire sur les fonctions de deux variables et à quatre périodes," *Mém. des Savans Etr.*, t. XI., p. 361; the memoir is dated 1846.

9. When the given quadric is a *hyperboloid of one sheet*, we have

$$\alpha > \beta > 0 > \gamma.$$

The roots of the equation

$$\frac{x^2}{\alpha-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1$$

must correspond to an ellipsoid and a hyperboloid of two sheets. For the former, we have

$$\gamma > \lambda_2,$$

both of course being negative; for the latter, we have

$$\alpha > \lambda_1 > \beta.$$

In order to have real geodesics, both $R\lambda_1$ and $R\lambda_2$ must be positive. The former is positive if $\delta < \lambda_1$, the latter if $\delta > \lambda_2$; so that

$$\lambda_1 > \delta > \lambda_2.$$

Combining the inequalities, we have

$$\alpha > \lambda_1 > \left\{ \begin{matrix} \beta > 0 > \gamma \\ \delta \end{matrix} \right\} > \lambda_2.$$

There are seven cases, viz.,

(I.) $\delta > \beta > 0 > \gamma$,

(II.) $\delta = \beta > 0 > \gamma$,

(III.) $\beta > \delta > 0 > \gamma$,

(IV.) $\beta > \delta = 0 > \gamma$,

(V.) $\beta > 0 > \delta > \gamma$,

(VI.) $\beta > 0 > \delta = \gamma$,

(VII.) $\beta > 0 > \gamma > \delta$.

10. To discriminate these cases, we consider the configuration of the surface in the immediate vicinity of x, y, z, and compare it with the central section by a plane parallel to the tangent plane at the point. The generators are parallel to the asymptotes of the central section; the angles between the generators are bisected by the lines of curvature, which are parallel to the axes of the central section; and that angle between the generators in which the ellipsoidal line of curvature lies corresponds to that angle between the asymptotes in which the real part of the curve of the central section lies, say, the *internal* angle of the asymptotes.

Now, by § 4, we have

$$\frac{1}{D^2} = \frac{\delta}{\lambda_1 \lambda_2};$$

and in the present case λ_1 is positive, λ_2 is negative. Hence, when δ is positive, D^2 is negative; and the direction of the geodesic lies within the external angle of the generators. When δ is zero, D is infinite; and the direction of the geodesic is one of the generators. When δ is negative, D^2 is positive; and the direction of the geodesic lies within the internal angle of the generators.

If a geodesic can cross the principal section in the plane $z = 0$, we have there

$$\lambda_2 = \gamma.$$

Now, at any point,

$$-\frac{2}{\Gamma^{\frac{1}{2}}} \frac{dz}{ds} = \left(\frac{\gamma-\lambda_2}{\gamma-\lambda_1}\right)^{\frac{1}{2}} \frac{d\lambda_1}{ds} + \left(\frac{\gamma-\lambda_1}{\gamma-\lambda_2}\right)^{\frac{1}{2}} \frac{d\lambda_2}{ds},$$

and $$\frac{d\lambda_1}{ds} = \frac{2\sqrt{R\lambda_1}}{\lambda_1(\lambda_1-\lambda_2)}, \quad \frac{d\lambda_2}{ds} = \frac{2\sqrt{R\lambda_2}}{\lambda_2(\lambda_2-\lambda_1)},$$

where the positive value has to be assigned to the real radicals $\sqrt{R\lambda_1}$ and $\sqrt{R\lambda_2}$, that is,

$$\sqrt{R\lambda_1} = \sqrt{\lambda_1(a-\lambda_1)(\lambda_1-\beta)(\lambda_1-\gamma)(\lambda_1-\delta)},$$

$$\sqrt{R\lambda_2} = \sqrt{-\lambda_2(a-\lambda_2)(\beta-\lambda_2)(\gamma-\lambda_2)(\delta-\lambda_2)}.$$

Substituting and then making $\lambda_2 = \gamma$, we have

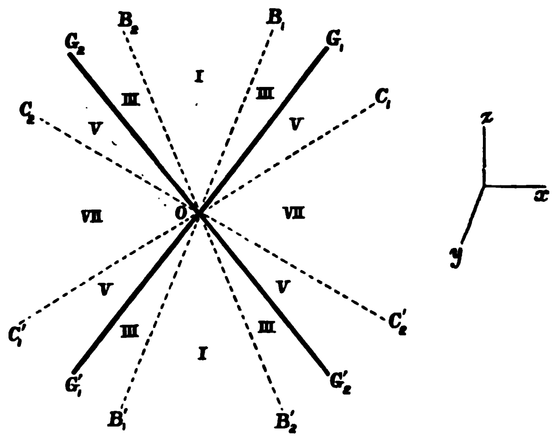
$$-\frac{dz}{ds} = \frac{\{\Gamma(\gamma-\lambda_1)\}^{\frac{1}{2}}}{\gamma(\gamma-\lambda_1)} \sqrt{-\gamma(a-\gamma)(\beta-\gamma)(\delta-\gamma)}.$$

Now Γ is negative, as is also $\gamma-\lambda_1$; thus the first radical on the right-hand side is real. Again $-\gamma$, $a-\gamma$, $\beta-\gamma$ are positive; hence, if $\delta > \gamma$, the value of $\frac{dz}{ds}$ is real. In this case, the geodesic crosses the principal section under consideration.

If $\delta = \gamma$, then $\frac{dz}{ds} = 0$ at the point; in this case the geodesic touches the principal section but does not cross it.

If $\delta < \gamma$, then $\frac{dz}{ds}$ is imaginary; that is, the geodesic cannot meet the principal section.

11. In the figure, G_1G_1' and G_2G_2' are the generators at the point O; they give the directions of the geodesics corresponding to $\delta = 0$. This is Case (IV.).



The lines B_1B_1' and B_2B_2' are lines equally inclined to the generators; they give the directions of the geodesics through O corresponding to $\delta = \beta$. This is Case (II.).

For any direction lying within the angles B_1OB_2 and $B_1'OB_2'$, we have $\delta > \beta$. Thus Case (I.) gives geodesics through O whose directions lie within one of the two regions marked (I.); one special line is the geodesic which touches the hyperboloidal line of curvature through O, the value of δ then being λ_1.

For any direction lying within one of the angles B_1OG_1, B_2OG_2, $B_2'OG_2'$, $B_1'OG_1'$, we have $\beta > \delta > 0$. Thus Case (III.) gives geodesics through O whose directions lie within one of the four regions marked (III.).

The lines C_1C_1' and C_2C_2' are lines equally inclined to the generators; they give the directions of the geodesics through O corresponding to $\delta = \gamma$. This is Case (VI.).

For any direction lying within one of the angles C_1OG_1, $C_1'OG_1'$, C_2OG_2, $C_2'OG_2'$, we have $O > \delta > \gamma$. Thus Case (V.) gives geodesics through O whose directions lie within one of the four regions marked (V.).

For any direction lying within the angles C_1OC_2' and C_2OC_1', we have $\delta < \gamma$. Thus Case (VII.) gives geodesics through O whose directions lie within one of the regions marked (VII.); one special line is the geodesic which touches the ellipsoidal line of curvature through O, the value of δ then being λ_2.

Geodesics through O whose directions lie within (but not on the boundary of) either of the angles C_1OC_2 and $C_1'OC_2'$ cross the principal elliptic section of the surface when they are continued.

The two geodesics through O whose directions are the lines C_1C_1' and C_2C_2' at that point touch, but do not cross, the principal elliptic section.

Geodesics through O whose directions lie within (but not on the boundary of) either of the angles C_1OC_2' and C_2OC_2' do not meet the principal elliptic section of the surface. Each of them touches an ellipsoidal line of curvature, determined by the value of δ; and extends, on either side of this point of contact, towards infinity away from the principal elliptic section. By this extension of the geodesic is implied a curve at every part of which the characteristic geodesic property is possessed; but the length of the arc of this curve between any two points of it is not necessarily the shortest surface-distance between the two points.

12. The course of the geodesic can be indicated by expressing the coordinates of any point on it in terms of a single parameter. The expressions in Cases (I.), (III.), (V.), (VII.) require hyper-elliptic functions as in two of the cases on the surface of the ellipsoid; in Cases (II.) and (VI.), elliptic functions and elliptic integrals of the third kind occur; in Case (IV.), the expressions are algebraical.

13. When the given quadric is a *hyperboloid of two sheets*, we have

$$\alpha > 0 > \beta > \gamma.$$

The roots, other than zero, of the equation

$$\frac{x^2}{\alpha-\theta}+\frac{y^2}{\beta-\theta}+\frac{z^2}{\gamma-\theta}=1$$

must correspond to an ellipsoid and a hyperboloid of one sheet. For the former, we have

$$\gamma > \lambda_2,$$

both of course being negative; for the latter, we have

$$\beta > \lambda_1 > \gamma.$$

In order to have real geodesics, we must have $R\lambda_1$ positive, a condition which is satisfied if $\delta < \lambda_1$; and we must have $R\lambda_2$ positive, a condition which is satisfied if $\delta > \lambda_2$, so that

$$\lambda_1 > \delta > \lambda_2.$$

Combining these inequalities, we have

$$\alpha > 0 > \beta > \lambda_1 > \left\{ \begin{matrix} \gamma \\ \delta \end{matrix} \right\} > \lambda_2.$$

There are three cases, viz.,

$$\text{(I.)} \quad \gamma = \delta,$$

$$\text{(II.)} \quad \gamma > \delta,$$

$$\text{(III.)} \quad \gamma < \delta.$$

14. The cases are similar to those that occur in the ellipsoid.

The first represents a geodesic passing through an umbilicus, but, with a single exception, not through the other umbilicus on the same sheet; beyond these points, it extends towards infinity.

The second represents a geodesic touching one ellipsoidal line of curvature and extending towards infinity in both directions.

The third represents a geodesic touching one line of curvature that lies upon a confocal hyperboloid of one sheet and extending towards infinity in both directions.

The last two require hyper-elliptic functions for the explicit expression of the variables along the course of the curve; the first, for the same purpose, requires elliptic integrals of the third kind.

15. It is unnecessary to consider, in any detail, geodesics on a *cone* or *cylinder;* their characteristic equation for such a surface can be deduced from the property that, when a developable surface is developed, the geodesic gives rise to a straight line on the developed surface. Thus, for instance, on a cone we should have

$$r \sin \phi = \text{constant};$$

where the constant is the parameter of the geodesic, r is the distance of any point on it from the vertex of the cone, and ϕ is the angle between the direction of the geodesic at the point and the generator through the point.

Non-Central Quadrics. §§ 16–22.

16. When the quadric is paraboloidal, its equation can be taken in the form

$$\frac{y^2}{a} + \frac{z^2}{c} = 4x.$$

When the paraboloid is *elliptic*, we have

$$a > c > 0;$$

when it is *hyperbolic*, we have

$$a > 0 > c.$$

The confocal paraboloids are given by

$$\frac{y^2}{a-k} + \frac{z^2}{c-k} = 4(x-k),$$

a cubic equation in k for each point x, y, z. One root is zero; let the others be k_1 and k_2, of which k_1 is assumed the greater. Then 0, k_1, k_2 are the roots of

$$4(a-k)(c-k)(x-k) - y^2(c-k) - z^2(a-k) = 0.$$

It is easily seen* that the roots are separated by ∞, a, c, $-\infty$. Hence in the case of the *elliptic paraboloid* we have

$$\infty > k_1 > a > k_2 > c > 0;$$

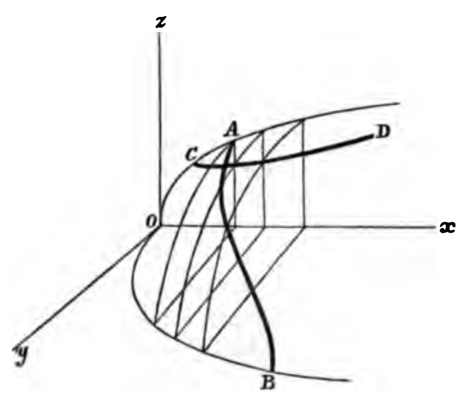
k_1 determines an elliptic paraboloid and k_2 a hyperbolic paraboloid. And in the case of the *hyperbolic paraboloid*, we have

$$\infty > k_1 > a > 0 > c > k_2;$$

k_1 and k_2 determine elliptic paraboloids.

17. The intersections of the confocal surfaces are lines of curvature on each of them.

Consider first the elliptic paraboloid.



* Frost's *Solid Geometry*, p. 138.

Its intersection with the confocal elliptic paraboloid is a curve one quarter of which is AB; when this curve is orthogonally projected on the plane of yz, it becomes the ellipse

$$\frac{y^2}{a(k_1-a)}+\frac{z^2}{c(k_1-c)}=4.$$

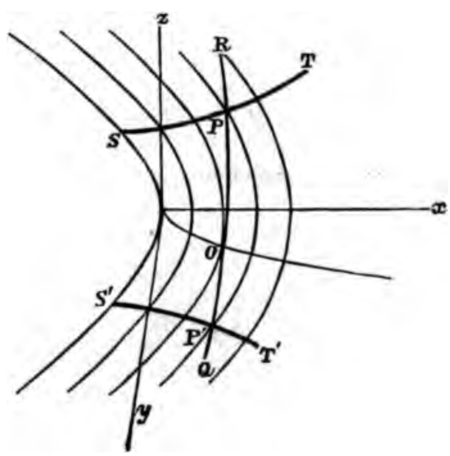
This curve is the whole of the real intersection with the confocal elliptic paraboloid.

The intersection with the confocal hyperbolic paraboloid consists of two curves. One half of one of them is CD, the other half of it being on the negative side of the plane zx; and the other curve is the reflexion of this curve in the plane of xy. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(a-k_2)}-\frac{z^2}{c(k_2-c)}=4.$$

The two real curves constitute the whole intersection with the confocal hyperbolic paraboloid.

Now consider the hyperbolic paraboloid. Its intersection with the confocal elliptic paraboloid determined by k_1 consists of two curves;



one is $QOPR$ ..., and the other is the reflexion of this curve in the plane of zx. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(k_1-a)}+\frac{z^2}{c(k_1-c)}=4.$$

These two (real) curves constitute the whole intersection.

The intersection with the confocal elliptic paraboloid determined by k_2 consists of two curves; $SPT\ldots$, $S'P'T'\ldots$ are halves of them, the other halves being their reflexion in the plane of xz. When these curves are orthogonally projected on the plane of yz, they become the two branches of the hyperbola

$$\frac{y^2}{a(a-k_2)}+\frac{z^2}{c(c-k_2)}=-4.$$

These two (real) curves constitute the whole intersection.

18. Take any point on a paraboloid and consider the geodesics through the point. If l, m, n denote the direction of the curve there, if p be the perpendicular from the vertex upon the tangent plane at the point, and if D denote the length of the chord through the vertex parallel to the geodesic direction, then* we have

$$\frac{lx^2}{p^2D}$$

constant along a curve. And, by an investigation similar to that contained in §§ 2 and 3, it can be proved—the analysis is not reproduced here—that the equation

$$\frac{d}{ds}\left(\frac{lx^2}{p^2D}\right)=0$$

determines upon the paraboloid either a geodesic or one of the lines of curvature through the point. If then the quantities k_1 and k_2 be introduced, the lines of curvature are given by

$$\frac{dk_1}{ds}=0,\quad \frac{dk_2}{ds}=0;$$

the equation $$\frac{d}{ds}\left(\frac{lx^2}{p^2D}\right)=0,$$

or $$\frac{lx^2}{p^2D}=\text{constant},$$

when transformed, will then represent a proper geodesic. Now

$$\frac{x^2}{p^2}=1+\frac{y^2}{4a^2}+\frac{z^2}{4c^2},$$

$$4\frac{l}{D}=\frac{m^2}{a}+\frac{n^2}{c};$$

* Frost's *Solid Geometry*, p. 320.

hence the equation characteristic of geodesics is

$$\left(\frac{m^2}{a}+\frac{n^2}{c}\right)\left(1+\frac{y^2}{4a^2}+\frac{z^2}{4c^2}\right) = \text{constant}$$

$$= \frac{1}{b}, \text{ say.}$$

Further, it is only upon the elliptic paraboloid that the umbilici are real. They are given by

$$x_1 = a-c, \quad y_1 = 0, \quad z_1 = 2\sqrt{c\,(a-c)};$$

also, for any direction in the tangent plane at an umbilicus, we have

$$-2l+\frac{nz_1}{c} = 0,$$

so that

$$l = n\sqrt{\frac{a-c}{c}}.$$

Thus

$$\frac{l^2}{a-c} = \frac{n^2}{c} = \frac{1-m^2}{a},$$

so that

$$\frac{m^2}{a}+\frac{n^2}{c} = \frac{1}{a}.$$

And

$$1+\frac{y_1^2}{4a^2}+\frac{z_1^2}{4c^2} = \frac{a}{c};$$

hence for a geodesic through an umbilicus the constant is

$$\frac{a}{c}\,\frac{1}{a} = \frac{1}{c}.$$

If therefore $b = c$, the geodesic passes through an umbilicus.

19. To use the parameters of the confocal paraboloids, we have

$$\frac{y^2}{a-k}+\frac{z^2}{c-k}-4\,(x-k) = 4\,\frac{k\,(k-k_1)(k-k_2)}{(a-k)(c-k)};$$

so that

$$y^2 = 4\,\frac{a\,(a-k_1)(a-k_2)}{c-a},$$

$$z^2 = 4\,\frac{c\,(c-k_1)(c-k_2)}{a-c};$$

and then
$$x = \frac{y^2}{4a} + \frac{z^2}{4c}$$
$$= k_1 + k_2 - a - c.$$

With these values, we have
$$1 + \frac{y^2}{4a^2} + \frac{z^2}{4c^2} = \frac{k_1 k_2}{ac},$$

so that the equation of the geodesic is
$$\frac{m^2}{a} + \frac{n^2}{c} = \frac{ac}{b}\,\frac{1}{k_1 k_2} = \frac{f}{k_1 k_2},$$

where
$$f = \frac{ac}{b};$$

and $f = a$ for a geodesic that passes through an umbilicus.

Now $l, m, n, = \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, respectively; thus
$$l = \frac{dk_1}{ds} + \frac{dk_2}{ds},$$
$$-m = \sqrt{\frac{a}{c-a}}\left\{\sqrt{\frac{a-k_2}{a-k_1}}\,\frac{dk_1}{ds} + \sqrt{\frac{a-k_1}{a-k_2}}\,\frac{dk_2}{ds}\right\},$$
$$-n = \sqrt{\frac{c}{a-c}}\left\{\sqrt{\frac{c-k_2}{c-k_1}}\,\frac{dk_1}{ds} + \sqrt{\frac{c-k_1}{c-k_2}}\,\frac{dk_2}{ds}\right\}.$$

Substituting these values in
$$l^2 + m^2 + n^2 = 1,$$
we find
$$\frac{k_1(k_1-k_2)}{(a-k_1)(c-k_1)}\left(\frac{dk_1}{ds}\right)^2 + \frac{k_2(k_2-k_1)}{(a-k_2)(c-k_2)}\left(\frac{dk_2}{ds}\right)^2 = 1;$$

and substituting them in
$$\frac{m^2}{a} + \frac{n^2}{c} = \frac{f}{k_1 k_2},$$
we find
$$\frac{(k_1-k_2)}{(a-k_1)(c-k_1)}\left(\frac{dk_1}{ds}\right)^2 + \frac{(k_2-k_1)}{(a-k_2)(c-k_2)}\left(\frac{dk_2}{ds}\right)^2 = \frac{f}{k_1 k_2}.$$

Let
$$\left.\begin{aligned} K_1 &= k_1(k_1-a)(k_1-c)(k_1-f) \\ K_2 &= k_2(k_2-a)(k_2-c)(k_2-f) \end{aligned}\right\};$$

then, when these equations are solved for $\left(\frac{dk_1}{ds}\right)^2$ and $\left(\frac{dk_2}{ds}\right)^2$, we have

$$\left.\begin{aligned} k_1(k_1-k_2)\frac{dk_1}{ds} &= \surd K_1 \\ k_2(k_2-k_1)\frac{dk_2}{ds} &= \surd K_2 \end{aligned}\right\}.$$

Hence the equations of a geodesic upon a paraboloid are

$$\left.\begin{aligned} \frac{k_1\,dk_1}{\surd K_1}+\frac{k_2\,dk_2}{\surd K_2} &= 0 \\ \frac{dk_1}{\surd K_1}+\frac{dk}{\surd K_2} &= du \\ -\frac{ds}{k_1k_2} &= du \end{aligned}\right\},$$

which correspond in form to those obtained in § 5 for a central quadric.

It would have been possible to deduce these results from the results in the case of a central quadric by changing the origin to a vertex of the latter and then passing to the limiting case, in which two of the semi-axes are made to increase without limit subject to the customary conditions.

20. In the case of the *elliptic paraboloid*, we have

$$k_1 > a > k_2 > c > 0.$$

Hence, in order that the geodesics may be real, we must have

$$k_1 > f,$$

$$k_2 < f,$$

that is,

$$k_1 > f > k_2;$$

and therefore the aggregate of conditions is

$$k_1 > \binom{a}{f} > k_2 > c > 0.$$

There are therefore three distinct classes to consider, viz.,

(I.) $f = a$,

(II.) $f < a$,

(III.) $f > a$.

They correspond to the three classes in the case of an unruled central quadric.

For the first of these classes, we have $f = a$; the geodesic passes through an umbilicus (but not through the other umbilicus) in the finite part of the surface.

To discriminate between the other classes, a simple method is to trace the course of a geodesic through the variations of k_1 and k_2. We have

$$\frac{k_1\,dk_1}{\sqrt{K_1}} + \frac{k_2\,dk_2}{\sqrt{K_2}} = 0,$$

$$\frac{dk_1}{\sqrt{K_1}} + \frac{dk_2}{\sqrt{K_2}} = du,$$

and therefore $$\frac{dk_1}{du} = \frac{k_2}{k_2 - k_1}\sqrt{K_1}, \quad \frac{dk_2}{du} = \frac{k_1}{k_1 - k_2}\sqrt{K_2}.$$

Thus k_1, for finite values of k_1, can be a maximum or a minimum, only when $K_1 = 0$; and, for all other values, K_1 must be positive. The only possible roots of K_1 are

$$k_1 = a, \quad k_1 = f;$$

and, for values of k_1 that are not roots,

$$(k_1 - a)(k_1 - f)$$

must be positive.

Hence when $f > a$, the only possible root is $k_1 = f$; and all other admissible values of k_1 must be greater than f. When $f < a$, the only possible root is $k_1 = a$; and all other admissible values of k_1 must be greater than a.

Again, k_2 can be a maximum or a minimum only when $K_2 = 0$; and, for all other values, K_2 must be positive. The only possible roots of K_2 are

$$k_2 = c, \quad k_2 = f;$$

and, for values of k_2 that are not roots,

$$(k_2 - c)(k_2 - f)$$

must be negative.

Hence when $f > a$, the only possible root is $k_2 = c$; all other admissible values of k_2 must lie between c and a. When $f < a$, both $k_2 = c$, $k_2 = f$ are possible roots; all other admissible values of k_2 lie between c and a.

Moreover, $k_2 = c$ refers to the (confocal) parabola in the plane $z = 0$, where the two curves of intersection of the given paraboloid with the confocal hyperbolic paraboloid coincide and become the (doubled) parabolic section OB... .

21. These inferences, when combined, lead to the main result, as follows.

When $f > a$, the geodesic touches the line of curvature AB, and, from the point of contact, extends to infinity in both directions away from the vicinity of the vertex O.

When $f < a$, the geodesic touches the line of curvature CD, and the other line of curvature which, with CD, forms the complete intersection of the given paraboloid with the confocal hyperbolic paraboloid. The geodesic undulates between these two lines, and, in passing from the vicinity of O, extends to infinity on both sides of the plane $y = 0$.

When $f = a$, the substitutions

$$k_1 = c\sec^2\phi_1, \quad \text{where} \quad \tfrac{1}{2}\pi > \phi_1 > \cos^{-1}\frac{c}{a},$$

$$k_2 = c\sec^2\phi_2, \quad ,, \quad \cos^{-1}\frac{c}{a} > \phi_2 > 0,$$

transform the equations

$$\frac{k_1\,dk_1}{\sqrt{K_1}} + \frac{k_2\,dk_2}{\sqrt{K_2}} = 0,$$

$$\frac{dk_1}{\sqrt{K_1}} + \frac{dk_2}{\sqrt{K_2}} = du,$$

so that the integrals can be expressed in terms of logarithmic and trigonometric functions.

When $f > a$, the transformation

$$k = f\,\frac{1-\theta^2}{1-\dfrac{f}{a}\theta^2},$$

and, when $f < a$, the transformation

$$k = a\,\frac{1-\theta^2}{1-\dfrac{a}{f}\theta^2},$$

transform the integrals into elliptic integrals (in Jacobi's form) having their moduli determined as the quantities

$$\left(\frac{fa-fc}{fa-ac}\right)^{\frac{1}{2}} \quad \text{and} \quad \left(\frac{af-ac}{af-fc}\right)^{\frac{1}{2}}$$

in the respective cases.

22. In the case of the *hyperbolic paraboloid*, we have

$$k_1 > a > 0 > c > k_2.$$

In order that the geodesics may be real, we must have

$$k_1 > f,$$

$$k_2 < f,$$

that is, $$k_1 > f > k_2;$$

and therefore the aggregate of conditions is

$$k_1 > \left\{ \begin{matrix} a > 0 > c \\ f \end{matrix} \right\} > k_2.$$

There are therefore seven distinct classes to consider, viz.,

$$\text{(I.)}\quad k_1 > f > a > 0 > c > k_2,$$

$$\text{(II.)}\quad k_1 > f = a > 0 > c > k_2,$$

$$\text{(III.)}\quad k_1 > a > f > 0 > c > k_2,$$

$$\text{(IV.)}\quad k_1 > a > f = 0 > c > k_2,$$

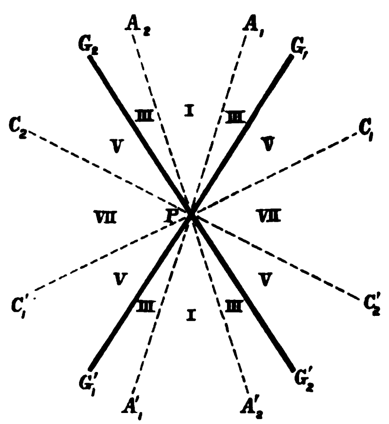
$$\text{(V.)}\quad k_1 > a > 0 > f > c > k_2,$$

$$\text{(VI.)}\quad k_1 > a > 0 > f = c > k_2,$$

$$\text{(VII.)}\quad k_1 > a > 0 > c > f > k_2,$$

They correspond to the seven classes in the case of a ruled central quadric.

The regions in the vicinity of a point on the surface to which the respective classes belong can be determined in the same manner as in the case of the hyperboloid of one sheet. Thus in the figure the



lines G_1G_1' and G_2G_2' are the generators through the point P; these give the geodesics corresponding to Case (IV.).

The lines $A_1'A_1$, A_2A_2' give directions through P on the surface that determine the geodesics corresponding to Case (II.).

The lines C_1C_1', C_2C_2' give directions through P on the surface that determine the geodesics corresponding to Case (VI.).

Every geodesic through P belonging to Class (I.) has its direction at P lying within (but not on the boundary of) one of the angles A_1PA_2, $A_1'PA_2'$.

Every geodesic through P belonging to Class (III.) has its direction at P lying within (but not on the boundary of) one of the angles A_1PG_1, A_2PG_2, $A_1'PG_1'$, $A_2'PG_2'$.

Every geodesic through P belonging to Class (V.) has its direction at P lying within (but not on the boundary of) one of the angles C_1PG_1, G_2PC_2, $C_1'PG_1'$, $G_2'PC_2'$.

Every geodesic through P belonging to Class (VII.) has its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2', C_2PC_1'.

Every geodesic through P that has its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2, $C_1'PC_2'$ will, when produced, cut and cross the principal section of the surface by the plane $z = 0$.

The two geodesics through P having C_1PC_1' and C_2PC_2' as their directions through P will, when produced, touch, but not cross, this principal section of the surface.

And, lastly, no geodesic through P having its direction at P lying within (but not on the boundary of) one of the angles C_1PC_2', C_2PC_1' can, however far it may be produced, meet this principal section of the surface.

The differential equations of Classes (I.), (III.), (V.), (VII.) require the introduction of elliptic integrals for their integration; those belonging to Classes (II.) and (III.) can be integrated by logarithmic and circular functions; and those belonging to Class (IV.) can be integrated by algebraical functions.

*The Continuity of Pressure in Vortex Motion.** *By* R. Hargreaves, M.A. Received February 3rd, 1896. Read February 13th, 1896. Received, in recast form, May 15th, 1896.

The constructive part of this paper is based on, and forms an extension of, the third section of Helmholtz's "Vortex Motion," which is entitled "Space Integration." In that section the velocities are obtained as solutions of

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz},$$

with two similar equations and the equation of continuity, where ξ, η, ζ are understood to be assigned functions of x, y, z which satisfy the condition

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0,$$

and at the surface of the vortex make

$$l\xi + m\eta + n\zeta = 0,$$

these conditions defining a kinematically possible spin. Dependence on time is not excluded, as the parameters occurring in the functions ξ, η, ζ, or those defining the configuration may be functions of time; but, as the time does not appear explicitly, it seems proper to describe the velocities obtained in this way as *instantaneous* values which may, under certain circumstances, become functionally complete. The special feature of Helmholtz's potential method is that it gives values of the velocities continuous at the surface, and with the discontinuity in their first differential coefficients proper to the change from rotational to irrotational motion; the use of the method is therefore exactly equivalent to the assumption that no slipping takes place at the vortex surface.

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* As originally read this paper formed part of a communication to the Society on an Ellipsoidal Vortex. The object of the section was to explain on general grounds why a certain process, directed to securing continuity of motion, gave also continuity of pressure in an identical manner. The theorem in its general shape not being essential to the work on the ellipsoidal vortex, it seemed desirable to remove it from an environment of ellipsoidal harmonics, and the author availed himself of the permission of the Council to print under a separate title, and to incorporate some additional matter.

I propose to apply the same method to the ensuing phase of the motion, and in this way to find the instantaneous values of $\frac{du}{dt}, \frac{dv}{dt}, \frac{dw}{dt}$, which will be found to be such as secure the continuity of pressure at the surface. In its simplest form we may enunciate the theorem as follows :—Corresponding to any kinematically possible distribution of spin within a closed surface at any instant, there exists a continuous motion of infinite liquid with continuous pressure, such that within the surface the motion is rotational and has the given spin at the instant, and outside the surface the motion is irrotational. In effect, we know the instantaneous rate of deformation of the vortex surface, and the instantaneous rate of alteration of vortex distribution; hence we infer the instantaneous acceleration, and so the pressure. The discontinuities in the differential coefficients of velocity and pressure are also examined directly from the hydrodynamical equations, and the results shown to be in conformity with the Helmholtz solution. As regards pressure, the discontinuity first appears in the second differential coefficients, and has the character which belongs to the discontinuity in the potential of a volume distribution of matter, where there is an abrupt change of density in crossing a surface. Finally, the results are applied to the oscillations of a vortex about a state of steady motion, and it is shown that, when a solution gives continuity of velocity at the disturbed surface and satisfies the condition that the surface of the vortex always contains the same particles, the continuity of pressure is *ipso facto* secured.

1. Denote components of velocity within the vortex by u, v, w and without it by u', v', w'; take L, M, N for internal and L', M', N' for external values of the vortex-potentials, and suppose the liquid to extend to infinity.

At a surface of separation of rotational and irrotational motions, the first differential coefficients of L, L', ... are continuous, and therefore also the components of velocity. The second differential coefficients present discontinuities of the type

$$\frac{d^2L}{dx^2} + 2\xi l^2 = \frac{d^2L'}{dx^2}, \quad \frac{d^2L}{dx\,dy} + 2\xi lm = \frac{d^2L'}{dx\,dy}, \; \ldots,$$

l, m, n being direction-cosines of the normal to the surface (see

Kirchhoff, *Vorlesungen über Mechanik*, p. 179). Hence we obtain, by differentiating

$$u = \frac{dN}{dy} - \frac{dM}{dz} \quad \text{and} \quad u' = \frac{dN'}{dy} - \frac{dM'}{dz},$$

$$\left.\begin{aligned} \frac{du}{dx} - \frac{du'}{dx} = -2l\,(m\zeta - n\eta), \quad \frac{du}{dy} - \frac{du'}{dy} = -2m\,(m\zeta - n\eta), \\ \frac{du}{dz} - \frac{du'}{dz} = -2n\,(m\zeta - n\eta) \end{aligned}\right\} \quad \ldots(1),$$

and similar expressions for v and w, representing the discontinuities of the first differential coefficients of the components of velocity. Hence immediately

$$u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} - \left\{ u'\frac{du'}{dx} + v'\frac{du'}{dy} + w'\frac{du'}{dz'} \right\}$$

$$= -2\,(lu + mv + nw)(m\zeta - n\eta) \ldots\ldots (2a),$$

or, if the origin of coordinates be moving with velocities u_0, v_0, w_0, we may write

$$(u - u_0)\frac{du}{dx} + (v - v_0)\frac{du}{dy} + (w - w_0)\frac{du}{dz}$$

$$- \left\{ (u' - u_0)\frac{du'}{dx} + (v' - v_0)\frac{du'}{dy} + (w' - w_0)\frac{du'}{dz} \right\}$$

$$= -2\left\{ l\,(u - u_0) + m\,(v - v_0) + n\,(w - w_0) \right\}(m\zeta - n\eta)$$

$$= -2U\,(m\zeta - n\eta), \text{ say} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2b).$$

The surface condition for translation of the vortex without change of shape is that U should vanish; but, if U does not vanish, a change of shape accompanies the translation, the rate of extension along the normal being measured by U. Denote by $u + f\,dt$, ... the velocities at x, y, z and time $t + dt$, due to the altered vortex, which has $\xi + \frac{d\xi}{dt}dt$ in lieu of ξ, and which, apart from mere translation, extends to a normal distance $U\,dt$ beyond its original surface. It is clear that f, g, h, the components of the instantaneous acceleration, are derived from vector potentials of volume distributions with densities $\frac{1}{2\pi}\frac{d\xi}{dt}$, $\frac{1}{2\pi}\frac{d\eta}{dt}$, $\frac{1}{2\pi}\frac{d\zeta}{dt}$, and surface distributions with densities $\frac{U\xi}{2\pi}$, $\frac{U\eta}{2\pi}$, $\frac{U\zeta}{2\pi}$, exactly as u, v, w from L, M, N. The volume distribution gives no discontinuity to f, f' at the surface, but only to their differential coefficients; but, if F, G, H be the potentials of the surface distributions, we have discontinuities in their first differential coefficients of the type

$$\frac{dF}{dx}-\frac{dF'}{dx}=2l\xi U.$$

As the parts contributed to f, f' are

$$\frac{dH}{dy}-\frac{dG}{dz} \quad \text{and} \quad \frac{dH'}{dy}-\frac{dG'}{dz},$$

we derive
$$f-f'=2U(m\zeta-n\eta) \quad \text{..........................(3)},$$

and two similar equations. Adding to (2*b*)

$$f+(u-u_0)\frac{du}{dx}+(v-v_0)\frac{du}{dy}+(w-w_0)\frac{du}{dz}$$
$$=f'+(u'-u_0)\frac{du'}{dx}+(v'-v_0)\frac{du'}{dy}+(w'-w_0)\frac{du'}{dz} \quad \text{...(4)},$$

which at once gives $\frac{dp}{dx}=\frac{dp'}{dx}$, ... at the surface. The adjustment of pressure then turns on the choice of a constant, and in fact assigns a value for the difference in the pressures at infinity and at some fixed point within the vortex, in terms of the constants of the configuration, analogous to the relations

$$\Pi-p_0=\frac{2\rho k^2a^4}{45}, \quad \Pi-p_0=\frac{2\rho\zeta^2a^2b^2}{(a+b)^2},$$

for Hill's spherical vortex and Kirchhoff's elliptical vortex. The value of $\frac{d\xi}{dt}$, the instantaneous change in ξ, is given by

$$\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}-u\frac{d\xi}{dx}-v\frac{d\xi}{dy}-w\frac{d\xi}{dz},$$

as a function of x, y, z, and, the densities of both volume and surface distributions being given, f and f' are fully defined.

To meet a case like that of Kirchhoff's rotating vortex, the axes may be taken as rotating with angular velocities θ_1, θ_2, θ_3. Then in

(2*b*) we must write $u_0 - y\theta_3 + z\theta_2$ in lieu of u_0, on both sides of the equation, so that the form of U is modified. As the new form of U occurs also in (3), (4) remains true with the corrected values of u_0, v_0, w_0. The new form of the hydrodynamical equation is

$$X - \frac{1}{\rho}\frac{dp}{dx} = f + (u - u_0 + y\theta_3 - z\theta_2)\frac{du}{dx} + \ldots - v\theta_3 + w\theta_2,$$

and the last terms obviously do not affect the continuity.

Helmholtz takes account of cases where the fluid does not extend to infinity, but has some external boundary, by adding a term $\frac{dP}{dx}$ to the value of u. P satisfies $\nabla^2 P = 0$, within and without the vortex, and has no discontinuity at the vortex surface. Its function is to reduce to zero or to an assigned value the normal velocities at the external boundary, and it is completely defined by this boundary condition and the equation $\nabla^2 P = 0$. Equations (1) and (2) are unaltered by this term. At the end of time dt its function is to reduce to an assigned value the normal velocities given by $u + f\,dt$ at this external boundary, f having the value obtained above; and therefore also $\frac{dP}{dt}$ is fully defined as a function of x, y, z which has no discontinuity at the vortex surface, f and f' having then the additional term $\frac{d^2P}{dx\,dt}$. Thus we can assign in this more general case also the proximate change in the shape of the vortex surface, and in the distribution of spin, and these lead to values of the instantaneous acceleration which give continuity of pressure at the vortex surface.

If an attempt is made from assigned forms for ξ, η, ζ as functions of x, y, z, with parameters which may depend on time, and with assigned forms for the vortex surface and the external boundary also with variable parameters, to construct a functionally complete solution, the criterion for this is that some time variation of parameters should exist which will change ξ to $\xi + \frac{d\xi}{dt}dt$, and u to $u + f\,dt$, where

$$\frac{d\xi}{dt} + u\frac{d\xi}{dx} + v\frac{d\xi}{dy} + w\frac{d\xi}{dz} = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz},$$

and f has the value obtained above; also this same variation must give to the vortex surface a normal extension $= U\,dt$.

[2. A direct expression for the pressure may be found in terms of certain potentials. Let D be the distance between the point xyz, within or without the vortex, to which the potentials refer, and the point $x'y'z'$ within the vortex, where the components of molecular rotation are ξ', η', ζ'. Then, denoting by $\dot{L}, \dot{M}, \dot{N}$ the potentials of the volume distributions, and remarking that

$$\frac{dD}{dx} = -\frac{dD}{dx'},$$

$$\begin{aligned}
\dot{L} &= \frac{1}{2\pi}\iiint \dot{\xi}' D^{-1}\,dx'dy'dz' \\
&= \frac{1}{2\pi}\iiint D^{-1}\left[\frac{d}{dy'}(u'\eta'-v'\xi') - \frac{d}{dz'}(w'\xi'-u'\zeta')\right] dx'dy'dz' \\
&= \frac{1}{2\pi}\iint D^{-1}\left[m\,(u'\eta'-v'\xi') - n\,(w'\xi'-u'\zeta')\right] dS' \\
&\qquad - \frac{1}{2\pi}\iiint \left[(u'\eta'-v'\xi')\frac{dD^{-1}}{dy'} - (w'\xi'-u'\zeta')\frac{dD^{-1}}{dz'}\right] dx'dy'dz' \\
&= -\frac{1}{2\pi}\iint \xi'\,(lu'+mv'+nw')\,dS' \\
&\qquad + \frac{1}{2\pi}\iiint \left[(u'\eta'-v'\xi')\frac{dD^{-1}}{dy} - (w'\xi'-u'\zeta')\frac{dD^{-1}}{dz}\right] dx'dy'dz',
\end{aligned}$$

or

$$\dot{L}+F = \frac{1}{2\pi}\iiint \left[(u'\eta'-v'\xi')\frac{dD^{-1}}{dy} - (w'\xi'-u'\zeta')\frac{dD^{-1}}{dz}\right] dx'dy'dz'.$$

Thus

$$\begin{aligned}
f &= \frac{d}{dy}(\dot{N}+H) - \frac{d}{dz}(\dot{M}+G) \\
&= \frac{1}{2\pi}\frac{d}{dx}\iiint \left[(v'\zeta'-w'\eta')\frac{dD^{-1}}{dx} + (w'\xi'-u\,\zeta)\frac{dD^{-1}}{dy}\right. \\
&\qquad\qquad \left. + (u'\eta'-v'\xi')\frac{dD^{-1}}{dz}\right] dx'dy'dz' \\
&\quad - \frac{1}{2\pi}\iiint (v'\zeta'-w'\xi')\,\nabla^2 D^{-1}\,dx'dy'dz',
\end{aligned}$$

and the last term is $2v\zeta - 2w\eta$ within the vortex, but vanishes outside. Taking this with the hydrodynamical equation

$$f + \frac{1}{\rho}\frac{dp}{dx} + \frac{dV}{dx} + \frac{d}{dx}\tfrac{1}{2}(u^2+v^2+w^2) = 2v\zeta - 2w\eta$$

within the vortex, and $=0$ outside, we have in each case

$$\frac{p}{\rho}+V+\tfrac{1}{2}(u^2+v^2+w^2)$$
$$+\frac{1}{2\pi}\iiint\left[(v'\zeta'-w'\eta')\frac{dD^{-1}}{dx}+(w'\xi'-u'\zeta')\frac{dD^{-1}}{dy}\right.$$
$$\left.+(u'\eta'-v'\xi')\frac{dD^{-1}}{dz}\right]dx'dy'dz',$$

equal to a constant. The triple integral, integrated by parts, gives $V_\sigma+V_\tau$, where

$$V_\sigma=-\frac{1}{2\pi}\iint\left[l(v'\zeta'-w'\eta')+m(w'\xi'-u'\zeta')+n(u'\eta'-v'\xi')\right]D^{-1}dS'$$
$$=\frac{1}{2\pi}\iint\left[u'(m\zeta'-n\eta')+v'(n\xi'-l\zeta')+w'(l\eta'-m\xi')\right]D^{-1}dS',$$

and $$V_\tau=\frac{1}{2\pi}\iiint\left[\frac{d}{dx'}(v'\zeta'-w'\eta')+\frac{d}{dy'}(w'\xi'-u'\zeta')\right.$$
$$\left.+\frac{d}{dz'}(u'\eta'-v'\xi')\right]D^{-1}dx'dy'dz'.$$

Hence $$\frac{p}{\rho}+\tfrac{1}{2}(u^2+v^2+w^2)+V+V_\sigma+V_\tau=\text{constant} \quad\text{.........(5)},$$

a relation which expresses the way in which the energy of unit volume of the liquid varies in the different parts of the field due to the vortex; or it states that this energy is the same throughout the field, when the terms V_σ and V_τ are included as parts of the potential energy.

Application of (1) will show that this expression makes not only p, but $\frac{dp}{dx}$, $\frac{dp}{dy}$, $\frac{dp}{dz}$, continuous at the vortex surface, the discontinuities due to first differential coefficients of V_σ and $\tfrac{1}{2}(u^2+v^2+w^2)$ cancelling. The result might be obtained therefore by forming

$$\nabla^2\left[p+\tfrac{1}{2}(u^2+v^2+w^2)\right]$$

directly from the hydrodynamical equations; this would give the volume integral, and the surface integral would then be inferred from the discontinuities just mentioned.

The form given to $\dot{L}+F$ shows at once that

$$\frac{d}{dx}(\dot{L}+F)+\frac{d}{dy}(\dot{M}+G)+\frac{d}{dz}(\dot{N}+H)=0,$$

but it is evident that the volume potentials alone do not form a solenoidal system, unless

$$l\dot{\xi}+m\dot{\eta}+n\dot{\zeta}=0$$

at the surface. In this case the surface potentials are also solenoidal, and their use may be replaced by that of a single function Ω within the vortex, Ω' without the vortex. The parts of f, g, h due to the surface potentials are given by

$$f=\frac{d\Omega}{dx}, \quad f'=\frac{d\Omega'}{dx},$$

and we have $\nabla^2\Omega=0$ within the vortex, and $\nabla^2\Omega'=0$ without the vortex.] It is then convenient to transform (3), multiplying in turn by (lmn), $(l'm'n')$, $(l''m''n'')$, a mutually orthogonal system, in which (lmn) is the direction of the normal. The result is

$$\left.\begin{aligned}
&\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)(\Omega-\Omega')=0\\
&\left(l'\frac{d}{dx}+m'\frac{d}{dy}+n'\frac{d}{dz}\right)(\Omega-\Omega')=-2U(l''\xi+m''\eta+n''\zeta)\\
&\left(l''\frac{d}{dx}+m''\frac{d}{dy}+n''\frac{d}{dz}\right)(\Omega-\Omega')=+2U(l'\xi+m'\eta+n'\zeta)
\end{aligned}\right\}\ldots(6).$$

These are readily interpreted if we choose $(l'm'n')$ as the direction of resultant molecular rotation τ in the surface. The right-hand member of the second equation then vanishes, and that of the third $=2U\tau$. Hence

$$\frac{d\Omega}{ds}-\frac{d\Omega'}{ds}=2U\tau$$

for a direction of ds in the surface at right angles to the molecular rotation, and vanishes for all directions at right angles to this. In the special case of motion in meridian planes through an axis z, if $\tau = -k\varpi$, ϖ being the distance from the axis ($k\varpi$ being taken to be positive when the rotation is from ϖ to z), then

$$\varpi U = -\frac{d\psi'}{ds},$$

ds being now an element of the meridian curve.

Hence at the surface

$$\Omega - \Omega' = 2k\psi' \quad \text{.............................(7)},$$

and ψ' is the surface value of the stream function corrected for the motion of translation.

[The application of Cauchy's integrals shows at once that, if τ is at any moment proportional to ϖ, it will remain so, and k will be constant, so that in this case the volume integrals disappear, and the accelerations are those due to Ω, Ω'. In fact, the internal condition for a steady motion, viz., that there should be no change in vorticity, is satisfied; but, unless $U = 0$, or the surface condition is also satisfied, the shape of the vortex is subject to change.]

In a more general case the solution may be advanced to the same stage by using curvilinear coordinates α, β, γ in which $\alpha =$ const. is the momentary shape of the vortex surface. With X, Y, Z for components of rotation, U, V, W of velocity, equations (5) become

$$\frac{d\Omega}{d\alpha} - \frac{d\Omega'}{d\alpha} = 0, \quad \frac{d\Omega}{d\beta} - \frac{d\Omega'}{d\beta} = -\frac{2UZ}{h_3}, \quad \frac{d\Omega}{d\gamma} - \frac{d\Omega'}{d\gamma} = +\frac{2UY}{h_2} \quad \text{...(8)}.$$

In this notation the equation

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0$$

transforms to $$\frac{1}{h_2 h_3}\frac{dX}{d\alpha} + \frac{d}{d\beta}\left(\frac{Y}{h_1 h_3}\right) + \frac{d}{d\gamma}\left(\frac{Z}{h_1 h_2}\right) = 0$$

at the surface where $X = 0$, and the vortex equation

$$\left(Xh_1\frac{d}{d\alpha} + Yh_2\frac{d}{d\beta} + Zh_3\frac{d}{d\gamma}\right)(Uh_1) = \left(Uh_1\frac{d}{d\alpha} + Vh_2\frac{d}{d\beta} + Wh_3\frac{d}{d\gamma}\right)(Xh_1)$$

gives at the surface

$$-Uh_1^2\frac{dX}{d\alpha}+Yh_3\frac{d}{d\beta}(Uh_1)+Zh_2\frac{d}{d\gamma}(Uh_1)=0.$$

Multiplying the first of these equations by Uh_1, the second by $\frac{1}{h_1h_2h_3}$, and adding, we have at the surface

$$\frac{d}{d\beta}\left(\frac{UY}{h_3}\right)+\frac{d}{d\gamma}\left(\frac{UZ}{h_2}\right)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9);$$

and therefore at the surface we can write

$$\frac{UY}{h_3}=\frac{d\psi}{d\gamma},\quad \frac{UZ}{h_2}=-\frac{d\psi}{d\beta},$$

where ψ is a function of β and γ, a generalized stream function on the surface $\alpha=\text{const}$. The surface conditions (7) reduce therefore to

$$\frac{d\Omega}{d\alpha}=\frac{d\Omega'}{d\alpha},\quad\text{and}\quad \Omega-\Omega'=2\psi(\beta,\gamma)\ldots\ldots\ldots\ldots(10),$$

ψ being a known function when U is found. If suitable coordinates can be found, the functions Ω and Ω' can be readily constructed.

3. If the discontinuity at a vortex surface is treated without the intervention of the Helmholtz integrals, it is necessary to state expressly that there is to be no slipping at the surface. This is the usual assumption of hydrodynamics, only departed from when, as in the case of flow past sharp edges, the solution yields infinite values of the velocities. We shall suppose the vortex to present no sharp points or edges as part of its boundary, so that the functions concerned have finite differential coefficients at the boundary, and any discontinuity arises from the difference in functional form on the two sides of the boundary.

The continuity of u with u' over the entire surface requires

$$l'\left(\frac{du}{dx}-\frac{du'}{dx}\right)+m'\left(\frac{du}{dy}-\frac{du'}{dy}\right)+n'\left(\frac{du}{dz}-\frac{du'}{dz}\right)=0,$$

$$l''\left(\frac{du}{dx}-\frac{du'}{dx}\right)+m''\left(\frac{du}{dy}-\frac{du'}{dy}\right)+n''\left(\frac{du}{dz}-\frac{du'}{dz}\right)=0.$$

Hence

$$\frac{du}{dx}-\frac{du'}{dx}=l\chi_1,\quad \frac{du}{dy}-\frac{du'}{dy}=m\chi_1,\quad \frac{du}{dz}-\frac{du'}{dz}=n\chi_1 \quad\text{.........(11).}$$

Similarly, $\dfrac{dv}{dx}-\dfrac{dv'}{dx}=l\chi_2, \ \ldots \quad \dfrac{dw}{dx}-\dfrac{dw'}{dx}=l\chi_3, \ \ldots,$

where χ_1, χ_2, χ_3 are to be determined. Now

$$2\xi=\frac{d\,(w-w')}{dy}-\frac{d\,(v-v')}{dz}=m\chi_3-n\chi_2,$$

and two similar equations from which we derive

$$2\,(m\zeta-n\eta)=-\chi_1+l\,(l\chi_1+m\chi_2+n\chi_3).$$

The equation $\dfrac{d\,(u-u')}{dx}+\dfrac{d\,(v-v')}{dy}+\dfrac{d\,(w-w')}{dz}=0$

then gives $l\chi_1+m\chi_2+n\chi_3=0,$

so that

$$\chi_1=-2(m\zeta-n\eta),\quad \chi_2=-2\,(n\xi-l\zeta),\quad \chi_3=-2\,(l\eta-m\xi)\text{...(12).}$$

These values are identical with (1), as we should expect; the result implying Helmholtz's interpretation of P as a potential of matter on or beyond the boundaries of the whole fluid dealt with, and having no discontinuity at the vortex surface.

The special feature of the velocities within and without a vortex, that they have different functional forms which have a common surface value, is also characteristic of the pressure. Without the vortex it is assigned in terms of the velocity potential; within, the condition for its existence is that

$$\frac{d\xi}{dt}+u\frac{d\xi}{dx}+v\frac{d\xi}{dy}+w\frac{d\xi}{dz}=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz},$$

and two similar vortex equations should be satisfied. This being admitted, the question arises: What conditions must be fulfilled to bring into agreement at the surface the two functional forms for pressure? Exactly as in proving (11), the continuity of p and p' over the entire surface requires

$$l'\frac{dp}{dx}+m'\frac{dp}{dy}+n'\frac{dp}{dz}=l'\frac{dp'}{dx}+m'\frac{dp'}{dy}+n'\frac{dp'}{dz},$$

$$l''\frac{dp}{dx}+m''\frac{dp}{dy}+n''\frac{dp}{dz}=l''\frac{dp'}{dx}+m''\frac{dp'}{dy}+n''\frac{dp'}{dz},$$

leading to

$$\frac{dp}{dx}-\frac{dp'}{dx}=-\rho\sigma l,\quad \frac{dp}{dy}-\frac{dp'}{dy}=-\rho\sigma m,\quad \frac{dp}{dz}-\frac{dp'}{dz}=-\rho\sigma n;$$

and therefore

$$\frac{Du}{Dt}-\frac{Du'}{Dt}=\sigma l,\quad \frac{Dv}{Dt}-\frac{Dv'}{Dt}=\sigma m,\quad \frac{Dw}{Dt}-\frac{Dw'}{Dt}=\sigma n,$$

where σ is an arbitrary function. But the normal equation of continuity

$$lu+mv+nw=lu'+mv'+nw',$$

always valid, gives, when $u=u',\ v=v',\ w=w'$,

$$l\frac{Du}{Dt}+m\frac{Dv}{Dt}+n\frac{Dw}{Dt}=l\frac{Du'}{Dt}+m\frac{Dv'}{Dt}+n\frac{Dw'}{Dt},$$

and this makes $\sigma=0$; and therefore

$$\frac{dp}{dx}=\frac{dp'}{dx}\quad\text{with}\quad\frac{Du}{Dt}=\frac{Du'}{Dt}.$$

That is, if we admit the continuity of $u, u', \ldots$, nothing short of

$$\frac{Du}{Dt}=\frac{Du'}{Dt},\ \ldots$$

will make p continuous with p', and then the *normal* variation of pressure is also the same on both sides of the surface.

It is perhaps hardly necessary to observe that, as the liquid once in a vortex surface always remains there, the interpretation of

$$\frac{Du}{Dt}=\frac{Du'}{Dt},\ \ldots\quad\text{with}\quad u=u',\ \ldots$$

is that the continuity of velocity existing at time t is maintained at time $t+dt$.

This condition, combined with (12) or (2*b*), to which (12) immediately leads, gives

$$\frac{du}{dt}-\frac{du'}{dt}=2U(m\zeta-n\eta) \qquad \text{(13)},$$

and we have two similar equations for v and w. These, then, are the requisite conditions for continuity of pressure, and we find that the time as well as the space differential coefficients of u, u', ... are discontinuous at a vortex surface. It is unnecessary to repeat from another point of view the reasoning which connects this discontinuity with the change of shape of the vortex when the conditions of steady motion are not fulfilled.

4. Again, the continuity of $\frac{dp}{dx}$ with $\frac{dp'}{dx}$ over the entire surface leads exactly as above to

$$\frac{d^2p}{dx^2}-\frac{d^2p'}{dx^2}+\rho\nu_1 l=0, \qquad \frac{d^2p}{dxdy}-\frac{d^2p'}{dx\,dy}+\rho\nu_1 m=0,$$

$$\frac{d^2p}{dx\,dz}-\frac{d^2p'}{dx\,dz}+\rho\nu_1 n=0,$$

ν_1 being an arbitrary function. Similarly,

$$\frac{d^2p}{dxdy}-\frac{d^2p'}{dx\,dy}+\rho\nu_2 l=0, \;\ldots$$

and

$$\frac{d^2p}{dx\,dz}-\frac{d^2p'}{dx\,dz}+\rho\nu_3 l=0, \;\ldots.$$

Writing $\nu_1=\nu l$, $\nu_2=\nu m$, $\nu_3=\nu n$ brings the two values of $\frac{d^2p}{dxdy}-\frac{d^2p'}{dxdy}$, ... into accord, and we get

$$\frac{d^2p}{dx^2}-\frac{d^2p'}{dx^2}+\rho\nu l^2=0, \qquad \frac{d^2p}{dxdy}-\frac{d^2p'}{dxdy}+\rho\nu lm=0 \qquad \text{(14)}.$$

These are discontinuities such as belong to the potential of a *volume* distribution, where the density has at the surface the abrupt increment of value $\frac{\rho\nu}{4\pi}$. It is easy to show that, if dn_1, dn_2, dn_3 be distance elements in the directions (l, m, n), (l', m', n'), (l'', m'', n''), the equations (14) amount to

$$\frac{d^2p}{dn_1^2}-\frac{d^2p'}{dn_1^2}+\rho\nu=0,\quad \frac{d^2p}{dn_2^2}-\frac{d^2p'}{dn_2^2}=0,\quad \frac{d^2p}{dn_1dn_2}-\frac{d^2p'}{dn_1dn_2}=0,$$

all second differential coefficients except the one first written vanishing. It remains to find ν. We have

$$\frac{d}{dx}\left(\frac{Du}{Dt}\right)-\frac{D}{Dt}\left(\frac{du}{dx}\right)=\left(\frac{du}{dx}\right)^2+\frac{dv}{dx}\frac{du}{dy}+\frac{dw}{dx}\frac{du}{dz}$$

and a similar formula for u'. Subtracting, at the surface

$$\frac{d}{dx}\left(\frac{Du}{Dt}-\frac{Du'}{Dt}\right)-\frac{D}{Dt}\left(\frac{du}{dx}-\frac{du'}{dx}\right)$$

$$=\left(\frac{du}{dx}\right)^2-\left(\frac{du'}{dx}\right)^2+\frac{dv}{dx}\frac{du}{dy}-\frac{dv'}{dx}\frac{du'}{dy}+\frac{dw}{dx}\frac{du}{dz}-\frac{dw'}{dx}\frac{du'}{dz}$$

$$=2\chi_1\left(l\frac{du'}{dx}+m\frac{du'}{dy}+n\frac{du'}{dz}\right)+2\zeta\frac{du'}{dy}-2\eta\frac{du'}{dz}\quad\text{..............(15)},$$

where (11) is used to express $\frac{du}{dx}$, ... in terms of $\frac{du'}{dx}$, ... and the χ's;

and $$2\zeta=l\chi_2-m\chi_1$$

inferred from (12) simplifies the form of the result.

Write down the corresponding equations in v and w, and add together; then, since

$$\frac{D}{Dt}\left(\frac{du}{dx}-\frac{du'}{dx}+\frac{dv}{dy}-\frac{dv'}{dy}+\frac{dw}{dz}-\frac{dw'}{dz}\right)=0,$$

$$\nu=\frac{d}{dx}\left(\frac{Du}{Dt}-\frac{Du'}{Dt}\right)+\frac{d}{dy}\left(\frac{Dv}{Dt}-\frac{Dv'}{Dt}\right)+\frac{d}{dz}\left(\frac{Dw}{Dt}-\frac{Dw'}{Dt}\right)$$

$$=2\chi_1\left(l\frac{du'}{dx}+m\frac{du'}{dy}+n\frac{du'}{dz}\right)+2\chi_2\left(l\frac{dv'}{dx}+m\frac{dv'}{dy}+n\frac{dv'}{dz}\right)$$

$$+2\chi_3\left(l\frac{dw'}{dx}+m\frac{dw}{dy}+n\frac{dw'}{dz}\right)\text{......(16)}.$$

To interpret this result more readily, let $x'y'z'$ be a new rectangular system, x' being the direction of the normal; then

$$\nu=2\chi_1\frac{dU'}{dx'}+2\chi_2\frac{dV'}{dx'}+2\chi_3\frac{dW'}{dx'},$$

and, since in the new position

$$l = 1, \quad m = 0, \quad n = 0,$$

$$\chi_1 = 0, \quad \chi_2 = 2Z, \quad \chi_3 = -2Y,$$

therefore
$$\nu = 4\left(Z\frac{dV'}{dx'} - Y\frac{dW'}{dx'}\right).$$

Further, if the direction of y' is taken to be that of the resultant molecular rotation τ, $Z = 0$, $Y = \tau$, and then

$$\nu = -4\tau\frac{dW'}{dx'} \quad \text{.............................(17)},$$

W' being the external velocity in the surface at right angles to the direction of resultant spin. Hence, finally,

$$\frac{d^2p}{dn_1^2} - \frac{d^2p'}{dn_1^2} = 4\rho\tau\frac{dW'}{dx'} = 4\rho\tau\left(\frac{dW}{dx} + 2\tau\right) \quad \text{...........(18)},$$

in terms of the internal velocity.

If the direction of W' is normal to the surface γ of an orthogonal system, and $1/R$ is the principal curvature of the surface γ in the normal section through $d\alpha$, then

$$\frac{dW}{dx'} = \frac{dW}{dn_1} + \frac{U}{R} = h_1\frac{dW}{d\alpha} - Uh_1h_3\frac{d}{d\gamma}\left(\frac{1}{h_1}\right)$$

(see Love's *Elasticity*, pp. 204, 205).

For the case of motion in meridian planes through the axis of z, W' is the velocity along the tangent to the meridian curve, which we may call V'_t, and if the rotation is $k\varpi$ from ϖ to z, $\tau = -k\varpi$; hence

$$\frac{d^2p}{dn_1^2} - \frac{d^2p'}{dn_1^2} = -4k\rho\varpi\frac{dV'_t}{dn_1} \text{.......................(19)}.$$

For example, in the case of the sphere

$$\frac{dV'_t}{dn_1} = \frac{dV'_t}{dr} = \frac{3V_0}{2a}\sin\theta, \quad k = -\frac{15V_0}{4a^2};$$

therefore
$$\frac{d^2p}{dr^2} - \frac{d^2p'}{dr^2} = \frac{45\rho V_0^2}{2a^2}\sin^2\theta,$$

which will be found to agree with the value deduced from Prof. Hill's results for the spherical vortex (*Phil. Trans.*, 1884).

The formula is true for rotating axes, where the more general value of $\frac{Du}{Dt}$ given on p. 285 is required. The extra terms which appear in this case on the right-hand side of equation (15) are

$$\theta_2\left(\frac{dw}{dx}+\frac{du}{dz}\right)-\theta_3\left(\frac{dv}{dx}+\frac{du}{dy}\right),$$

and on making the addition required for (16) such terms vanish. It is also true for the instantaneous values given in the first part of the paper, as appears from the fact that the proof only makes use of first differential coefficients of velocities with regard to t, and these are given completely as functions of x, y, z.

In two dimensions the result may be put in the forms

$$\left.\begin{aligned}\frac{d^2p}{dn^2}-\frac{d^2p'}{dn^2}&=4\rho\zeta\left(h\frac{dV_t}{d\xi}+V_n\frac{dh}{d\eta}\right)\\&=4\rho\zeta\left\{h^2\frac{d^2\psi}{d\xi^2}+\left(\frac{d\psi}{d\xi}\frac{d}{d\xi}-\frac{d\psi}{d\eta}\frac{d}{d\eta}\right)\frac{h^2}{2}\right\}\\&=4\rho\zeta\left\{h^2\frac{d^2\phi}{d\xi d\eta}+\left(\frac{d\phi}{d\eta}\frac{d}{d\xi}+\frac{d\phi}{d\xi}\frac{d}{d\eta}\right)\frac{h^2}{2}\right\}\end{aligned}\right\}\quad\text{.........(20)},$$

where ϕ and ψ are potential and current functions for the external motion, ξ and η conjugate functions of which $\xi=$ constant is the equation to the vortex. For the case of Kirchhoff's vortex, this gives

$$\frac{d^2p}{dn^2}-\frac{d^2p'}{dn^2}=\frac{8\rho\zeta^2}{(a+b)^2}(\varpi^2+ab),$$

where ϖ is the perpendicular from the centre on the tangent. This value and the results

$$\frac{dp}{dn}=\frac{dp'}{dn},\quad\frac{d^2p}{dnds}=\frac{d^2p'}{dnds},\quad\frac{d^2p}{ds^2}=\frac{d^2p'}{ds^2},$$

ds being an element of the arc, have been tested by the use of the values given in Mr. Love's paper "On the Stability of certain Vortex Motions."

5. I propose now to give an application suggested by a remark in the paper just quoted. The author says: "It appears, in fact, that, alike in this case and in all the remaining cases of steady motion and

small oscillations here investigated, the condition of continuity of pressure across the surface of the vortex reduces to an identity when the stream functions are adjusted to satisfy the conditions of continuity of tangential and normal velocity, and the condition that the surface of the vortex always contains the same particles." The cases alluded to are in two dimensions, the simple Kirchhoff vortex, or Hill's more general case when there is an external boundary, and there follows a suggestion that this is always true for axes moving in a uniform manner, and for small oscillations about such steady motions. This may be proved quite generally. Let $a = a_0$ be the undisturbed surface, a being a function of xyz, a_0 a constant; $a = a_0+\chi$ the equation to the disturbed surface, χ being small, a function of t and of β and γ coordinates orthogonal to a, if such can be found, but in any case $\chi = C$ is orthogonal to $a = a_0$. Let u, v, w be internal velocities in the undisturbed motion, $u+U$, ... in the disturbed motion, and use u', $u'+U'$, ... for the corresponding external velocities. We determine first the surface condition satisfied by U, U'. The vortex in the disturbed state extends to a normal distance χ/h_1 beyond its original surface, and we find the values of u, u' at this surface by differentiating the original functions, and writing in the differential coefficients the values of coordinates at the original surface. Thus at the new surface

$$u = u_a + \frac{\chi}{h_1}\left(l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}\right)u_a,$$

where u_a is the value at the original surface.

Write a similar equation for u' and subtract; then, using the values of $\frac{du_a}{dx}-\frac{du'_a}{dx}$, ... given in (11) or (1), we have

$$u-u' = -\frac{2\chi}{h_1}(m\zeta-n\eta),$$

and the condition of continuity then gives

$$U-U' = \frac{2\chi}{h_1}(m\zeta-n\eta) \quad\text{.......................(21)},$$

and we have similar equations for V and W. Hence

$$\frac{dU}{dt}-\frac{dU'}{dt} = \frac{2}{h_1}\frac{d\chi}{dt}(m\zeta-n\eta) \quad\text{..................(22)},$$

and, if ξ, η, ζ are altered in the oscillation, these results are unaffected to the first order. Also, u, u' being independent of time before the substitution of the particular surface values involving χ, $\frac{dU}{dt}$, $\frac{dU'}{dt}$ represent the complete values of the time differential coefficients of velocities within and without respectively.

But the condition for continuity of pressure [see (13)] prescribes for the discontinuity in $\frac{dU}{dt}$ the equation

$$\frac{dU}{dt}-\frac{dU'}{dt}=2N(m\zeta-n\eta),$$

and N, the relative normal velocity, in its general form has the value

$$l(u+U-u_0+y\theta_3-z\theta_2)+\dots .$$

The boundary condition

$$N=\frac{1}{h_1}\frac{d\chi}{dt} \qquad \text{.............................(23)}$$

reduces this at once to the form given in (22), securing continuity of pressure identically when U, U' are determined as in (21), so as to establish continuity of velocity at the disturbed surface.

Written at length, the boundary condition, with the terms omitted which vanish in virtue of the steady motion, viz.,

$$(u_a-u_0+y\theta_3-z\theta_2)\frac{da}{dx}+\dots=0,$$

is $$(u-u_a+U)\frac{da}{dx}+(v-v_a+V)\frac{da}{dy}+(w-w_a+W)\frac{da}{dz}$$

$$=\frac{d\chi}{dt}+(u_a-u_0+y\theta_3-z\theta_2)\frac{d\chi}{dx}+\dots \qquad \dots(24).$$

It is possible that the oscillation may involve a change in u_0, ..., say to u_0+u_0', and in θ_1 ..., say to $\theta_1+\theta_1'$, where u_0', θ_1' are small and functions of time. These will be represented by appropriate terms in χ, and on the left-hand side of (24) we must write $U-u_0'+y\theta_3'-z\theta_2'$ for U. This has obviously no effect on the discontinuity in (21). Also, it may be remarked that in (24), as surface values are in question, we may use either the external or internal functions.

Two cases may be mentioned where the whole solution is determined by the surface condition (24), viz., that of motion in two dimensions with constant vorticity, and that of motion in three dimensions in meridian planes through an axis, when the disturbance is zonal and the spin $= k\varpi$.

In the latter case, for example, we have the vortex equation satisfied by $\tau = k\varpi$, whatever the values of u, v; and the equation

$$l\xi + m\eta + n\zeta = 0$$

is also satisfied identically with a zonal disturbance.

In these cases U and U' are derivable from potentials ϕ, ϕ' satisfying $\nabla^2\phi = 0$ within, $\nabla^2\phi' = 0$ without; and the surface conditions admit of the reduction given on p. 288.

Where the vortex equations are not satisfied in this automatic way, and the surface condition

$$l\xi + m\eta + n\zeta = 0$$

is broken by the altered values of l, m, n at the disturbed surface, the oscillation will entail changes in ξ, η, ζ which, however, do not affect the surface conditions described above to the first order of small quantities. The difficulty of obtaining a solution may be greatly increased, but it may be remarked that the vortex equations are linear in this case, though with variable coefficients.

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*An Ellipsoidal Vortex.** *By* R. Hargreaves, M.A. Read February 13th, 1896. Received, in revised form, May 25th, 1896.

The vortex discussed here is in the form of an ellipsoid of revolution, is in motion in the direction of the axis of symmetry, and has a molecular rotation proportional to the distance from the axis. It will appear that, unlike Professor Hill's spherical vortex, it cannot move as a solid through the liquid unchanged in form, but experiences a deformation at the surface. We have therefore only a phase in the motion of the vortex, but this phase admits of exact treatment, the

* Some re-arrangement, with addition of details, has been made since the paper was read.

stream lines being given, the pressure, and the distribution of the different parts of the energy. Also the solution is in finite terms, and applies to ovary and planetary forms ranging from the rod at one extreme to the disk at the other. Numerical values are given in a few cases for the velocity of translation and the distribution of energy, and it is hoped that these may be of interest in connexion with the vortex theory of matter (see § 8).

The results were first obtained by a direct integration of the Helmholtz integrals, and this is retained on account of the intrinsic interest of the method, though the treatment by differential equations which follows is at once simpler and shorter.

I have also given most of the results in the notation of the attraction of ellipsoids, which is more convenient for some purposes than the harmonic analysis. These will be found on p. 322.

1. By way of introduction to the ellipsoid, I apply the Helmholtz integrals to the spherical vortex. For the stream function generally, we have

$$-\psi = \iint \frac{k\varpi\varpi'^2 dS'}{\pi} \int_0^\pi \frac{\cos\phi \, d\phi}{D},$$

$k\varpi'$ denoting the molecular rotation at distance ϖ' from the axis, D the distance between two points (r, μ), (r', μ') whose azimuth differs by ϕ. The integration of dS' extends over the area of a meridian plane contained between the axis and the curve, so that

$$-\psi = \frac{k\varpi}{\pi} \int_0^a \int_{-1}^{+1} r'^3 P_{1,1}(\mu')\, dr' \, d\mu' \int_0^\pi \frac{\cos\phi \, d\phi}{D}.$$

D^{-1} is to be expanded by spherical harmonics, and we require only those terms which contain $\cos\phi$. For the external case, therefore,

$$-\psi_e = \frac{k\varpi}{2} \int_0^a \int_{-1}^{+1} r'^3 P_{1,1}(\mu')$$
$$\times \left\{ \frac{r'}{r^3} P_{1,1}(\mu)\, P_{1,1}(\mu') + \frac{r'^2}{3r^3} P_{2,1}(\mu)\, P_{2,1}(\mu') + \dots \right\} dr' d\mu',$$

using the notation $P_{n,1}(\mu)$ for $\sqrt{1-\mu^2}\dfrac{dP_n}{d\mu}$. Since

$$\int_{-1}^{+1} P_{n,1}(\mu')\, P_{n',1}(\mu')\, d\mu' = 0,$$

only the first term is required, and

$$\psi_e = -\frac{2k\varpi}{3}\int_0^a P_{1,1}(\mu)\frac{r'^4}{r^3}dr' = -\frac{2k\varpi a^5}{15r^3}P_{1,1}(\mu)$$

$$= -\frac{2ka^5}{15r}\sin^2\theta = -\frac{2ka^5\varpi^2}{15r^3} = -\frac{2ka^2\varpi^2}{15} + \frac{2ka^2\varpi^2}{15}\left(1-\frac{r^3}{a^3}\right).$$

For the interior case

$$\psi_i = -\frac{2k\varpi}{3}P_{1,1}(\mu)\int r'^3 dr'\left(\frac{r'}{r^3} \text{ or } \frac{r}{r'^2}\right),$$

the first part used from $r' = 0$ to $r' = r$, and the second from $r' = r$ to $r' = a$,

$$\psi_i = -\frac{2k\varpi}{3}P_{1,1}(\mu)\left\{\frac{r^3}{5} + \frac{r(a^2-r^2)}{2}\right\} = -\frac{2k\varpi^2a^2}{15} - \frac{k\varpi^2}{5}(a^2-r^2).$$

The values agree at the surface, and a comparison with $\frac{1}{2}V_0\varpi^2$ gives for the velocity of translation

$$V_0 = -\frac{4ka^2}{15}.$$

The total energy of the vortex is

$$-2k\pi\rho\iint\varpi\psi_i\,dS = -2\pi k\rho\iint r^2\sin\theta\,\psi_i\,dr\,d\theta$$

$$= \frac{2\pi k^2\rho}{15}\int_0^a\int_0^\pi \sin^3\theta\,(5a^2-3r^2)\,r^4\,dr\,d\theta = \frac{32\pi k^2\rho a^7}{3\times7\times15} = \frac{15}{14}MV_0^2.$$

2. In the main argument the ovary ellipsoid is taken, and the changes required to pass to the planetary are given later. The appropriate coordinates are ν and μ, where

$$z = c\nu\mu, \quad \varpi = c\sqrt{(\nu^2-1)(1-\mu^2)},$$

so that the equation to an ellipsoid confocal with the boundary is

$$\frac{z^2}{\nu^2} + \frac{\varpi^2}{\nu^2-1} = c^2.$$

At the boundary $\nu = \nu_0$, increases to infinity outside, and inside decreases to $\nu = 1$ for the line joining the foci; μ varies from -1 to $+1$.

$$dS' = dz'\,d\varpi' = c^2\left(\frac{dz'}{d\nu'}\frac{d\varpi'}{d\mu'} - \frac{dz'}{d\mu'}\frac{d\varpi'}{d\nu'}\right)d\nu'\,d\mu' = \frac{c^2(\nu'^2-\mu'^2)\,d\nu'\,d\mu'}{\sqrt{(\nu'^2-1)(1-\mu'^2)}},$$

and $$\varpi'^2 dS' = c^4(\nu'^2-\mu'^2)\sqrt{(\nu'^2-1)(1-\mu'^2)}\,d\nu'\,d\mu'$$

$$=\frac{2c^4}{15}\{P_{3,1}(\nu')\,P_{1,1}(\mu')-P_{1,1}(\nu')\,P_{3,1}(\mu')\}.$$

Hence

$$-\psi=\frac{2kc^4\varpi}{15\pi}\iint\{P_{3,1}(\nu')\,P_{1,1}(\mu')-P_{1,1}(\nu')\,P_{3,1}(\mu')\}\,d\nu'\,d\mu'\int_0^{\pi}\frac{\cos\phi\,d\phi}{D} \qquad \text{.........} (1).$$

In the expansion of D^{-1} we require the terms containing $\cos\phi$, which are of the type $A_n P_{n,1}(\mu)\,Q_{n,1}(\nu)\,P_{n,1}(\mu')\,P_{n,1}(\nu')$; where ν and μ are coordinates of an external point, Q_n is the second integral of the Legendre equation, and $Q_{n,1}$ is written for $\sqrt{\nu^2-1}\,\frac{dQ_n}{d\nu}$. To determine the constant, compare with the spherical case, noting that $\nu c = r$ in the limit. With ν and ν' both large, but $\nu>\nu'$ for the external case, $A_n Q_{n,1}(\nu)\,P_{n,1}(\nu')$ has for its principal term

$$-\frac{(n+1)!}{1.3\ldots(2n+1)}\,\frac{1}{\nu^{n+1}}\times\frac{1.3\ldots(2n-1)}{(n-1)!}\times\nu'^n\times A_n,$$

or $$-\frac{n(n+1)}{2n+1}\,\frac{\nu'^n}{\nu^{n+1}}A_n.$$

In the spherical case the corresponding term arises from

$$\frac{r'^n}{r^{n+1}}P_n(\cos\gamma),$$

and the coefficient of $P_{n,1}(\mu)\,P_{2,1}(\mu')\cos\phi$ is

$$\frac{2}{n(n+1)}\,\frac{r'^n}{r^{n+1}}.$$

Hence $$A_n=-\frac{2(2n+1)}{n^2(n+1)^2c}.$$

Effecting the integration with regard to ϕ, we have

$$\psi_e=\frac{2k\varpi c^3}{15}\int_1^{\nu_0}\int_{-1}^{+1}\{P_{3,1}(\nu')\,P_{1,1}(\mu')-P_{1,1}(\nu')\,P_{3,1}(\mu')\}$$

$$\times\{\tfrac{3}{4}P_{1,1}(\mu)\,Q_{1,1}(\nu)\,P_{1,1}(\mu')\,P_{1,1}(\nu')$$

$$+\tfrac{7}{144}P_{3,1}(\mu)\,Q_{3,1}(\nu)\,P_{3,1}(\mu')\,P_{3,1}(\nu')\}\,d\nu'\,d\mu'\ldots(2).$$

writing only the terms in the second bracket, which survive the next integration. For this, we have

$$\int_{-1}^{+1} \{P_{n,1}(\mu')\}^2\, d\mu' = \frac{2n(n+1)}{2n+1},$$

$$\int_{-1}^{+1} P_{n,1}(\mu')\, P_{n',1}(\mu')\, d\mu' = 0;$$

therefore

$$\psi_e = \frac{2k\varpi c^3}{15}\int_1^{\nu_0}\{P_{1,1}(\mu)\,Q_{1,1}(\nu) - \tfrac{1}{6}P_{3,1}(\mu)\,Q_{3,1}(\nu)\}\,P_{1,1}(\nu')P_{3,1}(\nu')\,d\nu'$$

$$= \frac{k\varpi c^3 . \nu_0(\nu_0^2-1)^2}{5}\{P_{1,1}(\mu)\,Q_{1,1}(\nu) - \tfrac{1}{6}P_{3,1}(\mu)\,Q_{3,1}(\nu)\}$$

$$= \frac{k\varpi^2 c^3 \nu_0(\nu_0^2-1)^2}{5}\{Q_1'(\nu) - \tfrac{1}{6}P_3'(\mu)\,Q_3'(\nu)\} \quad \text{......................(3).}$$

3. Before finding the internal stream function it will be useful to set down some analytical results. Q_n is given by a formula

$$P_n Q_0 - \frac{2n-1}{1.n}P_{n-1} - \frac{2n-5}{3(n-1)}P_{n-3} \ldots,$$

where
$$Q_0 = \tfrac{1}{2}\log\frac{\nu+1}{\nu-1}.$$

In what follows Q_1 is the dominating term, and the results are expressed more concisely by reference to Q_1, which is $(\nu Q_0 - 1)$. We have

$$\nu Q_2 = P_2 Q_1 - \tfrac{1}{2}, \quad \nu Q_3 = P_3 Q_1 - \frac{5\nu}{6} \quad \text{.................(4).}$$

The differential coefficient may be got from

$$P_n Q_n' - P_n' Q_n = -\frac{1}{\nu^2-1}.$$

Thus
$$\left.\begin{aligned} \nu(\nu^2-1)\,Q_1' &= (\nu^2-1)\,Q_1 - 1 \\ \nu(\nu^2-1)\,Q_2' &= P_2'\{(\nu^2-1)\,Q_1 - \tfrac{1}{3}\} \\ \nu(\nu^2-1)\,Q_3' &= P_3'\{(\nu^2-1)\,Q_1 - \tfrac{1}{3}\} + 1 \end{aligned}\right\} \quad \text{..............(5).}$$

Multiplying by $\sqrt{\nu^2-1}$ will give the form

$$\nu(\nu^2-1)\,Q_{3,1} = P_{3,1}\{(\nu^2-1)\,Q_1 - \tfrac{1}{3}\} + P_{1,1}$$

Again, we require the integrals

$$\int P_{3,1}(\nu)\, Q_{1,1}(\nu)\, d\nu \quad \text{and} \quad \int P_{1,1}(\nu)\, Q_{3,1}(\nu)\, d\nu$$

between any assigned limits. We have generally

$$\int P_{m,1}(\nu)\, Q_{n,1}(\nu) = \int (\nu^2-1)\frac{dP_m}{d\nu}\frac{dQ_n}{d\nu}\, d\nu$$

$$= (\nu^2-1)\, P_m \frac{dQ_n}{d\nu} - n(n+1)\int P_m Q_n\, d\nu,$$

or

$$= (\nu^2-1)\, Q_n \frac{dP_m}{d\nu} - m(m+1)\int P_m Q_n\, d\nu,$$

using

$$\frac{d}{d\nu}(\nu^2-1)\frac{dQ_n}{d\nu} = n(n+1)\, Q_n.$$

Hence, eliminating the integral

$$\int P_m Q_n\, d\nu,$$

we have

$$(m-n)(m+n+1)\int_{\nu_0}^{\nu} P_{m,1}(\nu)\, Q_{n,1}(\nu)\, d\nu$$

$$= \left[(\nu^2-1)\left\{m(m+1)\, P_m \frac{dQ_n}{d\nu} - n(n+1)\, Q_n \frac{dP_m}{d\nu}\right\}\right]_{\nu_0}^{\nu}.$$

This may be cleared of differential coefficients on the right-hand side by using

$$(\nu^2-1)\frac{dP_m}{d\nu} = m(\nu P_m - P_{m-1}),$$

and a similar formula for Q_n. Hence

$$(m-n)(m+n+1)\int_{\nu_0}^{\nu} P_{m,1}(\nu)\, Q_{n,1}(\nu)\, d\nu$$

$$= mn\left[(m-n)\,\nu P_m Q_n + (n+1)\, P_{m-1} Q_n - (m+1)\, P_m Q_{n-1}\right]_{\nu_0}^{\nu}$$

$$= \frac{mn(m+1)}{2m+1}\left[(m-n)P_{m+1}Q_n + (m+n+1)P_{m-1}Q_n - (2m+1)P_m Q_{n-1}\right]_{\nu_0}^{\nu}$$

$$= \frac{mn(n+1)}{2n+1}\left[(m-n)\, P_m Q_{n+1} + (2n+1)\, P_{m-1} Q_n - (m+n+1)\, P_m Q_{n-1}\right]_{\nu_0}^{\nu}$$

............(6),

the last two requiring an application of the sequence formula

$$(n+1)\,Q_{n+1}+nQ_{n-1}=(2n+1)\,\nu Q_n,$$

or the same with P_n. This general theorem is, of course, true if P_n is written for Q_n. Hence

$$\int_{\nu_0}^{\nu} P_{3,1}(\nu)\,Q_{1,1}(\nu)\,d\nu=\tfrac{1}{5}\left[2P_3Q_2+3P_3Q_1-5P_3Q_0\right]_{\nu_0}^{\nu}$$

$$=\left[\tfrac{3}{2}(\nu^2-1)^2\,Q_1-3\,(\nu^2-1)-\tfrac{6}{5}\right]_{\nu_0}^{\nu}\quad\ldots(7),$$

and

$$\int_{\nu_0}^{\nu} P_{1,1}(\nu)\,Q_{3,1}(\nu)\,d\nu=\tfrac{1}{5}\left[2P_2Q_3+3P_1Q_2-5P_0Q_3\right]_{\nu_0}^{\nu}$$

$$=\left[\tfrac{3}{2}(\nu^2-1)^2\,Q_1-\tfrac{1}{2}(\nu^2-1)+\tfrac{1}{5}\right]_{\nu_0}^{\nu}\quad\ldots(8).$$

4. The function ψ_i divides into two parts, of which the first ψ_1 is due to the integration within the surface on which the point lies, viz.,

$$\psi_1=\frac{k\varpi c^3\nu\,(\nu^2-1)^2}{5}\left\{P_{1,1}(\mu)\,Q_{1,1}(\nu)-\tfrac{1}{6}P_{3,1}(\mu)\,Q_{3,1}(\nu)\right\}$$

$$=\frac{k\varpi c^3}{5}\Big[P_{1,1}(\mu)\,P_{1,1}(\nu)\left\{(\nu^2-1)^2\,Q_1-(\nu^2-1)\right\}$$

$$-\tfrac{1}{6}P_{3,1}(\mu)\,P_{3,1}(\nu)\left\{(\nu^2-1)^2Q_1-\tfrac{1}{3}(\nu^2-1)+\tfrac{2}{15}\right\}+\tfrac{2}{15}P_{3,1}(\mu)\,P_{1,1}(\nu)\Big]$$

$$\ldots\ldots\ldots\ldots(9).$$

The second form is the one prepared for addition to ψ_2, and is derived from the first by the use of (5), and

$$(\nu^2-1)\,P_{1,1}(\nu)=\tfrac{2}{15}P_{3,1}(\nu)-\tfrac{4}{5}P_{1,1}(\nu)\ldots\ldots\ldots\ldots\ldots(10).$$

For the part ψ_2 due to the shell between ν and ν_0, ν being internal, we must interchange ν and ν' in the second bracket of (2), the original integral for ψ. The integration with regard to μ' then gives

$$\psi_2=\frac{2k\varpi c^3}{15}\int_{\nu}^{0}\Big\{P_{1,1}(\mu)\,P_{1,1}(\nu)\,Q_{1,1}(\nu')\,P_{3,1}(\nu')$$

$$-\tfrac{1}{6}P_{3,1}(\mu)\,P_{3,1}(\nu)\,Q_{3,1}(\nu')\,P_{1,1}(\nu')\Big\}\,d\nu'$$

$$=\frac{k\varpi c^3}{5}\Big[P_{1,1}(\mu)\,P_{1,1}(\nu)\,\Big\{(\nu_0^2-1)^2\,Q_1(\nu_0)-2\,(\nu_0^2-1)$$

$$-(\nu^2-1)^2\,Q_1(\nu)+2\,(\nu^2-1)\Big\}$$

$$-\tfrac{1}{6}P_{3,1}(\mu)\,P_{3,1}(\nu)\,\Big\{(\nu_0^2-1)^2\,Q_1(\nu_0)-\tfrac{1}{3}(\nu_0^2-1)$$

$$-(\nu^2-1)^2\,Q_1(\nu)+\tfrac{1}{3}(\nu^2-1)\Big\}\Big]$$

[quoting (7) and (8) for the integration]

$$= \frac{k\varpi c^3}{5}\Big[P_{1,1}(\mu)P_{1,1}(\nu)\{(\nu_0^2-1)^2Q_1(\nu_0)-2(\nu_0^2-1)-(\nu^2-1)^2Q_1(\nu)$$
$$+(\nu^2-1)-\tfrac{4}{3}\}+\tfrac{2}{15}P_{1,1}(\mu)P_{3,1}(\nu)$$
$$-\tfrac{1}{6}P_{3,1}(\mu)P_{3,1}(\nu)\{(\nu_0^2-1)Q_1(\nu_0)-\tfrac{1}{3}(\nu_0^2-1)$$
$$-(\nu^2-1)^2Q_1(\nu)+\tfrac{1}{3}(\nu^2-1)\}\Big],$$

changing the form by use of (10). The addition to the value of ψ_1 in (9) gives immediately

$$\psi_i = \frac{k\varpi c^3}{5}\Big[P_{1,1}(\mu)P_{1,1}(\nu)\{(\nu_0^2-1)^2Q_1(\nu_0)-2(\nu_0^2-1)-\tfrac{4}{3}\}$$
$$+\tfrac{2}{15}P_{1,1}(\mu)P_{3,1}(\nu)+\tfrac{2}{15}P_{3,1}(\mu)P_{1,1}(\nu)$$
$$-\tfrac{1}{6}P_{3,1}(\mu)P_{3,1}(\nu)\{(\nu_0^2-1)^2Q_1(\nu_0)-\tfrac{1}{3}(\nu_0^2-1)+\tfrac{2}{15}\}\Big] \qquad \text{............(12).}$$

The continuity of ψ_i and ψ_a at the surface follows from the mode in which they were obtained, viz., of the parts ψ_1 and ψ_2, of ψ_i, the second vanishes when the shell between ν and ν_0 is reduced to nothing, and ψ_1 has a form of the same type as ψ_a. It may be established directly by the use of (4), (5), and (10).

5. The treatment of the problem from the point of view of differential equations is decidedly simpler. Putting $\psi = \varpi\chi$ in the equation

$$\frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} + \frac{d^2\psi}{dz^2} = 2\omega\varpi = 2k\varpi^2,$$

we get
$$\frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} + \frac{d^2\chi}{dz^2} - \frac{\chi}{\varpi^2} = 2k\varpi,$$

showing that $\chi\cos\phi$ is a potential satisfying the equation

$$\nabla^2(\chi\cos\phi) = 2k\varpi\cos\phi = 2kx,$$

or
$$\frac{d}{d\nu}(\nu^2-1)\frac{d\chi}{d\nu} + \frac{d}{d\mu}(1-\mu^2)\frac{d\chi}{d\mu} - \chi\left(\frac{1}{1-\mu^2} - \frac{1}{1-\nu^2}\right)$$
$$= 2k\varpi c^3(\nu^2-\mu^2) = \frac{4kc^3}{15}\{P_{1,1}(\mu)P_{3,1}(\nu) - P_{1,1}(\nu)P_{3,1}(\mu)\}$$

for the inner region, while for the outer the right-hand member vanishes.

Thus

$$\chi_i = AP_{1,1}(\mu)P_{1,1}(\nu) + \frac{2kc^3}{75}\{P_{1,1}(\mu)P_{3,1}(\nu)+P_{3,1}(\mu)P_{1,1}(\nu)\}$$
$$+BP_{3,1}(\mu)P_{3,1}(\nu),$$
$$\chi_a = CP_{1,1}(\mu)Q_{1,1}(\nu) + DP_{3,1}(\mu)Q_{3,1}(\nu) \qquad \text{..............................(13).}$$

The particular solution is readily written down, and determines the general terms admissible. If we equate in χ_i, χ_e coefficients of like harmonics of μ for the surface value ν_0, this at the same time makes all differential coefficients with regard to μ equal at the surface, and therefore makes the normal components of velocity equal. If we make the first differential coefficients with regard to ν equal for the coefficients of the separate harmonics, then the tangential velocities are also equal, and complete continuity is secured. No other term is admissible, for, if $EP_{5,1}(\mu)\,P_{5,1}(\nu)$, $E'P_{5,1}(\mu)\,Q_{5,1}(\nu)$, for example, were compared, we should require

$$EP_{5,1}(\nu) = E'Q_{5,1}(\nu)$$

and
$$E\frac{d}{d\nu}\{P_{5,1}(\nu)\} = E'\frac{d}{d\nu}\{Q_{5,1}(\nu)\}$$

at the surface, or, combining, we should have $P_5Q_5' - P_5'Q_5 = 0$ instead of $-\dfrac{1}{\nu^2-1}$, as it is known to be.

Rejecting common factors, a comparison of the coefficients of $P_{3,1}(\mu)$ gives

$$D\frac{dQ_3}{d\nu} = B\frac{dP_3}{d\nu} + m, \quad D\frac{d^2Q_3}{d\nu^2} = B\frac{d^2P_3}{d\nu^2}, \quad \text{for} \quad \nu = \nu_0,$$

m being $\dfrac{2kc^3}{75}$. Combining

$$D\left\{(\nu^2-1)\frac{d^2Q_3}{d\nu^2} + 2\nu\frac{dQ_3}{d\nu}\right\} = B\left\{(\nu^2-1)\frac{d^2P_3}{d\nu^2} + 2\nu\frac{dP_3}{d\nu}\right\} + 2m\nu,$$

or
$$DQ_3 = BP_3 + \frac{m\nu}{6},$$

which, with the first equation, gives

$$D(P_3Q_3' - P_3'Q_3) = m\left(P_3 - \frac{\nu}{6}P_3'\right),$$

or
$$D = -\frac{5m}{4}\nu_0(\nu_0^2-1)^2 = -\frac{kc^3\nu_0}{30}(\nu_0^2-1)^2 \quad \ldots\ldots\ldots\ldots(14),$$

writing the surface value ν_0. Hence

$$B\nu_0 P_3 = D\left(P_3 Q_1 - \frac{5\nu}{6}\right) - \frac{m\nu^3}{6} \quad \text{[quoting (4)]}$$

$$= DP_3 Q_1 + \frac{m\nu_0}{12}(5\nu_0^2 - 7) P_3(\nu_0),$$

or

$$B = -\frac{kc^3}{30}\left\{(\nu_0^2-1)^2 Q_1(\nu_0) - \tfrac{1}{3}(\nu_0^2-1) + \tfrac{2}{15}\right\} \text{............(15).}$$

Again, comparing coefficients of $P_{1,1}(\mu)$,

$$CQ_1' - mP_3' = A, \quad CQ_1'' - mP_3'' = 0,$$

from which, as above, $A\nu_0 = CQ_1 - 6mP_3$,

or

$$A\nu_0(\nu_0^2-1) = -6m(\nu_0^2-1)P_3 + C(\nu_0^2-1)Q_1,$$

while the first, transformed by (5), gives

$$A\nu_0(\nu_0^2-1) = -m\nu_0(\nu_0^2-1)P_3' + C(\nu_0^2-1)Q_1 - C.$$

Hence

$$C = m(\nu_0^2-1)(6P_3 - \nu_0 P_3') = \frac{15m\nu_0}{2}(\nu_0^2-1)^2 = \frac{kc^3}{5}\nu_0(\nu_0^2-1)^2 \text{...(16),}$$

and

$$A = \frac{kc^3}{5}(\nu_0^2-1)^2 Q_1 - \frac{2kc^3}{25}(5\nu_0^2-3) = \frac{kc^3}{5}\left\{(\nu_0^2-1)^2 Q_1 - 2(\nu_0^2-1) - \tfrac{4}{5}\right\}$$
$$\text{............(17).}$$

These values of the constants make the stream function within and without the vortex exactly what was got before by direct evaluation of the integrals.

6. The external motion has a potential ϕ, which we proceed to find,

$$\frac{d\phi}{d\mu} = \frac{1}{\varpi}\frac{d\psi}{d\nu}\sqrt{\frac{\nu^2-1}{1-\mu^2}} = \frac{1}{c}\sqrt{\frac{\nu^2-1}{1-\mu^2}}\left(\frac{d\chi}{d\nu} + \frac{\chi}{\varpi}\frac{d\varpi}{d\nu}\right)$$

$$= \frac{1}{c}\sqrt{\frac{\nu^2-1}{1-\mu^2}}\left(\frac{d\chi}{d\nu} + \frac{\nu\chi}{\nu^2-1}\right).$$

Now, for a term $Q_{3,1}(\nu)$ in χ, we have

$$\frac{dQ_{3,1}}{d\nu} = \frac{d}{d\nu}\left(\sqrt{\nu^2-1}\,\frac{dQ_3}{d\nu}\right) = \frac{1}{\sqrt{\nu^2-1}}\left(\frac{d}{d\nu}(\nu^2-1)\frac{dQ_3}{d\nu} - \nu\frac{dQ_3}{d\nu}\right)$$

$$= \frac{1}{\sqrt{\nu^2-1}}(12Q_3 - \nu Q_3'),$$

or
$$\frac{dQ_{3,1}}{d\nu}+\frac{\nu Q_{3,1}}{\nu^2-1}=\frac{12Q_3}{\sqrt{\nu^2-1}}.$$

Hence
$$\frac{d\phi}{d\mu}=\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5\sqrt{1-\mu^2}}\left\{Q_1(\nu)P_{1,1}(\mu)-Q_3(\nu)P_{3,1}(\mu)\right\}$$
$$=\frac{2kc^3\nu\,(\nu_0^2-1)^2}{5}\left\{Q_1(\nu)P_1'(\mu)-Q_3(\nu)P_3'(\mu)\right\},$$

and
$$\phi=\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5}\left\{Q_1(\nu)P_1(\mu)-Q_3(\nu)P_3(\mu)\right\}\quad\text{......(18)},$$

V_n and V_t denoting the normal and tangential components of the external motion,

$$\left.\begin{aligned}V_t&=\frac{1}{c}\sqrt{\frac{1-\mu^2}{\nu^2-\mu^2}}\,\frac{d\phi}{d\mu}=\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5\sqrt{\nu^2-\mu^2}}\left\{Q_1(\nu)P_{1,1}(\mu)-Q_3(\nu)P_{3,1}(\mu)\right\}\\V_n&=\frac{1}{c}\sqrt{\frac{\nu^2-1}{\nu^2-\mu^2}}\,\frac{d\phi}{d\nu}=\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5\sqrt{\nu^2-\mu^2}}\left\{Q_{1,1}(\nu)P_1(\mu)-Q_{3,1}(\nu)P_3(\mu)\right\}\end{aligned}\right\}$$
...............(19).

If θ denotes the angle between the axis and the normal to surface ν,

$$\cos\theta=\mu\sqrt{\frac{\nu^2-1}{\nu^2-\mu^2}},$$

and the first term in V_n is

$$\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5}\,Q_1'(\nu)\cos\theta,$$

corresponding therefore to a translation of the surface as the surface of a solid moving in the direction of the axis with a velocity

$$\frac{2kc^3\nu_0(\nu_0^2-1)^2}{5}\,Q_1'(\nu),$$

which becomes less as ν is increased, and ultimately vanishes, while at the surface it is

$$V_0=\frac{2kc^3\nu_0}{5}(\nu_0^2-1)^2Q_1'(\nu_0),\quad\text{or}\quad\frac{2kc^3}{5}(\nu_0^2-1)\left\{Q_1(\nu_0^2-1)-1\right\}.$$

For the internal motion, there is no potential, but very similar work gives

$$\left.\begin{aligned} {}_iV_n &= \frac{2kc^2}{5\sqrt{\nu^2-\mu^2}}\Big[P_1(\mu)P_{1,1}(\nu)\left\{(\nu_0^2-1)^2Q_1-2(\nu_0^2-1)-\tfrac{4}{5}\right\} \\ &\qquad +\tfrac{2}{15}P_1(\mu)P_{3,1}(\nu)+\tfrac{4}{5}P_3(\mu)P_{1,1}(\nu) \\ &\qquad -P_3(\mu)P_{3,1}(\nu)\left\{Q_1(\nu_0^2-1)^2-\tfrac{1}{3}(\nu_0^2-1)+\tfrac{2}{15}\right\}\Big] \\ \text{and}\quad {}_iV_t &= \frac{2kc^2}{5\sqrt{\nu^2-\mu^2}}\Big[P_{1,1}(\mu)P_1(\nu)\left\{\ldots\ldots-\tfrac{4}{5}\right\}+\tfrac{4}{5}P_{1,1}(\mu)P_3(\nu) \\ &\qquad +\tfrac{2}{15}P_{3,1}(\mu)P_1(\nu)-P_{3,1}(\mu)P_3(\nu)\left\{\ldots\ldots+\tfrac{2}{15}\right\}\Big] \end{aligned}\right\}\ldots(20),$$

V_t in the direction of μ increasing.

The equality of the values at the surface inside and outside depends on the relations (5) and (10).

We may also write down the velocities U and V in the directions of ϖ and z,

$$V=\frac{1}{\varpi}\frac{d\psi}{d\varpi}=\frac{1}{c^2(\nu^2-\mu^2)}\left(\nu\frac{d\psi}{d\nu}-\mu\frac{d\psi}{d\mu}\right),$$

$$U=-\frac{1}{\varpi}\frac{d\psi}{dz}$$

$$=-\frac{1}{c^2(\nu^2-\mu^2)\sqrt{(\nu^2-1)(1-\mu^2)}}\left\{\mu(\nu^2-1)\frac{d\psi}{d\nu}+\nu(1-\mu^2)\frac{d\psi}{d\mu}\right\},$$

$$\left.\begin{aligned} U_i &= 3kc^3(\nu_0^2-1)\left\{Q_1(\nu_0^2-1)-\tfrac{1}{3}\right\}\mu\nu\sqrt{(\nu^2-1)(1-\mu^2)} \\ &= 3k(\nu_0^2-1)\left\{Q_1(\nu_0^2-1)-\tfrac{1}{3}\right\}\varpi z \\ U_a &= 3kc^3\nu_0(\nu_0^2-1)^2\frac{\nu\mu\sqrt{1-\mu^2}}{\sqrt{\nu^2-1}}\left\{(\nu^2-1)Q_1(\nu)-\tfrac{1}{3}\right\} \\ &= kc^3\nu_0(\nu_0^2-1)^2\mu Q_2'(\nu)\sqrt{(\nu^2-1)(1-\mu^2)} \\ V_a &= -2kc^2\nu_0(\nu_0^2-1)^2P_2(\mu)Q_2(\nu) \\ V_i &= -\frac{4kc^2}{9}P_2(\nu_0)+\frac{4kc^2}{9}\left\{P_2(\nu)+P_2(\mu)\right\} \\ &\qquad -2kc^2\left\{Q_1(\nu_0^2-1)^2-\tfrac{1}{3}(\nu_0^2-1)+\tfrac{2}{9}\right\}P_2(\nu)P_2(\mu) \\ &= -\frac{4kc^2}{9}P_2(\nu_0)+\frac{4kc^2}{9}\left\{P_2(\nu)+P_2(\mu)\right\} \\ &\qquad -\frac{2kc^2P_2(\nu)P_2(\mu)}{P_2(\nu_0)}\left\{\nu_0(\nu_0^2-1)^2Q_2(\nu_0)+\tfrac{2}{9}\right\} \end{aligned}\right\}\ldots(21).$$

7. This last may be employed to find the momentum in the direction of z, viz.,

$$MV_0 = 2\pi\rho \iint \varpi' V_i\, dS' = 2\pi\rho c^3 \iint (\nu^2-\mu^2)\, V_i\, d\nu\, d\mu.$$

With $$V_i = a+\beta\left\{P_2(\nu)+P_2(\mu)\right\}+\gamma P_2(\nu)\,P_2(\mu),$$

an easy integration gives

$$V_0 = a-\frac{\gamma}{5}+\frac{\beta}{10}(9\nu_0^2-1) = \frac{2kc^2}{5}\left\{Q_1(\nu_0^2-1)-1\right\}(\nu_0^2-1)$$

$$= \frac{2kc^2}{5}\,\nu_0\,(\nu_0^2-1)^2\,Q_1'.$$

This is the value of V_0 given by the first term of the external motion. The total energy of the motion is given by

$$T = -2\pi k\rho \iint \varpi\psi\, dS = -2\pi k\rho \iint \chi_i \varpi^2\, dS$$

$$= -4\pi k\rho \iint \chi_i \left\{P_{3,1}(\nu)\,P_{1,1}(\mu)-P_{1,1}(\nu)\,P_{3,1}(\mu)\right\} d\nu\, d\mu,$$

or, substituting the value of χ_i from (13),

$$T = -\frac{16\pi k\rho}{15}\int_1^{\nu_0} d\nu \left\{ \frac{A}{3} P_{3,1}(\nu) P_{1,1}(\nu) + \frac{m}{3}(P_{3,1})^2 - \frac{6m}{7}(P_{1,1})^2 - \frac{6B}{7} P_{3,1}(\nu)\, P_{1,1}(\nu) \right\}.$$

But $$\int_1^{\nu_0} \left\{\tfrac{1}{3}(P_{3,1})^2-\tfrac{6}{7}(P_{1,1})^2\right\} d\nu = \tfrac{3}{35}\nu_0(\nu_0^2-1)^2(25\nu_0^2+1),$$

and $$\int_1^{\nu_0} P_{3,1}(\nu)\,P_{1,1}(\nu)\, d\nu = \tfrac{3}{5}\nu_0\,(\nu_0^2-1)^3;$$

therefore

$$T = -\frac{16\pi k\rho c^4}{15}\left\{ \tfrac{3}{5}\left(\frac{A}{3}-\frac{6B}{7}\right)+\frac{3m}{28}(25\nu_0^2+1) \right\}\nu_0(\nu_0^2-1)^2$$

$$= -\frac{16\pi\rho k^2 c^7 \nu_0 (\nu_0^2-1)^3}{105}\left\{Q_1(\nu_0^2-1)-1\right\}$$

$$= -\frac{4Mk^2c^4(\nu_0^2-1)^2}{35}\left\{Q_1(\nu_0^2-1)-1\right\}$$

$$= -\frac{4Mk^2c^4\nu_0(\nu_0^2-1)^3 Q_1'}{35} = -\frac{2Mkc^2(\nu_0^2-1)\,V_0}{7} \quad\ldots\ldots\ldots\ldots\ldots(22).$$

The energy of the external motion can be obtained separately thus:

$$T_a = -\pi\rho\int_{-1}^{+1}\varpi\phi\frac{d\phi}{dn}\frac{ds}{d\mu}d\mu = -\pi\rho c\,(\nu_0^2-1)\int_{-1}^{+1}\phi\frac{d\phi}{d\nu}d\mu$$

$$= -\frac{4\pi\rho c^7 k^2\nu_0^2(\nu_0^2-1)^5}{25}\int_{-1}^{+1}\{Q_1(\nu_0)P_1(\mu)-Q_3(\nu_0)P_3(\mu)\}$$
$$\times\{Q_1'(\nu_0)P_1(\mu)-Q_3'(\nu_0)P_3(\mu)\}\,d\mu$$

$$= -\frac{2Mk^2c^4\nu_0(\nu_0^2-1)^4}{25}\{Q_1(\nu_0)Q_1'(\nu_0)+\tfrac{3}{7}Q_3(\nu_0)Q_3'(\nu_0)\} \quad \text{.........(23).}$$

When ν_0 is very large the most important term in $Q_1(\nu_0)$ is $\frac{1}{3\nu_0^2}$, and the values of T and T_a become in the limit $\frac{15}{14}M\dot{V}_0^2$ and $\frac{MV_0^2}{4}$, while V_0 has the same value as for the sphere. These are the proper values for the sphere.

At this stage it may be well to give the principal changes required to pass to the solution for the planetary ellipsoid. The value of ϖ is $c\sqrt{(\nu^2+1)(1-\mu^2)}$, and so (ν^2+1) replaces (ν^2-1) wherever the latter occurs. In lieu of the P and Q functions of ν, we have p and q solutions of

$$(\nu^2+1)\frac{d^2p_n}{d\nu^2}+2\nu\frac{dp_n}{d\nu} = n(n+1)p^n,$$

starting with

$$p_0 = 1,\quad p_1 = \nu,\quad p_2 = \frac{3\nu^2+1}{2},\ \ldots,\quad q_0 = \cot^{-1}\nu,\quad q_1 = 1-\nu\cot^{-1}\nu,\ \ldots.$$

Also $\nu^2+\mu^2$ replaces $\nu^2-\mu^2$ in equation (12); hence in (13) the particular solution containing $p_{1,1}(\nu)$ has its sign changed. The coefficients given by (14), (15), (16), and (17) are altered to

$$A = \frac{kc^3}{5}\{(\nu_0^2+1)^2q_1-2(\nu_0^2+1)+\tfrac{4}{5}\},$$

$$B = \frac{kc^3}{30}\{(\nu_0^2+1)^2q_1-\tfrac{1}{3}(\nu_0^2+1)-\tfrac{2}{15}\},$$

$$C = \frac{kc^3\nu_0}{5}(\nu_0^2+1),\quad D = \frac{kc^3}{30}\nu_0(\nu_0^2+1)^2.$$

In the results for T, T_a and V_0 no changes beyond those mentioned are required, but, in the course of the work for T, $\frac{A}{3}+\frac{6B}{7}$ occurs in lieu of $\frac{A}{3}-\frac{6B}{7}$, and $25\nu_0^2-1$ in lieu of $25\nu_0+1$.

8. To compare the energy of the system with that of the spherical vortex of equal volume, we write

$$c^3\nu_0(\nu_0^2-1)=a^3,$$

and replace c by a. The comparison gives

$$T/T_s=-\tfrac{3}{2}\left\{(\nu_0^2-1)\,Q_1(\nu_0)-1\right\}\left(\frac{\nu_0^2-1}{\nu_0^2}\right)^{\frac{1}{3}}.$$

For the ovary		ellipsoid axes	5 : 1,	the ratio is	·166,
„	„	„	2 : 1,	„	·475,
„	planetary	„	2 : 1,	„	1·78,
„	„	„	5 : 1,	„	3·2.

Formulæ may be given for the distribution of energy. Thus with

T = total energy, T_a = external energy,

T_t = energy of translation $\tfrac{1}{2}MV_0^2$,

T_r = remaining internal energy,

so that $$T=T_a+T_t+T_r,$$

we have $$T_t/T=-\tfrac{7}{10}\left\{(\nu_0^2-1)\,Q_1-1\right\},$$

$$T_a/T=\frac{7\,(\nu_0^2-1)}{10}\left\{Q_1+\tfrac{3}{7}\,\frac{Q_3Q_3'}{Q_1'}\right\};$$

and therefore $$T_r/T=\tfrac{8}{10}\left\{1-\frac{(\nu_0^2-1)\;Q_3Q_3'}{Q_1'}\right\}.$$

As numerical examples, take the ratios 5 : 1 and 2 : 1 for the axes in ovary and planetary cases, and compare with the distribution in the spherical case.

T	T_t	T_r	T_a	
100	66·1	29·4	4·5	ovary ellipsoid 5 : 1
100	57·8	29·5	12·7	„ „ 2 : 1
100	46·7	30	23·3	sphere
100	33·1	25·7	41·2	planetary ellipsoid 2 : 1
100	17·5	20·1	62·4	„ „ 5 : 1.

Now, consider the limits to which these figures approach for the cases of rod and disk. For a disk ν_0 is small,

$$q_1=1,\quad q_1'=-\frac{\pi}{2},\quad q_3=\tfrac{2}{3},\quad q_3'=-\frac{3\pi}{4};$$

hence $$V_0=-\frac{\pi kc^3\nu_0}{5},\quad T=\frac{2M\pi k^2c^4\nu_0}{35},\quad T_a=\frac{2M\pi k^2c^4\nu_0}{35},$$

the internal energy being evanescent compared with the external. Using the comparison with a sphere of equal volume through

$$a^3 = c^3\nu_0(\nu_0^2+1) = c^3\nu_0 \text{ in limit,}$$

and therefore $$c^3 = a^3\nu_0^{-1},$$

we get $$V_0 = -\frac{\pi k a^2}{5}\nu_0^{\frac{1}{3}},\quad T = T_a = \frac{2M\pi k^2 a^4}{35}\nu_0^{-\frac{1}{3}},$$

the energy being ultimately indefinitely great compared with a sphere for which

$$V = -\frac{4ka^2}{15},\quad T = \frac{8Mk^2a^4}{105},\quad T_a = \frac{4Mk^2a^4}{225},\quad T_t = \frac{8Mk^2a^4}{225}.$$

The limit of the ovary ellipsoid in a rod is given by

$$\nu_0 = 1 + x,$$

where x is very small, $$Q_0 = \tfrac{1}{2}\log\frac{2}{x};$$

and therefore

$$V_0 = -\frac{4kc^2x}{5}(1-2xQ_0),\quad T = \frac{16Mk^2c^4x^2}{35}(1-2xQ_0),\quad T_a = \tfrac{32}{35}Mk^2c^4x^3Q_0.$$

The energy of the surrounding medium is of inferior order, and

$$T_t = \tfrac{7}{10}T.$$

To compare with a sphere,

$$a^3 = c^3\nu_0(\nu_0^2-1) = 2c^3x;$$

therefore $$V_0 = -\frac{2ka^2}{5}2^{\frac{1}{3}}x^{\frac{1}{3}},\quad T = \frac{8Mk^2a^4x^{\frac{2}{3}}}{2^{\frac{1}{3}}\times 35},$$

the energy being indefinitely small compared with the sphere.

Comparing the velocity of translation with that of a sphere,

$$\frac{V_0}{V_s} = -\tfrac{3}{2}\left\{Q_1(\nu_0^2-1)-1\right\}\left(\frac{\nu_0^2-1}{\nu_0^2}\right)^{\frac{1}{3}}.$$

In the ovary	ellipsoid	$\frac{V_0}{V_s} =$	·4844	for axes	in ratio	5 : 1,
„	„	„	„	·7715	„	„	2 : 1,
„	planetary	„	„	1·126	„	„	2 : 1,
„	„	„	„	1·094	„	„	5 : 1.

The ratio in each case is ultimately vanishing, but for the planetary there is a maximum got by writing

$$\frac{d}{d\nu_0}\nu_0^{\frac{1}{3}}(\nu_0^2+1)^{\frac{1}{3}}(\nu_0^2+1)\,q_1' = 0,$$

which reduces to
$$\cot^{-1}\nu_0 = \frac{\nu_0(9\nu_0^2+7)}{(\nu_0^2+1)(9\nu_0^2+1)},$$

satisfied by
$$\nu_0 = \cdot 395,$$

for which
$$\frac{V_0}{V_s} = 1{\cdot}168.$$

With fixed volume and fixed constant (k) of vorticity, the value of T ranges from an infinitesimal at the rod limit to an infinite value at the disk limit. Writing r for

$$-\tfrac{3}{2}\left\{Q_1(\nu_0^2-1)-1\right\} \quad \text{or} \quad -\tfrac{3}{2}\left\{q_1(\nu_0^2+1)-1\right\},$$

we have with different vorticities

$$\frac{T}{T_s} = \frac{rk^2}{k_s^2}\left(\frac{\nu_0^2\mp 1}{\nu_0^2}\right)^{\frac{2}{3}}, \quad \frac{V_0}{V_s} = \frac{rk}{k_s}\left(\frac{\nu_0^2\mp 1}{\nu_0^2}\right)^{\frac{1}{3}};$$

therefore
$$\frac{V_0}{V_s} = \sqrt{r\frac{T}{T_s}}.$$

With vorticity adjusted so as to make the energy the same in each case,
$$V_0 = V_s\sqrt{r}.$$

For the ovary case this ratio increases slowly from 1 to $\sqrt{\frac{3}{2}}$; for the planetary case diminishes from 1 to the indefinitely small value $\sqrt{\frac{3\pi\nu_0}{4}}$.

9. The form obtained for V_n in (19) suggested that the first term represented a pure translation, and this was confirmed by finding the momentum in § 7. The second term represents the rate of normal extension or contraction of the boundary. At the end of time dt, therefore, the stream-function, say $\psi+\dot{\psi}\,dt$, is that due to the original vortex shape, plus a shell of thickness $V_n'\,dt$. The extra part $\dot{\psi}\,dt$ is found by taking the original integral form for ψ, omitting the

integration with regard to ν, and writing for $d\nu$,

$$\frac{\sqrt{\nu_0^2-1}}{c\sqrt{\nu_0^2-\mu^2}}V_n'\,dt, \quad \text{or} \quad -\frac{2kc\nu_0\,(\nu_0^2-1)^2}{5\,(\nu_0^2-\mu^2)}\,Q_3'(\nu_0)\,P_3(\mu)\,dt, \quad \text{or} \quad \frac{\gamma\,dt\,P_3(\mu)}{\nu_0^2-\mu^2},$$

where $$\gamma = -\tfrac{2}{5}kc\nu_0\,(\nu_0^2-1)^2\,Q_3'(\nu_0) \quad \text{......................(24)}$$

for the ovary, or $$+\frac{2kc\nu_0}{5}(\nu_0^2+1)^2\,q_3'(\nu_0)$$

for the planetary case, and in the latter case we have also $\nu_0^2+\mu^2$ for $\nu_0^2-\mu^2$. Hence

$$\dot{\psi}_a = -\frac{k\varpi}{\pi}\int_{-1}^{+1} c^4\gamma\,P_3(\mu')\sqrt{(\nu_0^2-1)(1-\mu'^2)}\,d\mu'\int_0^{\pi}\frac{\cos\phi\,d\phi}{D}.$$

But $$P_3(\mu')\sqrt{1-\mu'^2} = \tfrac{1}{7}\left\{P_4'(\mu')-P_2'(\mu')\right\}\sqrt{1-\mu'^2}$$
$$= \tfrac{1}{7}\left\{P_{4,1}(\mu')-P_{2,1}(\mu')\right\} \quad \text{...............(25)},$$

and it is clear that in $\int_0^{\pi}\frac{\cos\phi\,d\phi}{D}$ the only terms which survive the following integration are those of second and fourth orders. Hence

$$\dot{\psi}_a = \frac{k\gamma c^3\sqrt{\nu_0^2-1}\,\varpi}{14}\int_{-1}^{+1} d\mu'\left\{P_{4,1}(\mu')-P_{2,1}(\mu')\right\}$$
$$\times\left\{\tfrac{5}{18}P_{2,1}(\mu)\,Q_{2,1}(\nu)\,P_{2,1}(\nu_0)\,P_{2,1}(\mu')\right.$$
$$\left.+\tfrac{9}{200}P_{4,1}(\mu)\,Q_{4,1}(\nu)\,P_{4,1}(\nu_0)\,P_{4,1}(\mu')\right\}$$
$$= \frac{k\gamma c^3(\nu_0^2-1)\varpi}{14}\left\{-\tfrac{2}{3}P_{2,1}(\mu)\,Q_{2,1}(\nu)\,P_2'(\nu_0)+\tfrac{1}{5}P_{4,1}(\mu)\,Q_{4,}\;(\nu)\,P_4'(\nu_0)\right\}$$
$$\text{............(26)}.$$

The values of $\dot{u}$, $\dot{v}$, the instantaneous accelerations, are derived from $\dot{\psi}_a$ exactly as u, v from the original ψ. Using $\dot{\phi}$ in the same way for the potential from which $\dot{u}$, $\dot{v}$ are derived, exactly as in § 6 we found ϕ_a from ψ_a, here we get

$$\dot{\phi}_a = -\frac{2kc^3\,(\nu_0^2-1)}{7}\left\{P_2(\mu)\,Q_2(\nu)\,P_2'(\nu_0)-P_4(\mu)\,Q_4(\nu)\,P_4'(\nu)\right\}\text{...(27)}.$$

Within the vortex $\dot{u}$, $\dot{v}$ are also derivable from a potential which similar work proves to be

$$\dot{\phi}_i = -\frac{2k\gamma c^3\,(\nu_0^2-1)}{7}\left\{P_2(\mu)\,P_2(\nu)\,Q_2'(\nu_0)-P_4(\mu)\,P_4(\nu)\,Q_4'(\nu_0)\right\}\text{...(28)}.$$

At the surface we have

$$\frac{d\dot{\phi}_a}{d\nu} = \frac{d\dot{\phi}_i}{d\nu}, \quad \text{and} \quad \frac{d\dot{\phi}_a}{d\mu} - \frac{d\dot{\phi}_i}{d\mu} = 2k\gamma c^3 P_3(\mu) \text{............} (29),$$

the latter depending on $P_n Q_n' - P_n' Q_n = -\dfrac{1}{\nu^2-1}$ and (25). Also

$$\dot{\phi}_a - \dot{\phi}_i = \frac{2k\gamma c^3}{7}\{P_4(\mu) - P_2(\mu)\} = -\frac{k\gamma c^3}{6}(1-\mu^2)P_3'(\mu)$$

$$= \frac{k^2c^4\nu_0(\nu_0^2-1)^3 Q_3'(\nu_0)}{15}(1-\mu^2)P_3'(\mu).$$

But at the surface the original value of ψ, with the term belonging to translation omitted (ψ' say), was [see (3)]

$$-\frac{k\varpi c^3\nu_0(\nu_0^2-1)^2}{30}P_{3,1}(\mu)\,Q_{3,1}(\nu_0),$$

or

$$-\frac{kc^4\nu_0(\nu_0^2-1)^3}{30}Q_3'(\nu_0)(1-\mu^2)P_3'(\mu).$$

Hence at the surface

$$\dot{\phi}_i - \dot{\phi}_a = 2k\psi' \text{..............................} (30),$$

a result wanted for the dynamical equations.

10. The continuity of the new velocities $u+\dot{u}\,dt$, $u'+\dot{u}'dt$, ... at the new vortex surface follows from the mode of derivation from the Helmholtz integral, but may be verified independently. One section is obtained by writing for ν, $\nu_0 + \dfrac{\gamma\,dt\,P_3(\mu)}{\nu_0^2-\mu^2}$ in the original expressions for internal and external velocity; the other is derived from the functions due to the shell just found. Normal velocities depend on $\dfrac{d\psi}{d\mu}$, and at the displaced surface on $\dfrac{d^2\psi}{d\nu\,d\mu}$. These agree within and without, because ψ_i and ψ_a, $\dfrac{d\psi_i}{d\nu}$, $\dfrac{d\psi_a}{d\nu}$ were made equal term for term as regards μ. As we have just shown that

$$\frac{d\dot{\phi}_a}{d\nu} = \frac{d\dot{\phi}_i}{d\nu},$$

both sections of normal component agree. Tangential velocities depend on first differential coefficients with regard to ν, and at the disturbed surface on second differential coefficients. The condition for agreement is

$$\frac{1}{c}\sqrt{\frac{\nu_0^2-1}{\nu_0^2-\mu^2}}\,\frac{1}{\varpi}\left(\frac{d^2\psi_a}{d\nu^2}-\frac{d^2\psi_i}{d\nu^2}\right)\frac{\gamma\,dt\,P_3(\mu)}{\nu_0^2-\mu^2}$$
$$+\frac{1}{c}\sqrt{\frac{1-\mu^2}{\nu_0^2-\mu^2}}\left(\frac{d\dot{\phi}_a}{d\mu}-\frac{d\dot{\phi}_i}{d\mu}\right)dt=0,$$

or $$\sqrt{\nu_0^2-1}\left(\frac{d^2\chi_a}{d\nu^2}-\frac{d^2\chi_i}{d\nu^2}\right)+2kc^3(\nu_0^2-\mu^2)\sqrt{1-\mu^2}=0 \quad \text{[quoting (29)]}$$

But, by (13),

$$\frac{d^2\chi_a}{d\nu^2}-\frac{d^2\chi_i}{d\nu^2}=\sqrt{(\nu_0^2-1)(1-\mu^2)}\left[CQ_1'''(\nu_0)-mP_0'''(\nu_0)\right.$$
$$\left.+P_3'(\mu)\left\{DQ_3'''(\nu_0)-BP_3'''(\nu_0)\right\}\right],$$

and $$DQ_3'=BP_3'+m,\quad DQ_3''=BP_3'';$$

hence, applying $$(1-\nu^2)\,Q_3'''-4\nu Q_3''+10Q_3'=0,$$

we have $$DQ_3'''-BP_3'''=\frac{10}{\nu_0^2-1}(DQ_3'-BP_3')=\frac{10m}{\nu_0^2-1}.$$

Similarly $$CQ_3'''-mP_3'''=-\frac{10mP_3'(\nu_0)}{\nu_0^2-1};$$

therefore $$\sqrt{\nu_0^2-1}\left(\frac{d^2\chi_a}{d\nu^2}-\frac{d^2\chi_i}{d\nu^2}\right)=-10m\sqrt{1-\mu^2}\left\{P_3'(\nu_0)-P_3'(\mu)\right\}$$
$$=-2kc^3(\nu_0^2-\mu^2)\sqrt{1-\mu^2},$$

which is the required condition, verified by the differential equation method used in § 5.

11. We are now in a position to apply the dynamical equations; for the interior

$$\frac{dp}{\rho\,dz}+\dot{v}+u\frac{dv}{d\varpi}+(v-V_0)\frac{dv}{dz}=0,$$

$$\frac{dp}{\rho\,d\varpi}+\dot{u}+u\frac{du}{d\varpi}+(v-V_0)\frac{du}{dz}=0,$$

or $$\frac{dp}{\rho\,dz}+\dot{v}+\frac{d}{dz}(\tfrac{1}{2}q^2)-2k\frac{d\psi'}{dz}=V_0\frac{dv}{dz},$$

$$\frac{dp}{\rho\,d\varpi}+\dot{u}+\frac{d}{d\varpi}(\tfrac{1}{2}q^2)-2k\frac{d\psi'}{d\varpi}=V_0\frac{dv}{d\varpi},$$

where $$q^2 = u^2+v^2,$$

ψ' belongs to relative motion, and we have used

$$\frac{dv}{d\varpi}-\frac{du}{dz} = 2k\varpi.$$

Integrating, for the interior,

$$\left.\begin{aligned} &\frac{p}{\rho}+\dot{\phi}_i+\tfrac{1}{2}q^2-V_0v-2k\psi_i' = \frac{\Pi}{\rho} \\ &\text{while, for the exterior,} \\ &\frac{p'}{\rho}+\dot{\phi}_a+\tfrac{1}{2}q'^2-V_0v' \qquad = \frac{\Pi}{\rho} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(31),$$

where Π is the pressure at an infinite distance from the vortex.

The two pressures agree at the surface in virtue of the agreement of velocities and the relation (30).

At the centre of the ellipsoid $\nu = 1$, $\mu = 0$; therefore

$$\psi' = 0, \quad u = 0, \quad v = \frac{5V_0}{2}.$$

Also $$\dot{\phi}_i = \frac{kc^3\gamma\,(\nu_0^2-1)}{7}\left\{Q_2'(\nu_0)+\tfrac{3}{4}Q_4'(\nu_0)\right\} = \frac{kc^3\gamma\nu_0\,(\nu_0^2-1)}{4}\,Q_3'(\nu_0)$$

in virtue of $$nQ_{n+1}'+(n+1)\,Q_{n-1}' = (2n+1)\,\nu Q_n';$$

therefore $$\dot{\phi}_i = -\frac{k^2c^4\nu_0^2\,(\nu_0^2-1)^4}{10}\left\{Q_3'(\nu_0)\right\}^2$$

at the centre.

$$\left.\begin{aligned} &\text{Therefore } \frac{p_0}{\rho}-\frac{k^2c^4\nu_0^2(\nu_0^2-1)^4}{10}\left\{Q_3'(\nu_0)\right\}^2+\frac{5V_0^2}{8} = \frac{\Pi}{\rho} \\ &\text{or} \qquad \frac{p_0}{\rho}+\frac{k^2c^4\nu_0^2(\nu_0^2-1)^4}{10}\left[\left\{Q_1'(\nu_0)\right\}^2-\left\{Q_3'(\nu)\right\}^2\right] = \frac{\Pi}{\rho} \\ &\text{or} \qquad \frac{p_0}{\rho}+\frac{5V_0^2}{8}\left[1-\left\{\frac{Q_3'(\nu_0)}{Q_1'(\nu_0)}\right\}^2\right] = \frac{\Pi}{\rho} \end{aligned}\right\} \quad \ldots(32),$$

which for the sphere takes the well known form

$$\frac{p_0}{\rho}+\tfrac{5}{8}V_0^2 = \frac{\Pi}{\rho}.$$

It will be noticed that in the dynamical equations and in the original definition of $\dot{\phi}$, $\dot{\psi}$ reference to an origin moving with the vortex is assumed. We may describe $\dot{\phi}$, $\dot{\psi}$ as initial or instantaneous values of $\frac{d\phi}{dt}$, $\frac{d\psi}{dt}$; and a comparison of their forms with those of ϕ and ψ will show that they cannot for a spheroid as boundary be made functionally complete by taking c and ν_0 to depend on time. We had no right to expect this, for it is known that, when a spheroid changes to another of equal volume by a time variation of the axes, the value of $\frac{d\phi_n}{d\nu}$ at the surface is a harmonic of the second order (see asset's *Hydrodynamics*, Vol. II., p. 11); whereas for the present motion, when the term expressing pure translation is removed, it is a harmonic of the third order. The shape of the boundary surface is, at the end of time Δt,

$$\nu = \nu_0 + \frac{P_3(\mu)\,\gamma\Delta t}{\nu_0^2 - \mu^2},$$

and here also we have to be content with initial or instantaneous rate of deformation. In the diagrams (Figs. 1 and 2) showing this,

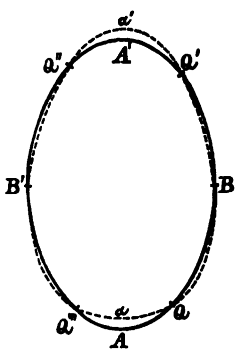

FIG. 1.

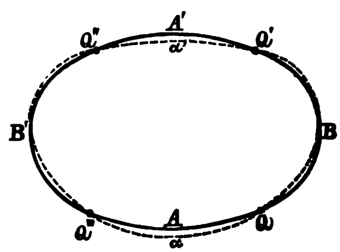

FIG. 2.

the motion of the vortex is downwards towards the foot of the page, and the dotted lines give the deformed shape. The ovary form presents the broader end in the direction of the motion; the contrary is the case for the planetary form.

It may be mentioned that the method of differential equations can be readily applied to find $\dot{\phi}_a$, $\dot{\phi}_i$. We should then use the argument of § 10, to show that continuity of motion at the surface requires

$$\frac{d\dot{\phi}_a}{d\nu} = \frac{d\dot{\phi}_i}{d\nu}, \quad \text{and} \quad \dot{\phi}_i - \dot{\phi}_a = 2k\psi';$$

these surface conditions, with

$$\nabla^2\dot{\phi}_i = 0, \quad \nabla^2\dot{\phi}_a = 0,$$

leading readily to (27) and (28).

These surface conditions make

$$\frac{dp}{d\mu} = \frac{dp'}{d\mu} \quad \text{and} \quad \frac{dp}{d\nu} = \frac{dp'}{d\nu},$$

the real discontinuity appearing only in the second differential coefficient of p with regard to ν. On this point and also in connexion with the following paragraph, a reference may be made to the preceding paper on "Continuity of Pressure in Vortex Motion," pp. 289, 293.

In Professor Hill's paper "On the Spherical Vortex," the velocities for the spheroid also, are calculated from the Helmholtz integrals, but no account is taken of the accelerations. It is clear that the omission of $\dot{\phi}_i$, $\dot{\phi}_a$ in the dynamical equations leaves the definite discontinuity $2k\rho\psi'$ in the surface values of p. Nor is it possible, starting from $\omega = k\varpi$ as initial distribution of spin, and the spheroid as initial boundary, to satisfy the conditions of steady motion by adding to u, v terms of the form $\frac{dP}{d\varpi}, \frac{dP}{dz}$. For then P would have to satisfy the conditions $\nabla^2 P = 0$ within and without the vortex, the derived velocities must vanish at infinity, be finite within the vortex, and yet present no discontinuity at the surface. These conditions reduce P to a constant. This is in accordance with Helmholtz's use of P, viz., to correct the values of velocities due to the potential functions L, M, N in the case where there is an external boundary. P is accordingly defined as the potential of matter on or beyond this external boundary, and does not exist for infinite liquid, as L, M, N give vanishing velocities at infinity. Also, if $\omega = k\varpi$ initially, this relation remains true so long as the motion is in meridian planes through the axis of z; thus the vorticity is constant, but the vortex boundary changes shape. The general internal condition for steady motion is known to be $\omega = \varpi f(\psi_i)$; attempts I have made with this more general form, to satisfy the boundary condition, have not hitherto proved successful.

12. It may be useful to write some of the results in the notation of the attraction of ellipsoids. For the ovary ellipsoid the connecting formulæ are

$$\left.\begin{aligned} &A = \sigma\left(Q_0 - \frac{1}{\nu}\right), \quad B = \frac{\sigma}{2}\left(\frac{\nu}{\nu^2-1} - Q_0\right), \quad H = \frac{\sigma c^2}{2} Q_0 \\ &\text{with } Q_0 = \tfrac{1}{2}\log\frac{\nu+1}{\nu-1}, \quad \sigma = 4\pi\nu_0(\nu_0^2-1) = \frac{4\pi a_0 b_0^2}{c^3}, \quad c^2 = a^2 - b^2 \\ &Aa^2 + 2Bb^2 = 2H, \quad A + 2B = \frac{\sigma}{\nu(\nu^2-1)} \; (= 4\pi \text{ at the boundary}) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(33).$$

When a and b are used they refer to a variable confocal outside, $a_0 b_0$ being used for the surface of the vortex; a is in the direction of the axis of symmetry, V the velocity in this direction, U at right angles to it. With

$$\beta = \frac{k(\nu_0^2-1)}{2\pi} = \frac{kb_0^2}{2\pi c^2},$$

$$\left.\begin{aligned} U_a &= \beta\varpi z(A-B), \qquad U_i = \beta\varpi z(A_0 - B_0) \\ V_a &= -\beta\left\{\tfrac{3}{2}(Az^2 + B\varpi^2) - H\right\} \\ V_i &= -\beta\left\{\tfrac{3}{2}(A_0 z^2 + B_0\varpi^2) - H_0\right\} + 2\pi\beta a_0^2\left(\frac{z^2}{a_0^2} + \frac{\varpi^2}{b_0^2} - 1\right) \\ \psi_a &= -\frac{\beta B c^2\varpi^2}{5} + \frac{\beta\varpi^2}{10}\left\{B(2a^2+3b^2) - 5Aa^2\right\}\left(\frac{z^2}{a^2} - \frac{\varpi^2}{4b^2}\right) \\ \psi_i &= -\frac{\beta B_0 c^2\varpi^2}{2} - \frac{\beta\varpi^2 z^2}{2}(A_0 - B_0) + \frac{\beta\varpi^4}{8b_0^2}\left\{a_0^2(A_0 + 2B_0) - 3B_0 b_0^2\right\} \\ V_0 &= -\frac{kB_0 b_0^2}{5\pi} = -\frac{2\beta B_0 c^2}{5} \\ \gamma &= \frac{\beta}{5c}\left\{B_0(2a_0^2 + 3b_0^2) - 5A_0 a_0^2\right\} \qquad \text{[deformation constant (24)]} \\ T &= \frac{2Mk^2 b_0^4}{35\pi} B_0 = \frac{2M\beta^2}{35} B_0(A_0 + 2B_0)c^4 \qquad \text{[total energy]} \\ T_i &= \frac{M\beta^2}{70}\left\{A_0^2 a_0^2(3a_0^2 + 2b_0^2) - 10A_0 B_0 a_0^2 b_0^2 + B_0^2(8a_0^4 - 14a_0^2 b_0^2 + 11b_0^4)\right\} \\ &\qquad\qquad\qquad \text{[internal energy]} \\ T_a &= \frac{M\beta^2}{70}\left\{2A_0 B_0(2a_0^4 + 2b_0^4 + a_0^2 b_0^2) - A_0^2 a_0^2(3a_0^2 + 2b_0^2) - B_0^2 b_0^2(2a_0^2 + 3b_0^2)\right\} \\ &\qquad\qquad\qquad \text{[external energy]} \\ T_t &= \frac{2M\beta^2 B_0^2 c^4}{25} \qquad \text{[energy of translation].} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(34$$

To reduce to the spherical case we write $r_0 = c\nu_0$, and make ν_0 infinite. That they reduce to correct values for the sphere has been tested, but some of the work is laborious.

For the planetary ellipsoid we write ν_0^2+1 for ν_0^2-1 in β; c^2 stands for b^2-a^2,

$$A = \sigma\left(\frac{1}{\nu}-\cot^{-1}\nu\right),\quad B = \frac{\sigma}{2}\left(\cot^{-1}\nu-\frac{\nu}{\nu^2+1}\right),\quad 2H = \sigma c^2\cot^{-1}\nu.$$

The forms for U, V are readily tested, and are suggested by the equations

$$\nabla^2 U = \frac{U}{\varpi^2},\quad \nabla^2 V = 4k \text{ or } 0,$$

which follow from $\dfrac{d^2\psi}{d\varpi^2}+\dfrac{d^2\psi}{dz^2}-\dfrac{1}{\varpi}\dfrac{d\psi}{d\varpi} = 2k\varpi^2$ or 0,

by differentiation with regard to ϖ and z. V_0 is then obtained by an integration through the spheroid for momentum, T_i in the same way, T requires the use of ψ_i, and T_a is got by subtracting these values.

13. We may use this form of ψ to trace the stream lines of the relative motion, correcting to ψ'' by the addition of $-\frac{1}{2}V_0\varpi^2$. This gives for the interior

$$\psi_i' = \frac{\beta\varpi^2}{2}\Big[-\tfrac{3}{5}B_0(a_0^2-b_0^2)-z^2(A_0-B_0)$$
$$+\frac{\varpi^2}{4b_0^2}\{(A_0+2B_0)a_0^2-3B_0b_0^2\}\Big]\ldots(34).$$

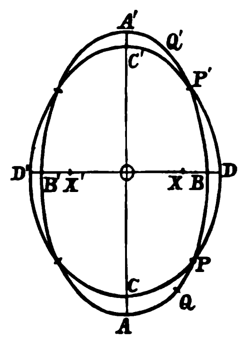

FIG. 3.

In the limit when $a_0 = b_0$, the bracket varies as $(r_0^2-z^2-\varpi^2)$ as for the sphere. To a zero value of ψ'' correspond the axis and the ellipsoid got by equating the bracket to zero: this for the ovary case has its major axis along the axis of symmetry but less than OA, while the minor axis is greater than OB. But outside the ellipsoid the surface must be continued with ψ_a'' instead of ψ_i'', this continuation departing slightly from true ellipsoidal form. We may speak of this surface which separates the stream lines which run in loops from those which go from $z = -\infty$ to $z = +\infty$ as the critical ovoid. The outer lines run roughly parallel to the axis, but bulge out in passing the ovoid. Contact is first made by a line touching the ellipsoid at Q, Q' points given by $\mu^2 = \frac{3}{5}$ roots of

$$P_3(\mu) = 0,\quad \left(\text{or } z^2 = \frac{3a_0^2}{5},\ \varpi^2 = \frac{2b_0^2}{5}\right).$$

Within this stream lines cross AQ, emerge through QP, skirt the ovoid, reenter the ellipsoid by $P'Q'$ and leave it by $Q'A'$. The inner stream lines are loops about X, X' points for which $V_i = V_0$, which gives

$$\varpi^2 \left\{a_0^2 (A_0+2B_0)-3B_0 b^2\right\} = \tfrac{6}{5} B_0 b_0^2 (a_0^2-b_0^2).$$

For the sphere the limiting position is given by

$$\varpi^2 = \frac{r_0^2}{2},$$

for the rod by
$$\varpi^2 = \frac{3b^2}{5},$$

for the disk by
$$\varpi^2 = \frac{2b^2}{5}.$$

One of these loops touches the ellipsoid at B, for the third root of $P_3(\mu) = 0$; loops outside this cross the ellipsoid along PB, and reenter along BP'. For the major axis of the ovoid we have generally

$$OC^2 = \frac{3B_0 c^2}{5(B_0 - A_0)},$$

and for the rod limit $\qquad OC^2 = \tfrac{3}{5} OA^2.$

If the ellipsoidal shape were continued outside, we should have

$$OD^2 = \tfrac{6}{5} OB^2$$

at this limit, but, when ψ_a'' is used,

$$OD^2 = \tfrac{5}{4} OB^2,$$

the correct value differing very little even in this extreme case from that got by continuing CP as an ellipsoid beyond P. The general value is given by
$$OD^2 = c^2(\nu^2 - 1),$$

where
$$B(\nu^2+1) - A\nu^2 = \frac{8B_0}{5}$$

determines the value of ν.

In the planetary case, the outer stream lines encroach on the ellipsoid; one of them touches at B, for one root of $P_3(\mu) = 0$; of the loops one touches at Q, Q', for which $\mu^2 = \frac{3}{5}$ the remaining roots of $P_3(\mu) = 0$, and the loops outside this cross AQ and reenter by QP. The critical surface is, within the range PP', a planetary ellipsoid more

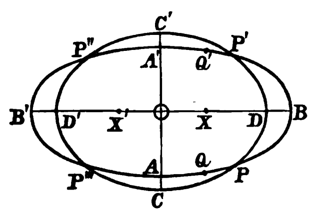

FIG. 4.

spherical than the original; without it the continuation departs slightly from the ellipsoidal shape. For the limiting case of disk shape

$$OD^2 = \tfrac{4}{5} OB^2;$$

and, OC being $c\nu$, ν is given by

$$B(\nu^2+1) - A\nu^2 = \frac{2B_0}{5},$$

in the general case, while for the limit

$$(3\nu^2+1)\cot^{-1}\nu - 3\nu = \frac{\pi}{5},$$

satisfied by $\nu = {\cdot}356$ approx. The extent of the loop circulation is here finite in both directions, and therefore embraces a mass of liquid infinite relatively to the rotational section; this according well with the statement in § 8 that, for this limit, all but a relatively infinitesimal amount of the energy is external. The encroaching of outer stream lines on the rotational territory, with the complementary encroaching of the loop circulation on the irrotational, is an essential feature of the motion, and gives rise to the deformation, which clearly shows the character proper to a harmonic of the third order. In each case the short diameter gives the region of strong attraction; thus in the planetary form a small body of non-rotating fluid, fore and aft, is dragged into the closed circulation, while a small section of rotating fluid amidships is flowing down the open stream lines, the parts being reversed in the ovary form. The extent of this region of cross flow is really very small except in the extreme cases; to prevent crowding of lines a sensible exaggeration was necessary in the diagrams. Figs. 5 and 6 (p. 326) show the stream lines; the dotted lines represent the vortex boundary in each case, the heavy lines the critical ovoid and planetoid. The vortex is moving towards the foot of the page, and the way in which the deformation is brought about by stream lines crossing the surface appears from a comparison with Figs. 1 and 2.

14. A good general view of the whole motion may be obtained by extending ϕ_a, the potential of the external motion, continuously within the rotational part. If ϕ_i has the continuity with ϕ_a due to any mass distribution within the vortex, $\frac{d\phi_i}{d\varpi}$, $\frac{d\phi_i}{dz}$ agree with the external values at the surface, *i.e.*, with U_a, V_a, *i.e.*, again with U_i, V_i.

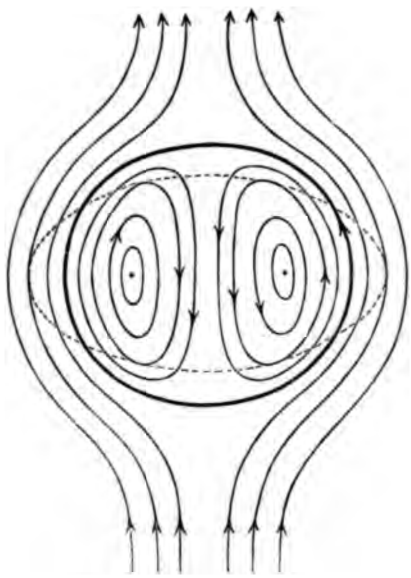

FIG. 5.

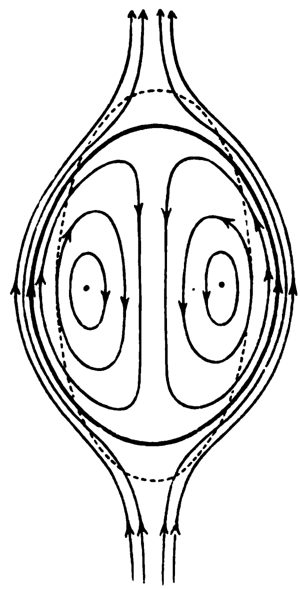

FIG. 6.

Now these are quadratic functions of z and ϖ, and a reference to (34) suggests the forms

$$\frac{d\phi_i}{dz} = -\beta\left\{\tfrac{3}{2}\left(A_0 z^2 + B_0 \varpi^2\right) - H_0\right\} + \alpha\left(\frac{z^2}{a_0^2} + \frac{\varpi^2}{b_0^2} - 1\right)$$

$$\frac{d\phi_i}{d\varpi} = \beta\varpi z\left(A_0 - B_0\right) + \gamma\left(\frac{z^2}{a_0^2} + \frac{\varpi^2}{b_0^2} - 1\right).$$

Comparing values of $\dfrac{d^2\phi_i}{d\varpi\, dz}$ derived from the two,

$$\gamma = 0, \quad \alpha = \frac{\beta b_0^2}{2}\left(A_0 + 2B_0\right) = 2\pi\beta b_0^2.$$

and then, if
$$4\pi\rho + \nabla^2\phi_i = 0,$$

we get
$$\rho = \frac{\beta\left(a_0^2 - b_0^2\right)}{a_0^2} z = \frac{k\left(\nu_0^2 - 1\right)}{2\pi\nu_0^2} z,$$

for the density of the matter within the ellipsoid which gives rise to ϕ_i within and ϕ_e without. We have then

$$V_i = \frac{d\phi_i}{dz} + kb_0^2\left(\frac{z^2}{a_0^2} + \frac{\varpi^2}{b_0^2} - 1\right), \quad U_i = \frac{d\phi_i}{d\varpi} \quad \text{............(35).}$$

Also

$$\phi_i = \beta z\left[-\left\{\tfrac{1}{2}(A_0 z^2 + 3B_0\varpi^2) - H_0\right\} + \frac{b_0^2}{2}(A_0 + 2B_0)\left(\frac{z^2}{3a_0^2} + \frac{\varpi^2}{b_0^2} - 1\right)\right]$$

The external value is

$$\phi_e = \frac{\beta z}{6}\left\{A(2\varpi^2 - 3z^2) - 5B\varpi^2 + a^2(A + 2B) + 2c^2(A - B)\right\} \qquad \text{...........(36).}$$

Hence the whole external motion is represented by the potential of a magnetic mass-distribution varying as z; the internal motion requires the addition of the simple term given in (35) to the component of velocity along the axis.

I have applied these methods to the ring vortex of finite circular section, and obtained results corresponding to §§ 1–7. It appears at once that the circular section is appropriate for steady motion only as a limiting case for indefinitely small section.

Thursday, March 12th, 1896.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

Mr. Horatio Scott Carslaw, B.A., Scholar of Emmanuel College, Cambridge, Lecturer in Mathematics in the University of Glasgow, and Miss Frances Hardcastle, Girton College, Cambridge, were elected members.

The President read an abstract of a paper by Prof. Lloyd Tanner, entitled "On the Enumeration of Groups of Totitives."

Prof. Greenhill read two papers, viz., (1) "The Motion of the Top," (2) "The Catenary on the Paraboloid and Cone." The President and Mr. Love joined in a discussion of the papers.

Lt.-Col. Cunningham stated that he had recently extended the evidence of the new criterion of 2 as a 16-ic residue so as to include *all* the primes less than 25 million for which the 16-ic character of 2 is known from Lucas' Table of Divisors of $(2^n - 1)$ in his paper "Sur la Série recurrente de Fermat." He also gave a proof that

$\frac{1}{4}(5^{11}-1) = 12207031$ and $\frac{1}{8}\cdot\frac{1}{23}(7^{11}+1) = 10746341$ are both *prime* numbers.

The following presents to the Library were received :—

"Proceedings of the Royal Society," Vol. LIX., No. 354.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 2; Leipzig, 1896.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 1; 1896.

"Proceedings of the Physical Society of London," Vol. XIV., Pts. 2, 3; February, March, 1896.

"Nyt Tidsskrift for Matematik," A. Aargang 6, Nr. 8, Aargang 7, Nr. 1; B. Aargang 6, Nr. 4; Copenhagen, 1895, 1896.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XII., No. 4; Coimbra, 1895.

"Monatshefte für Mathematik und Physik," Jahrgang 7, Hefte 1, 2, 3; Wien, 1896.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXIX., Liv. 4, 5; Harlem, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 5; New York, February, 1896.

"Bulletin de la Société Mathématique de France," Tome XXIV., No. 1; Paris, 1896.

"Bulletin des Sciences Mathématiques," Tome XX.; Paris, January, 1896.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," 1895, Heft 4; Göttingen, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 3, Vol. II., Fasc. 1; Napoli, 1896.

"Rendiconti del Circolo Matematico di Palermo," Tomo X., Fasc. 1, 2; 1896.

"Journal of the College of Science, Japan," Vol. VIII., Pt. 2, Vol. IX., Pt. 1.

Drude, P.—"Über die Anomale elektrische Dispersion von Flüssigkeiten," Roy. 8vo; Leipzig, 1896.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 3, 4; Roma, 1896.

"Indian Engineering," Vol. XIX., Nos. 4, 5, 6, 7, January 25th to February 15th, 1896.

"Educational Times," March, 1896.

On the Enumeration of Groups of Totitives. *By* Prof. H. W. Lloyd Tanner.

Received and read March 12th, 1896.

This paper explains a method of determining how many groups of given order can be formed with the totitives of any integer n. In the investigation use is made of a function formed from a binomial coefficient by replacing each factor, say r, of the numerator or denominator by $p^r - 1$, so that the binomial coefficient is, in fact, the limiting value of the function as p approaches 1.

There are indications of the existence of a reciprocity theorem [namely, that the number of groups of order ν is equal to the number of groups of order $\tau(n)/\nu$], but this theorem is not proved. The attempt to establish the theorem has led to the discovery of some notable properties of the functions—a Vandermonde theorem, for instance—which will be found in the paper.

The functions in question are well known. They were used by Euler as generating functions for the number of partitions, and by Cayley.* Jacobi in a memoir "Ueber einige der Binomialreihe analoge Reihe," 1846 (*Crelle*, XXXII.; *Ges. Werke*, Bd. II., 163–173), starting with a more general function, obtained a number of formulæ which appear to be different from those used in this paper. Gauss in the famous *Summatio serierum quarundam singularium* used these functions, the base being a complex number of modulus 1. They have been used too (in Schellbach's treatise) as a means of forming the theta-functions. The present application is of a different kind. As in Euler's theory, they are used for enumeration; but the number sought is given by the actual value of the function when the base p is a prime factor of τn.

A portion of the following (Arts. 1–17) was communicated to the Society just five years ago as a part of the paper under the title "Some Theorems concerning Groups of Totitives of n." When revising that paper for the press I found (thanks to the criticism of my colleague Mr. Pinkerton) a flaw in the proof of the reciprocity theorem, and therefore asked permission to withdraw it. Although I am not yet able to give a proof of the theorem, I hope that the

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* "Researches on the Partition of Numbers," *Phil. Trans.*, CXLV., *C. M. P.*, Vol. II., p. 243, &c.

[redacted]

method of enumeration and the properties of the functions employed will be acceptable to the Society.

1. The group of all the totitives of n is expressible in one way only (I. 22, 23)* as a product of sub-groups G_p, G_q, ..., G_r, where G_p consists of all the totitives whose orders are powers of p; G_q, ..., G_r have the like meaning with respect to q, ..., r; and p, q, ..., r are all the prime factors of rn.

Similarly any sub-group, G', of totitives of n is expressible in one way only as a product

$$G'_p . G'_q, \ldots, G'_r,$$

where G'_p includes all elements of G' whose orders are powers of p, and so for G'_q, ..., G'_r.

We shall consider two groups, say G', G'', to be different if either contains an element that is not contained in the other. Thus, if G'_p and G''_p are different groups, the two products

$$G'_p . G'_q, \ldots, G'_r \quad \text{and} \quad G''_p . G'_q, \ldots, G_r$$

are different. Now suppose that there are P different sub-groups G'_p, which are factors of G_p, counting G_p itself and 1 as sub-groups; and, similarly, Q different groups G'_q which are factors of G_q, ..., and R different groups G'_r factors of G_r. Then it is clear that the number of all the different groups G' (including G and 1) which are factors of G is the product $PQ \ldots R$. The problem of enumerating the different groups G' of the totitives of n is thus reduced to the simpler problem of enumerating the different groups G'_p which are factors of G_p, p being any one of the prime factors of rn.

2. The group G_p naturally presents itself as a product of simple independent groups which may be obtained in a canonical form in the following way.

Let
$$n = p^{\alpha} . q^{\beta} \ldots r^{\gamma},$$

where p, q, ..., r are different primes, and α, β, ..., γ are positive integers, but α may be zero. Then the q^{β} numbers

$$knq^{-\beta}+1, \quad k = 0, 1, 2, \ldots, q^{\beta}-1$$

* The reference is to § I., Arts. 21, 22, of a memoir "On Cyclotomic Functions," *Proc. Lond. Math. Soc.*, Vol. xx., pp. 63–83. Similar abbreviations are used in the sequel.

are all prime to $p, \ldots, r$, but $1/q$ of them are multiples of q. The rest, $(q-1)\, q^{\beta-1}$ in number, make up a group of totitives of n isomorphous with the group of all the totitives of q^{β}. Hence, if p^{λ} is the highest power of p which is a factor of $q-1$ (so that λ may be zero), this group of totitives of n contains as a factor a simple group of order p^{λ}, and no other elements of G_p. The same process applies to each prime factor of n except p itself. If α, the exponent of p in n, is greater than 1, then there is in general a group of order $p^{\alpha-1}$ consisting of totitives of the form $kpq^{\beta} \ldots r^{\gamma}+1$ $(k = 0, 1, 2, \ldots, p^{\alpha-1}-1)$. There is an exception when $p = 2$, in which case the simple group of order $p^{\alpha-1}$ is replaced by the product of two simple groups of orders 2 and $2^{\alpha-1}$, respectively.

3. Using the notation of (I. 33), we shall represent by h_i the number of simple groups of order p^i contained in the canonical form of G_p just obtained, where $i = 1, 2, \ldots, \mu$ and p^{μ} is the order of the largest simple group present.

By k_i we denote how many factor groups are of order p^i at least. Thus

$$k_i = h_i + h_{i+1} + \ldots + h_{\mu}.$$

K_i is defined by the equation

$$K_i = k_1 + k_2 + \ldots + k_i;$$

so that $$K_i = h_1 + 2h_2 + \ldots + (i-1)\, h_{i-1} + i\,(h_i + h_{i+1} + \ldots + h_{\mu}).$$

It follows that $$K_i - K_{i-1} = k_i;$$

and, since $K_1 = k_1$, this gives $K_0 = 0$.

4. The numbers h_i, k_i, K_i have been obtained from a special form of the group G_p; but they have a significance with respect to G_p itself, and are independent of the particular mode in which G_p is represented. For p^{K_i} is the number of elements in G_p of order not greater than p^i (I. 33). This number, and therefore K_i, is evidently independent of the way in which G_p may be factorized. Hence also k_i, $= K_i - K_{i-1}$, and h_i, $= k_i - k_{i+1}$, are invariants of G_p as well as K_i.

5. It will be convenient, before attacking the general case, to consider the particular case in which G_p is the product of h_1 independent groups of order p. Then G_p contains p^{h_1} elements of which one is of order 1, and p^{h_1} elements of which one is of order 1 and $p^{h_1}-1$ are of order p. Now any group of order p contains $p-1$

elements of order p, and none of these elements can be common to two different groups of order p. Hence the number of different groups of order p which can be formed from the elements of G_p is

$$(p^{h_1}-1)/(p-1).$$

A group of order p^2 must, in this particular case, be the product of two groups of order p—say A, B. Now A may be any one of the $(p^{h_1}-1)/(p-1)$ different groups formed from the elements of G_p; but A being selected removes $p-1$ of the elements of order p, and leaves only $p^{h_1}-p$ available for the making of B. Thus there are $(p^{h_1}-1)/(p-1)$ different groups B for each group A, and therefore there are $(p^{h_1}-1)(p^{h_1}-p)/(p-1)^2$ different products A, B: if by "different products" we understand products in which either the first factors are different, or the second factors are different, or both different. Now clearly "different products" do not necessarily constitute different groups; in fact, taking a G_p of order p^2, so that $h_1 = 2$, we find that any product of two simple groups of order p is represented by

$$(p^2-1)(p^2-p)/(p-1)^2$$

different products. Hence the number of different groups of order p^2 that can be formed with elements of G_p is found, on dividing the number of different products A, B by this last number, to be

$$\frac{(p^{h_1}-1)(p^{h_1-1}-1)}{(p^2-1)(p-1)}.$$

This function will be denoted by $\left\{ {h_1 \atop 2} \right\}$ in imitation of a well-known notation for binomial coefficients.

Similarly, we shall write

$$\left\{ {h \atop i} \right\} \quad \text{for} \quad \frac{(p^h-1)(p^{h-1}-1)\dots(p^{h-i+1}-1)}{(p^i-1)(p^{i-1}-1)\dots(p-1)}$$

(h, i being positive integers); from which it follows that

$$\left\{ {h \atop i} \right\} = \left\{ {h \atop h-i} \right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots \ (1),$$

and that $$\left\{ {h \atop i} \right\} \Big/ \left\{ {h \atop i-1} \right\} = \frac{p^{h-i+1}-1}{p^i-1} = \left\{ {h-i+1 \atop 1} \right\} \Big/ \left\{ {i \atop 1} \right\} \quad \dots \ (2).$$

6. It can now be shown, by induction from $i-1$ to i, that $\left\{ {h_1 \atop i} \right\}$ different groups of order p^i can be formed from the elements of G_p.

In fact, we may represent such a group as a product of two groups, A of order p^{i-1} and B of order p. By hypothesis, there are $\left\{ {h_1 \atop i-1} \right\}$ different groups A, and each takes up $p^{i-1}-1$ of the elements of order p, leaving $p^{h_1}-p^{i-1}$ such elements available for the formation of B. The number of different products thus made is

$$\left\{ {h_1 \atop i-1} \right\} \cdot \frac{(p^{h_1-i+1}-1)}{p-1} p^{i-1} = \left\{ {h_1 \atop i-1} \right\} \left\{ {h_1-i+1 \atop 1} \right\} p^{i-1}$$

But, on putting $h_1 = i$ in this formula, it is seen that one group of order p^i is represented by

$$\left\{ {i \atop i-1} \right\} p^{i-1} = \left\{ {i \atop 1} \right\} p^{i-1}$$

different products. Hence on division it appears that there are

$$\left\{ {h \atop i-1} \right\} \left\{ {h-i+1 \atop 1} \right\} \Big/ \left\{ {i \atop 1} \right\} = \left\{ {h \atop i-1} \right\} \left\{ {h \atop i} \right\} \Big/ \left\{ {h \atop i-1} \right\} \quad \text{[by (2)]}$$

$$= \left\{ {h \atop i} \right\}$$

different groups of order p_i formed from the elements of G_p.

Hence the theorem is true for i if true for $i-1$; and, having been proved for $i-1 = 1, 2$, it is established generally.

7. A consequence of (1), Art. 5, is that, for this particular form of G_p, there are as many different groups of order p^a as there are of order p^{h_1-a}. The converse is also true—viz., if the groups of order p^a, p^β are equi-numerous, then either $a = \beta$ or $a+\beta = h$. For it is clear from (2) of Art. 5 that $\left\{ {h_1 \atop i} \right\}$ always increases with i when $i+1 < h_1-i$, and decreases as i increases when $h_1-i < i+1$.

General Case. (Arts. 8–15.)

8. In the general case, G_p is a product of k_1 simple independent groups—viz., h_i groups of order p_i when $i = 1, 2, \ldots, \mu$. A group G'_p, which is a factor of G_p, is a product of k'_1 simple independent groups, of which h'_i are of order p^i ($i = 1, 2, \ldots, \mu$). It is convenient to extend to accented letters the notation of Art. 3, so that h', k', K' have the same meanings in reference to G'_p as h, k, K in reference to G. We proceed to enumerate the number of different groups G'_p for which all the h' are given.

9. G_p contains p^{K_i} elements of order not greater than p^i, and therefore the number of elements of order p_i is

$$p^{K_i}-p^{K_{i-1}} = p^{K_{i-1}}(p^{k_i}-1).$$

Each of these elements can serve as base of a group of order p^i; no one of them can be common to two different groups of this order, and every such group contains p^i-p^{i-1} of them. Therefore the number of different monobasic groups of order p^i contained in G_p is

$$p^{K_{i-1}}(p^{k_i}-1)/p^{i-1}(p-1) = p^{K_{i-1}-i+1}\left\{\begin{matrix} k_i \\ 1 \end{matrix}\right\}.$$

10. The investigation might be continued on the lines of Art. 5, but it seems better to use a different process suggested by the form of the last result, which has the advantage of interpreting the two factors.

Let A denote any simple group of order p^i formed from the elements of G_p, having a for its base, and write a for $a^{p^{i-1}}$, so that the group

$$(\mathrm{a}, \mathrm{a}^2, \ldots, \mathrm{a}^p = 1)$$

is the group of order p that is included in A. This group will be called the penultimate group of A (or of a). It will be observed that a is characterized by being of order p and at the same time a power $a^{p^{i-1}}$. The penultimate group, including all the a's, is of order p^{k_i}, for it is the penultimate of the product of the k_i groups (order $\overline{\overline{>}}\, p^i$). Thus the factor

$$\left\{\begin{matrix} k_i \\ 1 \end{matrix}\right\}$$

of the result in Art. 9 expresses the number of different penultimate groups in G_p each of which can be the penultimate of a simple group of order p^i. The other factor shows how many different groups of order p^i belong to one and the same penultimate. (*Cf.* Art. 12.)

11. The number of different penultimate groups in G_p which can be penultimates of G'_p is

$$\prod_i \left\{\begin{matrix} k_i-k'_i+h'_i \\ h'_i \end{matrix}\right\} = \prod_i \left\{\begin{matrix} k_i-k'_{i+1} \\ k'_i-k'_{i+1} \end{matrix}\right\}.$$

Here h'_i, k'_i are supposed to be known for all values of i; but G'_p represents any group which is (i.) a factor of G_p, (ii.) has the given constants h'_i, k'_i, but is otherwise unconditioned.

In forming the above expression, we begin with the h'_μ groups of order p^μ in G_p, and then consider in succession the groups of lower order. Now the penultimate of the h'_μ groups of G'_p is necessarily a factor of the penultimate of the k_μ groups (order p_μ) of G_p. Hence (Art. 6) there are

$$\left\{ \begin{matrix} k_\mu \\ h'_\mu \end{matrix} \right\}$$

different penultimates available for the h'_μ groups of G'_p. [This is what the specimen factor of Π becomes when $i = \mu$, since $k'_\mu = h'_\mu$ and $h'_{\mu+1} = 0$. If $h'_\mu = 0$, the value 1 is to be assigned to the symbol.]

Consider now the penultimate of the h'_i groups of G'_p which are of order p^i. It must be a factor of the penultimate of the k_i groups of G_p whose orders are $\geqq p^i$. But, of these k_i groups, h'_{i-1} have already been taken up by the p^{i-1}-ordinal groups of G'_p, h'_{i+2} by the groups of order p^{i+2}, and the penultimate of G_p still available for a penultimate of the p^i-ordinal groups of G'_p is a product of

$$k_i - h'_{i+1} - h'_{i+2} - \ldots - h'_\mu = k_i - k'_{i+1} = k_i - k'_i + h'_i$$

groups of order p.

From the elements of this product, we can obtain (Art. 6)

$$\left\{ \begin{matrix} k_i - k'_i + h'_i \\ h'_i \end{matrix} \right\} = \left\{ \begin{matrix} k_i - k'_{i-1} \\ h'_i - k'_{i+1} \end{matrix} \right\}$$

different groups of order p^i, each of which is a penultimate of h'_i p^i-ordinal groups.

The product of the numbers just obtained—i being replaced by 1, 2, ..., μ in turn—gives the number of different penultimates of G_p which are penultimates of a G'_p with the given h'_i, k'_i, and this is the statement that was to be proved.

12. To every penultimate group of G_p there correspond

$$\Pi p^{k'_i(K_{i-1}-K'_{i-1})} = p^{\Sigma h'_i(K_{i-1}-K'_{i-1})}$$

different groups G'_p.

For let a be the base of one of the simple groups in the selected penultimate, and let this simple group be the penultimate of a group of order p^i in G'_p. If a is the base of the last-named group in any one of the groups G'_p, then a is a solution of the congruence

$$a^{p^{i-1}} \equiv \text{a}, \bmod n;$$

and ab is a solution if, and only if, b is of order p^{i-1} at most. The number of multipliers b (including unity) is therefore $p^{K_{i-1}}$. The same thing may be done with each of h'_i groups of order p^i, and thus it is seen that, for a given penultimate, we have

$$\Pi p^{h'_i K_{i-1}}$$

representations of G'_p which differ from each other in the bases of the several groups. It is obvious, however, that different representations do not mean different groups G'_p; and, by putting $h_i = h'_i$ (and therefore $K_{i-1} = K'_{i-1}$) for all values of i, we find, from the above formula, that G'_p admits of

$$\Pi p^{h'_i K'_{i-1}}$$

different representations. Dividing the former by the latter number, it is found that there are

$$\Pi_i p^{h'(K_{i-1}-K'_{i-1})}$$

different groups G'_p the penultimates of which are identical.

13. Combining the results of Arts. 11, 12, we find that the number of different groups G'_p (with given values of h'_i) that can be formed from the elements of a group G_p is

$$p^{\Sigma_i h'_i(K_{i-1}-K'_{i-1})} \cdot \Pi_i \left\{ \begin{matrix} k_i - k'_i + h'_i \\ h'_i \end{matrix} \right\},$$

where $i = 1, 2, \ldots, \mu$. But, since the specimen factor becomes 1 when $h'_i = 0$, it is only needful to consider those values of i for which h'_i does not vanish.

14. To obtain the number of different groups G'_p of given order $p^{\omega'}$, we must form all the sets $h'_1, h'_2, \ldots, h'_\mu$ which satisfy the equation

$$\omega' = h'_1 + 2h'_2 + \ldots + \mu h'_\mu,$$

subject to the condition that for every value of i

$$k'_i = h'_i + h'_{i+1} + \ldots + h'_\mu \leqq k_i,\text{*}$$

and then calculate the number of groups G'_p for each system. The sum of these numbers is the number of different groups of the proposed order that are contained in G_p.

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* If any set of h' is used in which the condition is violated, the number involves a vanishing factor $\{\ \}$ in which the lower symbol is greater than the upper. Accordingly the condition may be ignored if convenient.

15. For the sake of reference, the formulæ are here given for $\varpi' = 1, 2, 3, 4$. An example which follows (Art. 17) will sufficiently explain the details of the calculations.

Order of G'_p.	Number of different groups G'_p.	Sets of h'.
p'	$\begin{Bmatrix} k_1 \\ 1 \end{Bmatrix}$	1
p^2	$\begin{Bmatrix} k_1 \\ 2 \end{Bmatrix} + p^{K_1-1} \begin{Bmatrix} k_2 \\ 1 \end{Bmatrix}$	20 ; 01
p^3	$\begin{Bmatrix} k_1 \\ 3 \end{Bmatrix} + p^{K_1-2} \begin{Bmatrix} k_1-1 \\ 1 \end{Bmatrix} + p^{K_2-2} \begin{Bmatrix} k_3 \\ 1 \end{Bmatrix}$	300 ; 110 ; 001
p^4	$\begin{Bmatrix} k_1 \\ 4 \end{Bmatrix} + p^{K_1-3} \begin{Bmatrix} k_1-1 \\ 2 \end{Bmatrix} \begin{Bmatrix} k_2 \\ 1 \end{Bmatrix} + p^{2K_1-4} \begin{Bmatrix} k_2 \\ 2 \end{Bmatrix}$	400; 210; 020
	$+ p^{K_2-3} \begin{Bmatrix} k_1-1 \\ 1 \end{Bmatrix} \begin{Bmatrix} k_3 \\ 1 \end{Bmatrix} + p^{K_3-3} \begin{Bmatrix} k_4 \\ 1 \end{Bmatrix}$	101 ; 0001.

16. As an example of the mode of working, I take G_p to be the product of four simple groups of orders p^4, p^4, p^2, p, respectively; so that $h_1 = 1$, $h_2 = 1$, $h_3 = 0$, $h_4 = 2$. It will sometimes be useful to symbolize such a group by 4421.

Instances of this group are given by the totitives of

$$n = 5440 = 2^6 . 5 . 17 \quad (\tau n = 2 . 2^4 \times 2^2 \times 2^4),$$

or by the group G_2 for

$$n = 28815 = 3 . 5 . 17 . 113 \quad (\tau n = 2 \times 2^2 \times 2^4 \times 7 . 2^4),$$

or, again, by the group G_3 for

$$n = 5267997 = 3^5 . 7 . 19 . 163 \quad (\tau n = 2 . 3^4 \times 2 . 3 \times 2 . 3^2 \times 2 . 3^4).$$

In each of these instances the other G_p are of a simple character, so that the complete enumeration of the groups of totitives of any proposed order for each of these values of n is effected with ease.

17. It is convenient to arrange the values of h_1, k_1, K in a table, thus—

i	h_i	k_i	K_{i-1}
1	1	4	.
2	1	3	4
3	.	2	7
4	2	2	9

[redacted]

where k_i is the sum of h_i and all the h that follow it, while K_{i-1} is the sum of all the k above k_i.

Let us find the number of groups G'_p of order p^4. For this purpose, 4 is partitioned, and for each partition of 4 a table is formed like the above. For example, for the partition 31, the table is the first of these—

i	h'_i	k'_i	K'_{i-1}
1	1	2	.
2	.	1	2
3	1	1	3

i	h'_i	$k_i-k'_i$	$K_{i-1}-K'_{i-1}$
1	1	2	.
2	.	2	2
3	1	1	4

But it is more convenient to form at once a table like the second, which shows the differences $k_i-k'_i$, $K_{i-1}-K'_{i-1}$. Using this second table, the number of different groups G'_p of this type is written down. The exponent of p is the sum of the products of entries (on the same line) in the second and fourth columns. The lower numbers in the $\{\ \}$ are the entries in the second column; the upper numbers are found by adding to the lower number the corresponding third-column entry. Thus, for this partition, the number of G'_p of the type considered is

$$p^4 \left\{\begin{matrix}2\\1\end{matrix}\right\} \left\{\begin{matrix}3\\1\end{matrix}\right\} = p^4+2p^5+2p^6+p^7.$$

This is the second line of the following scheme, which shows the number of G'_p of order p^4:—

			p^0	p	p^2	p^3	p^4	p^5	p^6	p^7
4	$p^6\left\{\begin{matrix}2\\1\end{matrix}\right\}$	=	.	.	.	.	.	.	1	1
31	$p^4\left\{\begin{matrix}2\\1\end{matrix}\right\}\left\{\begin{matrix}3\\1\end{matrix}\right\}$	=	.	.	.	.	1	2	2	1
22	$p^4\left\{\begin{matrix}3\\2\end{matrix}\right\}$	=	.	.	.	.	1	1	1	.
211	$p\left\{\begin{matrix}3\\1\end{matrix}\right\}\left\{\begin{matrix}3\\2\end{matrix}\right\}$	=	.	1	2	3	2	1	.	.
1111	1	=	1	.	.	.	.	.	.	.

$$\text{Sum} = 1+p+2p^2+3p^3+4p^4+4p^5+4p^6+2p^7$$

In the same way the number of groups of order p^7 is found as in the following table:—

		p^0	p	p^2	p^3	p^4	p^5	p^6	p^7
43	$p^6\begin{Bmatrix}2\\1\end{Bmatrix}$	.	.	.	.	.	.	1	1
421	$p^4\begin{Bmatrix}2\\1\end{Bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}$	.	.	.	.	1	3	3	1
4111	$p^3\begin{Bmatrix}2\\1\end{Bmatrix}$	.	.	.	1	1	.	.	.
331	$p^4\begin{Bmatrix}2\\1\end{Bmatrix}$	.	.	.	.	1	1	.	.
322	$p^3\begin{Bmatrix}2\\1\end{Bmatrix}$	.	.	.	1	1	.	.	.
3211	$p\begin{Bmatrix}2\\1\end{Bmatrix}\begin{Bmatrix}2\\1\end{Bmatrix}$	.	1	2	1	.	.	.	.
2221	1	1	.	.	.	.	.	.	.

$$\text{Sum} = 1+p+2p^2+3p^3+4p^4+4p^5+4p^6+2p^7.$$

It will be observed that the number of different groups G'_p of order p^4 is equal to the number of groups G'_p of order p^7, these orders being complementary ($p^4 \times p^7 = p^{11}$, the order of G_p).

18. As a second example I take G_p to be a product of two simple groups of orders p^α, p^β, where $\alpha \leqq \beta$. The table for G_p is

i	h_i	k_i	K_{i-1}
⋮	⋮	⋮	⋮
i	0	2	$2i-2$
⋮	⋮	⋮	⋮
α	1	2	$2\alpha-2$
⋮	⋮	⋮	⋮
i	0	2	$\alpha+i-1$
⋮	⋮	⋮	⋮
β	1	1	$\alpha+\beta-1$,

and it will appear that the number of groups of order p_s is

$$\left\{ \begin{matrix} s+1 \\ 1 \end{matrix} \right\} \quad \text{or} \quad \left\{ \begin{matrix} \alpha+1 \\ 1 \end{matrix} \right\} \quad \text{or} \quad \left\{ \begin{matrix} \alpha+\beta-s+1 \\ 1 \end{matrix} \right\},$$

when $$s \leqq \alpha \quad \text{or} \quad \alpha \leqq s \leqq \beta \quad \text{or} \quad \beta \leqq s.$$

In the boundary cases the two formulæ applicable are equivalent. And here again the complementary theorem is true.

Clearly G'_p is a product of 2 or 1 simple groups, and in the following enumeration I have indicated the orders of these groups in the first column, and the number in the second; the three cases being separately considered.

Case I. $s \leqq \alpha$.

	Type of G'_p.	No. of G'_p.	
	$0,\ s$	$p^{s-1}\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$	
	$1,\ s-1$	$p^{s-3}\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$	
	⋮		
	$t,\ s-t$	$p^{s-2t-1}\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$	
	⋮		
	$s/2,\ s/2$	1	when s is even
or	$(s-1)/2,\ (s+1)/2$	$\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}$	when s is odd.

For the type $t,\ s-t$, the auxiliary table gives

$$h'_t = 1, \quad k_t - k'_t = 0, \quad K_{t-1} - K'_{t-1} = 0,$$

and $$h'_{s-t} = 1, \quad k_{s-t} - k'_{s-t} = 1,$$

$$K_{s-t-1} - K'_{s-t-1} = 2s - 2t - 2 - (s-1) = s - 2t - 1.$$

The sum of the numbers in the second column is

$$p^s + p^{s-1} + \ldots + 1 = \left\{ \begin{matrix} s+1 \\ 1 \end{matrix} \right\}.$$

Case II. $\alpha \leqq s \leqq \beta$.

Type of G'_p.	No. of G'_p.	
$0,\ s$	p^s	
$\vdots$		
$t,\ s-t$	p^{s-t}	where $s-t>\alpha$.
$\vdots$		
$s-\alpha,\ \alpha$	$p^{2\alpha-s-1}\left\{\begin{matrix}2\\1\end{matrix}\right\}$	
$s-\alpha+t,\ \alpha-t$	$p^{2\alpha-s-2t-1}\left\{\begin{matrix}2\\1\end{matrix}\right\}$	
$\vdots$	$\vdots$	
$s/2,\ s/2$	1	when s is even
or $(s-1)/2,\ (s+1)/2$	$\left\{\begin{matrix}2\\1\end{matrix}\right\}$	when s is odd.

The total number here is

$$p^s+p^{s-1}+\ldots+1=\left\{\begin{matrix}\alpha+1\\1\end{matrix}\right\}.$$

Case III. $\beta \leqq s$.

Type of G'_p.	No. of G'_p.	
$s-\beta,\ \beta$	$p^{\alpha+\beta-s}$	
$\vdots$		
$s-\beta+t,\ \beta-t$	$p^{\alpha+\beta-s-t}$	$(\beta-t>\alpha$
$\vdots$	$\ldots$	
$s-\alpha,\ \alpha$	$p^{2\alpha-s-1}\left\{\begin{matrix}2\\1\end{matrix}\right\}$	
$\vdots$		
$s-\alpha+t,\ \alpha-t$	$p^{2\alpha-s-2t-1}\left\{\begin{matrix}2\\1\end{matrix}\right\}$	
$\vdots$		
$s/2,\ s/2$	1	if s is even
or $(s-1)/2,\ (s+1)/2$	$\left\{\begin{matrix}s\\2\end{matrix}\right\}$	if s is odd,

and the number is $\left\{\begin{matrix}\alpha+\beta-s+1\\1\end{matrix}\right\}$.

This table is formed on the supposition that s is not greater than 2α. If s is greater than 2α, the final group is of the type $\alpha,\ s-\alpha$ (the latter the greater number) because G'_p cannot contain two groups of order greater than a, and therefore is only one group G'_p of this type. The total number therefore is the same as before.

19. The examples already given suggest the existence of a theorem that the number of different groups G'_p of a given order $p^{\omega'}$ that can be formed with elements of a group G_p of order $p^{\omega'+\omega''}$ is the same as the number of different groups G''_p of order $p^{\omega''}$ that can be formed with elements of G_p. In seeking a direct proof—or the conditions of failure—of this theorem (which still remain to find), some interesting results have come to light which are given in the sequel. It will furnish opportunities of illustrating these incidental theorems if I now explain how they were to be used.

20. A reference to Art. 15 will show that the denumerant of Art. 13 rapidly increases in complication with the order of G'_p. In order to verify the theorem by the use of this formula it would be needful to prove the identity of two expressions, one of which becomes so much the more complicated as the other becomes simpler. Obviously it was desirable to transform the denumerant so that it would assume a simple form when the order of G'_p is nearly equal to that of G_p; and then compare the denumerants of the complementary groups each in its most manageable shape.

For this we introduce a new set of constants γ, δ, Δ characterizing G'_p, and connected with h', k', K' by the equations

$$h'_i = h_i - \gamma_i, \quad k'_i = k_i - \delta_i, \quad K'_i = K_i - \Delta_i,$$

so that

$$\delta_i = \gamma_i + \gamma_{i+1} + \ldots + \gamma_\mu,$$

$$\Delta_i = \delta_1 + \delta_2 + \ldots + \delta_i,$$

recalling the equations of Art. 3. The γ, however, unlike the h', may be negative. For instance, a simple group of order p^2 $(h_1 = 0,\ h_2 = 1)$ has factor groups of order p $(h'_1 = 1,\ h'_2 = 0)$, so that in this case $\gamma_1 = -1$, $\gamma_2 = 1$. It follows then that, unlike the k', the δ_i are not arranged in order of magnitude. For

$$\delta_i - \delta_{i+1} = \gamma_i$$

may be either negative or positive. The δ must all be positive (or zero) because

$$\delta_i = k_i - k'_i,$$

and the latter expression cannot be negative.

21. The original form of the denumerant was

$$p^{\Sigma h'_i (K_{i-1} - K'_{i-1})} \Pi \begin{Bmatrix} k_i - k'_i + h'_i \\ h'_i \end{Bmatrix}.$$

The second factor becomes, when $k_i - k'_i$, h'_i are replaced,

$$\begin{Bmatrix} \delta_i + h_i - \gamma_i \\ h_i - \gamma_i \end{Bmatrix} = \begin{Bmatrix} \delta_i + h_i - \gamma_i \\ \delta_i \end{Bmatrix} = \begin{Bmatrix} h_i + \delta_{i+1} \\ \delta_i \end{Bmatrix}.$$

The exponent of $p = \Sigma h'_i (K_{i-1} - K'_{i-1}) = \Sigma (k'_i - k'_{i+1})(K_{i-1} - K'_{i-1})$

$$= \Sigma k'_i (K_{i-1} - K_{i-2}) - k'_i (K'_{i-1} - K'_{i-2})$$

$$= \Sigma k'_i (k_{i-1} - k'_{i-1})$$

$$= \Sigma (k_i - \delta_i) \delta_{i-1}.$$

Hence the transformed denumerant is

$$p^{\Sigma \delta_{i-1} k_i - \delta_i \delta_{i-1}} \Pi \begin{Bmatrix} h_i + \delta_{i+1} \\ \delta_i \end{Bmatrix}.$$

The number of groups G'_p of given order $p^{\omega'}$ is the sum of the denumerants for all G'_p such that

$$\omega' = K'_\mu = k'_1 + k'_2 + \dots + k'_\mu,$$

and every partition of ω' determines one type of G'_p. The number of groups G''_p of complementary order $p^{K_\mu - \omega'}$ $(= p^{\Delta_\mu})$ is the sum of the denumerants for all G''_p such that

$$\Delta_\mu = K_\mu - (K_\mu - \omega') = \omega',$$

that is to say, $\delta_1 + \delta_2 + \dots + \delta_\mu = \omega'.$

Here the δ are positive integers, but their sequence is not determined, and each *composition* of ω' corresponds to a set of groups of order $p^{\omega''}$.

22. Some examples of the developed formulæ for the numbers of groups of given order nearly equal to the order of G_p follow. (*Cf.* Art. 15.)

I. For factors of G_p of order $p^{\omega-1}$ we require the compositions of 1,

$$\delta_1 + \delta_2 + \dots + \delta_\mu = 1.$$

These are given by $\delta_i = 1$, and every other $\delta = 0$. The number of groups for which $\delta_i = 1$ is

$$p^{k_{i+1}} \begin{Bmatrix} h_i \\ 1 \end{Bmatrix},$$

and the total number is $\sum_i p^{k_{i+1}} \begin{Bmatrix} h_i \\ 1 \end{Bmatrix}$ $(i = 1, 2, \ldots \mu)$.

II. When the order of G_p'' is $p^{\mu-2}$, the compositions

$$\delta_1 + \ldots \delta_i + \ldots + \delta_\mu = 2$$

are of two sets $\delta_i = 2$ and $\delta_i = 1$, $\delta\ = 1$,

the other δ being 0.

The first set contributes

$$\Sigma p^{2p_{i+1}} \begin{Bmatrix} h_i \\ 2 \end{Bmatrix}$$

to the total number.

The second set answers to different numbers according as i, j are or are not consecutive. In the general case when i, j are not consecutive the contribution is

$$\sum_{i,j} p^{k_{i+1} + k_{j+1}} \begin{Bmatrix} h_i \\ 1 \end{Bmatrix} \begin{Bmatrix} h_j \\ 1 \end{Bmatrix} \qquad (i+1<j).$$

But, if $\delta_i = 1$, $\delta_{i+1} = 1$, the number of corresponding groups is

$$\Sigma p^{k_{i+1} - 1 + k_{i+2}} \begin{Bmatrix} h_i + 1 \\ 1 \end{Bmatrix} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix}.$$

III. For the order $p^{\mu-3}$, we have the compositions

$(\delta_i = 3)$;

$(\delta_i = 2,\ \delta_j = 1)$; $(\delta_i = 2,\ \delta_{i+1} = 1)$; $(\delta_i = 1,\ \delta_{i+1} = 2)$;

$(\delta_i = 1,\ \delta_j = 1,\ \delta_k = 1)$; $(\delta_i = 1,\ \delta_{i+1} = 1,\ \delta_j = 1)$;

$(\delta_i = 1,\ \delta_{i+1} = 1,\ \delta_{i+2} = 1)$;

and the number of groups is

$$\Sigma p^{3k_{i+1}} \begin{Bmatrix} h_i \\ 3 \end{Bmatrix}$$

$$+ \Sigma p^{2k_{i+1} + k_{j+1}} \begin{Bmatrix} h_i \\ 2 \end{Bmatrix} \begin{Bmatrix} h_j \\ 1 \end{Bmatrix} \qquad (j<i-1 \text{ or } i+1<j)$$

$$+\Sigma p^{2k_{i+1}+k_{i+2}-2}\left\{ {h_i+1 \atop 2} \right\}\left\{ {h_{i+1} \atop 1} \right\}$$

$$+\Sigma p^{k_{i+1}+2k_{i+2}-2}\left\{ {h_i+2 \atop 1} \right\}\left\{ {h_{i+1} \atop 2} \right\}$$

$$+\Sigma p^{k_{i+1}+k_{j+1}+k_{l+1}}\left\{ {h_i \atop 1} \right\}\left\{ {h_j \atop 1} \right\}\left\{ {h_l \atop 1} \right\} \qquad (i+1<j,\ j+1<l)$$

$$+\Sigma p^{k_{i+1}+k_{i+2}+k_{j+1}-1}\left\{ {h_i+1 \atop 1} \right\}\left\{ {h_{i+1} \atop 1} \right\}\left\{ {h_j \atop 1} \right\} \qquad (j<i-1 \text{ or } i+2<j)$$

$$+\Sigma p^{k_{i+1}+k_{i+2}+k_{i+3}-2}\left\{ {h_i+1 \atop 1} \right\}\left\{ {h_{i+1}+1 \atop 1} \right\}\left\{ {h_{i+2} \atop 1} \right\}.$$

23. The first theorem that assists in the identification of the two total denumerants is analogous to the Vandermonde theorem for binomial coefficients and is thus stated.

If r and s be two positive integers less than n (also a positive integer),

$$\left\{ {n \atop r} \right\} = \left\{ {n-s \atop r-s} \right\} + \ldots + p^{t(r-s+t)}\left\{ {s \atop t} \right\}\left\{ {n-s \atop r-s+t} \right\} + \ldots + p^{rs}\left\{ {n-s \atop r} \right\}$$
............(s).

The last term written corresponds to $t = s$ and implies that $r \leqslant n-s$, that is, $r+s \leqslant n$. If $n<r+s$, the actual last term is that for which $t = n-r$, namely,

$$p^{(n-r)(n-s)}\left\{ {s \atop n-r} \right\},$$

and, if $n = r+s$, the actual last term is that for which $t = s = n-r$, namely, the term p^{rs}. It is, however, indifferent which is written, because the only effect of writing the wrong one is to add so many zero terms.

The theorem can be expressed in other forms, because of the identities such as

$$\left\{ {s \atop t} \right\} = \left\{ {s \atop s-t} \right\},$$

by means of which we can make either or both of the lower elements in each term decrease as t increases. The above form seems to be the standard, because in it the index of p is the product of the two lower elements. Besides these changes we may replace r by $n-r$, since this does not alter the value of the left side.

24. When $s=1$, the theorem becomes

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \begin{Bmatrix} n-1 \\ r-1 \end{Bmatrix} + p^r \begin{Bmatrix} n-1 \\ r \end{Bmatrix} \quad \text{..................(1)},$$

and is at once verified by the definition. For

$$\begin{Bmatrix} n \\ r \end{Bmatrix} - \begin{Bmatrix} n-1 \\ r-1 \end{Bmatrix}$$

$$= \frac{(p^n-1)(p^{n-1}-1)\dots(p^{n-r+1}-1)}{(p^r-1)(p^{r-1}-1)\dots(p-1)} - \frac{(p^{n-1}-1)\dots(p^{n-r+1}-1)}{(p^{r-1}-1)\dots(p-1)}$$

$$= \frac{(p^{n-1}-1)\dots(p^{n-r+1}-1)}{(p^r-1)\dots(p-1)} \{p^n-1-(p^r-1)\}$$

$$= \text{..............................} \; (p^{n-r}-1)\,p^r$$

$$= \begin{Bmatrix} n-1 \\ r \end{Bmatrix} p^r.$$

The proof for any other value of s is effected by induction from $s-1$ to s. Assuming the theorem to be true for $s-1$,

$$\begin{Bmatrix} n \\ r \end{Bmatrix} = \Sigma p^{t(r-s+1+t)} \begin{Bmatrix} s-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-s+1 \\ r-s+1+t \end{Bmatrix} \quad \text{.........}(s-1).$$

But, by (1),

$$\begin{Bmatrix} n-s+1 \\ r-s+1+t \end{Bmatrix} = \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix} + p^{r-s+1+t} \begin{Bmatrix} n-s \\ r-s+1+t \end{Bmatrix}.$$

Thus the term of $(s-1)$ written above—say the term t—is separated into two parts

$$p^{t(r-s+1+t)} \begin{Bmatrix} s-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix} + p^{(t+1)(r-s+1+t)} \begin{Bmatrix} s-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-s \\ r-s+1+t \end{Bmatrix} \quad \text{............}(t).$$

Now, if we combine the first part of the term t with the second part of the term $(t-1)$ similarly analysed, we obtain

$$p^{t(r-s+1+t)} \begin{Bmatrix} s-1 \\ t \end{Bmatrix} \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix} + p^{t(r-s+t)} \begin{Bmatrix} s-1 \\ t-1 \end{Bmatrix} \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix}$$

$$= p^{t(r-s+t)} \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix} \left[p^t \begin{Bmatrix} s-1 \\ t \end{Bmatrix} + \begin{Bmatrix} s-1 \\ t-1 \end{Bmatrix} \right]$$

$$= p^{t(r-s+t)} \begin{Bmatrix} n-s \\ r-s+t \end{Bmatrix} \begin{Bmatrix} s \\ t \end{Bmatrix} \quad \text{by (1)},$$

and this is the term t of the expansion (s). In this way every term of (s) is obtained except the first and the last. But, by putting $t=0$ in (t) it is verified that the first part of the first term of $(s-1)$ is the first term of (s); and, by putting $t=s-1$ in (t), that the second part of the last term of $(s-1)$ is the last term of (s). Hence

$$(s)=(s-1)=\ldots=(1)=\left\{\begin{matrix}n\\r\end{matrix}\right\}.$$

25. In the identity (s) replace $\left\{\begin{matrix}n\\r\end{matrix}\right\}$ by $\left\{\begin{matrix}n\\n-r\end{matrix}\right\}$ and $\left\{\begin{matrix}n-s\\r-s+t\end{matrix}\right\}$ by $\left\{\begin{matrix}n-s\\n-r-t\end{matrix}\right\}$ for all values of t. Then write r for $n-r$. The identity then becomes

$$\left\{\begin{matrix}n\\r\end{matrix}\right\}=\left\{\begin{matrix}n-s\\r\end{matrix}\right\}+\ldots+p^{t(n-r-s+t)}\left\{\begin{matrix}s\\t\end{matrix}\right\}\left\{\begin{matrix}n-s\\r-t\end{matrix}\right\}+\ldots+p^{s(n-r)}\left\{\begin{matrix}n-s\\r-s\end{matrix}\right\}.$$

Further, to introduce a notation more convenient for our immediate purpose, we write x, h for s, $n-s$ respectively. Thus

$$\left\{\begin{matrix}h+x\\r\end{matrix}\right\}=\left\{\begin{matrix}h\\r\end{matrix}\right\}+\ldots+p^{t(h-r+t)}\left\{\begin{matrix}h\\r-t\end{matrix}\right\}\left\{\begin{matrix}x\\t\end{matrix}\right\}+\ldots+p^{x(h+x)}\left\{\begin{matrix}h\\r-x\end{matrix}\right\}.$$

26. We are now in a position to prove the following theorem:— Let h_i, k_i have the meanings assigned in Art. 3, so that

$$k_i=h_i+h_{i+1}+\ldots+h_\mu \qquad (i=1, 2, \ldots \mu),$$

and

$$k_i=0, \quad \text{if} \quad i>\mu.$$

Also let a_i be positive integers or zeros, such that

$$a_1+a_2+\ldots+a_\mu=r.$$

Then I say that

$$\left\{\begin{matrix}k_1\\r\end{matrix}\right\}=\Sigma p^{a_{\mu-1}k_\mu+\ldots+a_ik_{i+1}+\ldots+a_1k_2-A}\left\{\begin{matrix}h_\mu\\a_\mu\end{matrix}\right\}\ldots\left\{\begin{matrix}h_i\\a_i\end{matrix}\right\}\ldots\left\{\begin{matrix}h_1\\a_1\end{matrix}\right\},$$

where the summation includes all the compositions $(a_1, a_2, \ldots a_\mu)$ of r, and $A=\Sigma a_ia_j$ $(i\neq j)$.

By the theorem of Art. 25,

$$\left\{\begin{matrix}k_1\\r\end{matrix}\right\}=\left\{\begin{matrix}h_\mu+(k_1-h_\mu)\\r\end{matrix}\right\}=\sum_t p^{t(h_\mu-r+t)}\left\{\begin{matrix}h_\mu\\r-t\end{matrix}\right\}\left\{\begin{matrix}k_1-h_\mu\\t\end{matrix}\right\},$$

the sum including the terms $t=0, 1, 2, \ldots r$.

In this write a_ρ for $r-t$, so that $t=r-a_\rho$, and the equation becomes

$$\left\{\begin{matrix} k_1 \\ r \end{matrix}\right\} = \Sigma p^{(r-a_\rho)(h_\rho-a_\rho)} \left\{\begin{matrix} h_\rho \\ a_\rho \end{matrix}\right\} \left\{\begin{matrix} k_1-h_\rho \\ r-a_\rho \end{matrix}\right\},$$

where a_ρ in the several terms on the right is $r, r-1, \dots 1, 0$ respectively.

The last factor of the specimen term can be expanded in the same way, giving

$$\left\{\begin{matrix} k_1-h_\rho \\ r-a_\rho \end{matrix}\right\} = \left\{\begin{matrix} h_{\rho-1}+(k_1-h_\rho-h_{\rho-1}) \\ r-a_\rho \end{matrix}\right\}$$

$$= \Sigma p^{t(h_{\rho-1}-r+a_\rho+t)} \left\{\begin{matrix} h_{\rho-1} \\ r-a_\rho-t \end{matrix}\right\} \left\{\begin{matrix} k_1-h_\rho-h_{\rho-1} \\ t \end{matrix}\right\}$$

$$= \Sigma p^{(r-a_\rho-a_{\rho-1})(h_{\rho-1}-a_{\rho-1})} \left\{\begin{matrix} h_{\rho-1} \\ a_{\rho-1} \end{matrix}\right\} \left\{\begin{matrix} k_1-h_\rho-h_{\rho-1} \\ r-a_\rho-a_{\rho-1} \end{matrix}\right\},$$

where $a_{\rho-1}=r-a_\rho-t$ has in successive terms the values $r-a_\rho$, $r-a_\rho-1, \dots 1, 0$. Continuing in this way, we obtain ultimately

$$\left\{\begin{matrix} k_1 \\ r \end{matrix}\right\} = \Sigma p^I \left\{\begin{matrix} h_\rho \\ a_\rho \end{matrix}\right\} \left\{\begin{matrix} h_{\rho-1} \\ a_{\rho-1} \end{matrix}\right\} \dots \left\{\begin{matrix} k_1-h_\rho-h_{\rho-1}-\dots-h_2 \\ r-a_\rho-a_{\rho-1}-\dots-a_2 \end{matrix}\right\}$$

$$= \Sigma p^I \left\{\begin{matrix} h_\rho \\ a_\rho \end{matrix}\right\} \left\{\begin{matrix} h_{\rho-1} \\ a_{\rho-1} \end{matrix}\right\} \dots \left\{\begin{matrix} h_1 \\ a_1 \end{matrix}\right\},$$

where the summation includes all the compositions $(a_1, a_2, \dots a_\rho)$ of r.

The exponent of p-factor,

$$I = (r-a_\rho)(h_\rho-a_\rho)+\dots+(r-a_\rho-a_{\rho-1}-\dots-a_{i+1})(h_{i+1}-a_{i+1})+\dots$$

$$= (a_{\rho-1}+a_{\rho-2}+\dots+a_1)(h_\rho-a_\rho)+\dots$$

$$+(a_i+a_{i-1}+\dots+a_1)(h_{i+1}-a_{i+1})+\dots$$

$$= a_{\rho-1}h_\rho+a_{\rho-2}(h_\rho+h_{\rho-1})+\dots+a_j(h_\rho+h_{\rho-1}+\dots+h_{j+1})+\dots$$

$$\dots+a_1(h_\rho+\dots+h_2)-\Sigma a_i a_j \qquad (j>i)$$

$$= a_{\rho-1}k_\rho+a_{\rho-2}k_{\rho-1}+\dots+a_1 k_2-\Sigma a_i a_j,$$

and thus the theorem is established.

27. Some particular examples are added in illustration—

$$\left\{ {k_1 \atop 1} \right\} = \Sigma p^{k_{i+1}} \left\{ {h_i \atop 1} \right\}.$$

Here the compositions of r (= 1) are 100 ..., 010 ..., 001 ..., &c.,

$$\left\{ {k_1 \atop 2} \right\} = \Sigma p^{2k_{i+1}} \left\{ {h_i \atop 2} \right\} + \Sigma p^{k_{i+1}+k_{j+1}-1} \left\{ {h_i \atop 1} \right\} \left\{ {h_j \atop 1} \right\} \dots$$

$$\left\{ {k_1 \atop 3} \right\} = \Sigma p^{3k_{i+1}} \left\{ {h_i \atop 3} \right\} + \Sigma p^{2k_{i+1}+k_{j+1}-2} \left\{ {h_i \atop 2} \right\} \left\{ {h_j \atop 1} \right\}$$

$$+ \Sigma p^{k_{i+1}+k_{j+1}+k_{l+1}-3} \left\{ {h_i \atop 1} \right\} \left\{ {h_j \atop 1} \right\} \left\{ {h_l \atop 1} \right\},$$

and so on.

28. These results enable us to compare the denumerants of Art. 15 with the denumerants of groups of complementary order (Art. 22).

The number of groups of order p contained in G_p is $\left\{ {k_1 \atop 1} \right\}$ (Art. 15), the number of groups of order $p^{\varpi-1}$ (Art. 22) is $\Sigma p^{k_{i+1}} \left\{ {h_i \atop 1} \right\}$, and these two expressions are equal by the preceding article.

For orders p^2, $p^{\varpi-2}$ the equality to be proved is

$$\left\{ {k_1 \atop 2} \right\} + p^{k_1-1} \left\{ {k_2 \atop 1} \right\}$$

$$= \Sigma p^{2k_{i+1}} \left\{ {h_i \atop 2} \right\} + \Sigma p^{k_{i+1}+k_{j+1}} \left\{ {h_i \atop 1} \right\} \left\{ {h_j \atop 1} \right\}$$

$$+ \Sigma p^{k_{i+1}+k_{i+2}-1} \left\{ {h_i+1 \atop 1} \right\} \left\{ {h_{i+1} \atop 1} \right\}.$$

Here the left side

$$= \Sigma p^{2k_{i+1}} \left\{ {h_i \atop 2} \right\} + \Sigma p^{k_{i+2}+k_{j+1}-1} \left\{ {h_{i+1} \atop 1} \right\} \left\{ {h_j \atop 1} \right\} + \Sigma p^{k_1+k_{i+2}-1} \left\{ {h_{i+1} \atop 1} \right\}.$$

Now $$p^{k_{i+2}+k_{j+1}-1} = p^{k_{i+2}+k_{j+1}} - (p-1)\, p^{k_{i+2}+k_{j+1}-1};$$

and therefore the second term

$$= \Sigma p^{k_{i+1}+k_{j+1}} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix} \begin{Bmatrix} h_j \\ 1 \end{Bmatrix} - \Sigma \sum_1^{i-1} p^{k_{i+2}+k_{j+1}-1} (p^{h_j}-1) \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix}$$

$$= \;\dots\;\dots\;\dots\;\dots\;\dots\; - \Sigma p^{k_{i+2}-1} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix} \sum^{i-1} (p^{k_j}-p^{k_{j+1}})$$

$$= \;\dots\;\dots\;\dots\;\dots\;\dots\; - \;\dots\;\dots\;\dots\;\dots\; (p^{k_1}-p^{k_i})$$

$$= \;\dots\;\dots\;\dots\;\dots\;\dots\; - \Sigma (p^{k_{i+2}+k_1-1} - p^{k_{i+2}+k_{i+1}-1}) \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix},$$

which cancels the third term; and proves the left side

$$= \Sigma p^{2k_{i+1}} \binom{h_i}{2} + \Sigma p^{k_{i+2}+k_{j+1}} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix} \begin{Bmatrix} h_j \\ 1 \end{Bmatrix} + \Sigma p^{k_{i+2}+k_{i+1}-1} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix}.$$

The third term of the right side contains a factor

$$\begin{Bmatrix} h_i+1 \\ 1 \end{Bmatrix} = 1 + p \begin{Bmatrix} h_i \\ 1 \end{Bmatrix},$$

so that the term may be written

$$\Sigma p^{k_{i+2}+k_{i+1}-1} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix} + \Sigma p^{k_{i+2}+k_{i+1}} \begin{Bmatrix} h_{i+1} \\ 1 \end{Bmatrix} \begin{Bmatrix} h_i \\ 1 \end{Bmatrix},$$

and then the right side becomes identical with the last written value of the left side.

The comparison for orders p^3, p^{a-3} is still more laborious, and it is hopeless to attack the general theorem in this way.

29. The theorem of Art. 23 is useful in the enumeration of the groups which are factors of G_p, a product of h simple independent groups of order p^a. The order of G'_p will be taken to be not greater than p^a, and to avoid accents will be denoted by p^s.

We have in this case $h_i = 0$ for $i = 1, 2, \dots a-1$, $k_i = h$ and $K_{i-1} = (i-1)\,h$. Thus the number of groups G'_p of order p^s is

$$\Sigma p^{\Sigma h'_i [(i-1)h - K'_{i-1}]} \Pi \begin{Bmatrix} h - k'_i + h'_i \\ h'_i \end{Bmatrix},$$

the summation extending to all positive (or zero) values of h' which satisfy the relations

$$h'_1 + h'_2 + \dots + h'_s \leqq h,$$

$$h'_1 + 2h'_2 + \dots + s h'_s = s.$$

The first of these will be replaced by an equation wherein h'_0 denotes a positive integer or 0, namely,

$$h'_0+h'_1+h'_2+\ldots+h'_s = h.$$

Then the Π-factor above is

$$=\frac{h!}{h'_s!\ldots h'_i!\ldots h'_0!},$$

where the symbols $h!$ represent the product

$$(p^h-1)(p^{h-1}-1)\ldots(p-1).$$

For the factor exhibited is

$$\frac{(h-k'_i+h'_i)!}{h'_i!\,(h-k'_i)!}=\frac{(h+k'_i+h'_i)!}{h_i!\,(h-k'_{i-1}+h'_{i-1})!}.$$

The exponent of p

$$=\Sigma h'_i\left[(i-1)\,h-(h'_1+2h'_2+\ldots+(i-1)\,h'_i+\ldots+(i-1)\,h'_s\right]$$

$$=\Sigma h'_i\left[(i-1)\,h'_0+(i-2)\,h'_1+\ldots+h'_{i-2}\right]$$

$$=\Sigma\,(i-j-1)\,h'_ih'_j\quad (j=0,\,1,\,\ldots\,i-2\,;\ i-2,\,3,\,\ldots\,s).$$

The total number of groups G'_p will be shown to be

$$\left\{\begin{matrix} s+h-1 \\ s \end{matrix}\right\}.$$

This is at once verified for $s = 1, 2$, &c., and the general theorem is established by induction, on the assumption that the theorem is true for all smaller values of s; and the method is to take together all terms of the sum that have the same value of h_0.

The type term is

$$=p^{\sum\limits_{j=1}(i-j-1)\,h'_ih'_j}\,p^{h'_0[h'_2+2h'_3+\ldots+(s-1)\,h'_s]}\left\{\begin{matrix} h \\ h'_0 \end{matrix}\right\}\frac{(h-h'_0)!}{h'_1!\,h'_2!\ldots h'_s!},$$

where $$h'_0+h'_1+\ldots+h'_s = h,$$

and $$h'_1+2h'_2+sh'_s = s,$$

so that $$h'_2+2h'_3+\ldots+(s-1)\,h'_s = s-h+h'_0.$$

The term is therefore

$$=p^{h'_0(s-h+h'_0)}\left\{\begin{matrix} h \\ h'_0 \end{matrix}\right\}\times p^{\Sigma(i-j-1)\,h'_ih'_j}\frac{(h-h'_0)!}{h_1!\ldots h'_i!\ldots h'_s!}.$$

Here the first factor is constant when h_0 is constant as well as h, s, and the sum of these terms is the product of this factor into the sum of

$$p^{\Sigma(i-j-1)h'_i h'_j} \frac{(h-h'_0)!}{h'_1! \ldots h'_s!},$$

where

$$h'_1+h'_2+\ldots+h'_s = h-h'_0,$$
$$h'_2+2h'_3+\ldots+(s-1)\,h'_s = s-h+h'_0.$$

This summation is similar to the original summation, but the new s is smaller. Assuming the theorem to be true for this smaller s, we have

$$\Sigma p^{\Sigma(i-j-1)h'_i h'_j} \frac{(h-h'_0)!}{h'_1! \ldots h'_s!} = \left\{ \frac{s-h+h'_0+(h-h'_0)-1}{s-h+h'_0} \right\}$$
$$= \left\{ \begin{matrix} s-1 \\ s-h-h'_0 \end{matrix} \right\},$$

and the sum of those terms of the denumerant which have a given h'_0 is

$$p^{h'_0(s-h+h'_0)} \left\{ \begin{matrix} h \\ h'_0 \end{matrix} \right\} \left\{ \begin{matrix} s-1 \\ s-h+h'_0 \end{matrix} \right\}.$$

But the sum of these terms for $h'_0 = h-s,\ h-s+1, \ldots h-1$ is, by Art. 23,

$$\left\{ \begin{matrix} h+s-1 \\ s \end{matrix} \right\},$$

and thus the proposition is proved.

It will be observed that the order p^s of the simple groups of which G_p is the product does not appear anywhere in the conditions, save as limiting the order p^s of G'_p. Hence the theorem will still be true if G_p is the product of h simple groups of orders not less than p^s. (*Cf.* the example of Art. 18.)

The failure of the formula when $s > a$ is easily explained. When s has this too large value there are

$$\left\{ \begin{matrix} s+h-1 \\ s \end{matrix} \right\}$$

groups G'_p that can be formed from the elements of an extended G_p, whose constituent groups are of sufficiently high order. But some of these G'_p will contain simple groups of higher order than are present in the original G_p. Thus the number given by the formula errs in excess, and it may be noted that the excess is in many cases expressed by a somewhat similar formula.

Thursday, April 23rd, 1896.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

The President read portions of abstracts of papers, viz., "On the Isomorphism of a Group with itself," by Prof. W. Burnside; and "Division of the Lemniscate," by Prof. G. B. Mathews.

Dr. Hobson read a paper "On some General Formulæ for the Potentials of Ellipsoids, Shells, and Discs."

The President made some remarks on "The Compensation for Difference of Capital in Gambling *à outrance.* A contribution to the Theory of the Duration of Play."

Mr. Basset read a paper "On the Stability of a Frictionless Liquid —Theory of Critical Planes." The author and Mr. Love joined in a discussion on the communication.

The following presents to the Library were received :—

"Proceedings of the Royal Society," Vol. LIX., No. 355.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 40, Heft 3–4 ; Zürich, 1895.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 3 ; Leipzig, 1896.

"Queen's College, Galway, Calendar for 1895–6," 8vo ; Dublin, 1896.

"Mathematical Questions with their Solutions," edited by W. J. C. Miller, Vol. LXIV., 8vo ; London, 1896.

"Journal of the Institute of Actuaries, Index to Vols. XXI.—XXX. (inclusive)"; London, 1896.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. X., No. 1 ; Manchester, 1896.

Robertson, J. A.—"On a New Method of performing approximately certain Operations in Multiplication and Division" (read before the Institute of Actuaries, 29th April, 1895).

"Berichte über die Verhandlungen der K. Sachs. Gesellschaft der Wissenschaften zu Leipzig," 1895, 5, 6 ; Leipzig, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 6 ; New York, 1896.

"Proceedings of the Physical Society," Vol. XIV., Pt. 4 ; April, 1896.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. II., Fasc. 2, 3 ; Napoli, 1896.

"Bulletin des Sciences Mathématiques," Tome XX. ; Paris, Fév., Mars, 1896.

"Tōkyō Mathematical-Physical Society," Maki No. 7, Dai 4 ; 1896.

"Rendiconti del Circolo Matematico di Palermo," Tomo x., Fasc. 3; May–June, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. v., Fasc. 5, 6; Roma, 1896.

Zeuthen, H. G.—"Om den Historiske Udvikling af Mathematiken som exakt Videnskab indtil Udgangen af det 18de Aarhunderte" (Indbydelsesskrift til Kjøbenhavns Universitets Aarsfest i anledning af Hans Majestat Kongens Fødselsdag den 8de April 1896).

Zeuthen, H. G.—"Die Geometrische Construction als 'Existenzbeweis' in der Antiken Geometrie" (from Band xlvii. of "Math. Annalen," pp. 222–228).

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 1895, 39–53, and Verzeichniss der Eingegangenen Druckschriften, Titel, Inhalt, &c.

"Educational Times," April, 1896.

"Annals of Mathematics," Vol. ix., No. 6; Vol. x., No. 1; Virginia, 1896.

"Annales de la Faculté des Sciences de Marseille," Tome iv., Fasc. 4, Tome v., Fasc. 1, 2, 3; 1894–5.

"Annales de la Faculté des Sciences de Toulouse," Tome ix., Fasc. 4; Paris, 1895.

"Indian Engineering," Vol. xix., Nos. 8–13, February 22nd to March 28th.

On the Isomorphism of a Group with itself. *By* W. Burnside.

Read April 23rd, 1896. Received May 11th, 1896.

The conception of the isomorphism of a group with itself is not a new one, but it is only recently that it has been developed in any detail. The importance of the conception, both for the general theory of groups and for the actual construction of groups from given factor-groups, is obvious. Indeed the latter problem cannot be undertaken till all possible isomorphisms with themselves of the groups involved are determined. It is to Herr O. Hölder* and Herr G. Frobenius† that the development of the general theory of

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* O. Hölder: "Bildung zusammengesetzter Gruppen," *Math. Ann.*, xlvi., pp. 321–422. The subject is also referred to in a paper by the same author in *Math. Ann.*, xliii.

† G. Frobenius: "Ueber endliche Gruppen," *Berliner Sitzungsberichte*, 1895, pp. 163–194; and "Ueber auflösbare Gruppen II.," *id.*, pp. 1027–1044.

the isomorphism of a group with itself, so far as it has been at present carried, is due.

In the first part of the present paper I have given the necessary definitions and general explanations to make what follows self-contained; in the second part three general theorems connected with the isomorphism of a group with itself are proved; and in the third part I have determined the groups of isomorphisms of the classes of simple groups, some of whose properties I have already investigated in Vol. xxv. of the Society's *Proceedings*.

I.

Let $S_1\,(=1)$, S_2, S_3, ... S_N be the operations of a group of finite order N. Then it is in general* possible to arrange these operations in a different way,

$$S_1',\; S_2',\; S_3',\; \ldots\; S_N',$$

so that, if

$$S_pS_q = S_r,$$

then

$$S_p'S_q' = S_r',$$

whatever p and q may be.

Each rearrangement of the operations of the group which possesses this property is said to define an *isomorphism* of the group with itself. An isomorphism of a group with itself is thus defined by a permutation of the operations of the group which leaves the multiplication table unchanged. The product of the two permutations (regarded as substitutions performed on the symbols of the operations) which correspond to two isomorphisms is a third permutation which must correspond to a third isomorphism, since the multiplication table remains unchanged. This third isomorphism may be regarded as the product of the two that give rise to it, taking account of the order in which they are carried out. An isomorphism of a group with itself may then be regarded as an operation performed on the group; and the complete set of isomorphisms form a group, since they have the group property that the product of any two is one of the set. This group is known as the group of isomorphisms of the given group.

* The one exception is the group of order 2.

If S_x be any operation of the group, then, from

$$S_p S_q = S_r,$$

$$S_x^{-1} S_p S_x \,.\, S_x^{-1} S_q S_x = S_x^{-1} S_r S_x$$

follows; and therefore

$$\begin{pmatrix} S_1, & S_2, & \dots & S_N \\ S_x^{-1} S_1 S_x, & S_x^{-1} S_2 S_x, & \dots & S_x^{-1} S_N S_x \end{pmatrix}$$

defines an isomorphism.

The permutation of the operations of the group in this case is given by transforming them all by one of themselves. Such an isomorphism is known as a *cogredient* isomorphism, and any isomorphism which is not capable of being generated in this way is called a *contragredient* isomorphism.

The totality of the cogredient isomorphisms clearly form a group, and Herr Hölder shows that this group is contained self-conjugately in the group of isomorphisms. Thus, if

$$\begin{pmatrix} S \\ S' \end{pmatrix}$$

be used as an abbreviated symbol for the isomorphism defined by

$$\begin{pmatrix} S_1, & S_2, & \dots & S_N \\ S_1', & S_2', & \dots & S_N' \end{pmatrix},$$

then
$$\begin{pmatrix} S \\ S' \end{pmatrix}^{-1} \begin{pmatrix} S \\ S_x^{-1} S S_x \end{pmatrix} \begin{pmatrix} S \\ S' \end{pmatrix} = \begin{pmatrix} S' \\ (S_x^{-1} S S_x)' \end{pmatrix}$$

$$= \begin{pmatrix} S' \\ S_x'^{-1} S' S_x' \end{pmatrix};$$

for the isomorphism that changes S into S' must change $S_x^{-1} S S_x$ into $S_x'^{-1} S' S_x'$. Hence, if a cogredient isomorphism be transformed by any isomorphism, the result is another cogredient isomorphism; and therefore the cogredient isomorphisms form, as stated, a self-conjugate sub-group of the group of isomorphisms.*

If
$$S_x S_y = S_z,$$

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* O. Hölder (*loc. cit.*), p. 326.

then
$$\begin{pmatrix} S \\ S_x^{-1}SS_x \end{pmatrix}\begin{pmatrix} S \\ S_y^{-1}SS_y \end{pmatrix} = \begin{pmatrix} S \\ S_x^{-1}SS_x \end{pmatrix}\begin{pmatrix} S_x^{-1}SS_x \\ S_y^{-1}S_x^{-1}SS_xS_y \end{pmatrix}$$
$$= \begin{pmatrix} S \\ S_z^{-1}SS_z \end{pmatrix};$$

and therefore the group of cogredient isomorphisms of a group G is isomorphous with the group G itself. This isomorphism between the two groups may be holohedric or merihedric. If G contains no self-conjugate operation, except identity, the two isomorphisms

$$\begin{pmatrix} S \\ S_x^{-1}SS_x \end{pmatrix} \text{ and } \begin{pmatrix} S \\ S_y^{-1}SS_y \end{pmatrix}$$

can clearly only be the same when

$$S_x = S_y.$$

In this case, then, G is holohedrically isomorphous, or essentially identical, with its group of cogredient isomorphisms. When G contains self-conjugate operations they will form a self-conjugate sub-group H, and the group of cogredient isomorphisms of G is identical with the factor group $\dfrac{G}{H}$.

If
$$\begin{pmatrix} S \\ S' \end{pmatrix}$$

be any isomorphism of a group G, and g any sub-group of G, the operations of g will by the isomorphism be changed into another set of operations which form a group holohedrically isomorphous with g. For, if r, s, t be any three operations of g, such that

$$rs = t,$$

then
$$r's' = t'.$$

The group g' into which g is thus changed by the isomorphism may be either (i.) identical with g, (ii.) identical with another sub-group of the conjugate set to which g belongs within G, or (iii.) it may be one of a set of conjugate sub-groups of G which does not contain g.

When g' coincides with g for every possible isomorphism of the group G, g is called a characteristic sub-group of G.* A characteristic sub-group of G is thus necessarily a self-conjugate sub-group; but a self-conjugate sub-group is clearly not necessarily characteristic.

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* *Cf.* Herr Frobenius, "Ueber endliche Gruppen," p. 183.

II.

1. If p^m is the highest power of a prime p that divides the order of the group G, all sub-groups of G of order p^m are conjugate within G. If, then, g be a sub-group of G of order p^m, every isomorphism of G must change g into a sub-group of G conjugate with g.

Suppose, now, that g is one of $n\,(=kp+1)$ conjugate sub-groups, and that no operation of G transforms each of these n sub-groups into itself, so that G can be expressed as a transitive substitution group of n symbols. Every isomorphism of G must interchange the n sub-groups of order p^m among themselves; and hence, unless an isomorphism of G, other than identity, changes each of these sub-groups into itself, the group of isomorphisms of G can be expressed as a transitive substitution group of degree n. Suppose, if possible, that an isomorphism

$$\binom{S}{S'}$$

of G does transform each of the n sub-groups into itself. Then, if S transforms any one of them, say g, into g_1, S' also transforms g into g_1. Hence S and S', when expressed as substitutions performed on the n symbols

$$g,\ g_1,\ g_2,\ \ldots\ g_{n-1},$$

are identical; and therefore

$$\binom{S}{S'}$$

is the identical isomorphism.

If, therefore, p^m is the highest power of a prime p that divides the order of G, and if G contains no operation which is permutable with each of the $kp+1$ sub-groups of order p^m, then the group of isomorphisms of G can be expressed as a transitive group of degree $kp+1$.

2. An isomorphism of a group G which changes S into S' must change the set of conjugate operations of G of which S is one into the set to which S' belongs. The cogredient isomorphisms of G change each set of conjugate operations into itself. If now I is any isomorphism of G which changes each set of conjugate operations of G into itself, and J is any isomorphism whatever of G, the isomorphism $J^{-1}IJ$ must change each set of conjugate operations into itself. Hence those isomorphisms of G which change each conjugate

set of operations into itself form a self-conjugate sub-group of the complete group of isomorphisms; and this sub-group is either identical with or contains the group of cogredient isomorphisms. If this sub-group of the group of isomorphisms be denoted by K, and the group of isomorphisms itself by L, the factor group $\frac{L}{K}$ is the (intransitive) substitution group of the conjugate sets of operations of G (each set being regarded as a single entity) which arises from carrying out all possible isomorphisms of G.

It is easy to show that K, supposing, if possible, that it is not identical with the group of cogredient isomorphisms of G, cannot contain any operation whose order is relatively prime to the order of G. Thus, if K contains such an operation, there must be an isomorphism

$$\begin{pmatrix} S \\ S' \end{pmatrix}$$

of G, of prime order q (not a factor of N, the order of G), which changes each conjugate set of operations of G into itself. This isomorphism may be regarded as an operation Q which transforms G into itself, so that the group $\{Q, G\}$ generated by Q and G is of order qN, and contains G self-conjugately.

If S is one of a set of m conjugate operations in G, it is also one of a set of m conjugate operations in $\{Q, G\}$, and therefore in $\{Q, G\}$ it must be permutable with some operation of order q. Hence every operation of G is permutable in $\{Q, G\}$ with some operation of order q. It follows that the sub-group of $\{Q, G\}$ which is permutable with a given operation of order q must contain operations belonging to every conjugate set in G, and therefore* must contain G itself. Hence Q is permutable with every operation of G, and the isomorphism given by Q is the identical isomorphism, contrary to supposition.

It follows at once that an isomorphism of G whose order contains a factor relatively prime to the order of G must interchange some of the conjugate sets of operations of G.

* A sub-group H of a group G cannot contain operations belonging to every conjugate set in G. For, if it did, the set of conjugate groups of which H is one would contain all the operations of G. This is impossible, since the number of distinct operations in this set of conjugate sub-groups is less than the order of G.

3. A simple group has no self-conjugate sub-group, and therefore no characteristic sub-group. Let G be a group which is not simple, and at the same time has no characteristic sub-group, and let H be a minimum self-conjugate sub-group of G, *i.e.*, a self-conjugate sub-group of G such that no self-conjugate sub-group of G of order less than the order of H is contained in H. Since H is not a characteristic sub-group of G, there must be an isomorphism of G which changes H into another sub-group H_1, also necessarily self-conjugate. Now H and H_1 can have no common sub-group except identity; for, if they had such a sub-group, it would be a self-conjugate sub-group of G and would be contained in H, contrary to supposition. Since H and H_1 are both self-conjugate sub-groups of G, every operation of H transforms H_1 into itself, and conversely. Hence, by a theorem due to Herr W. Dyck,* every operation of H is permutable with every operation of H_1. Now, let

$$H,\ H_1,\ H_2,\ \dots\ H_{n-1}$$

be the complete set of groups into which H is changed by the isomorphisms of G. Then every operation of any one of these groups is permutable with every operation of any other, and G must consist of the direct product of such of these groups as are independent, for, if $\{H, H_1, \dots H_{n-1}\}$ is not G, it is a characteristic sub-group of G. If, further, H were not a simple group, a self-conjugate sub-group h of H would be a self-conjugate sub-group of G; for h is transformed into itself by every operation of H, while every operation of h is permutable with every operation of H_r $(r = 1, 2, \dots n-1)$. Hence, if H were not a simple group, it could not be a minimum self-conjugate sub-group of G.

It follows therefore that:—

A group which has no characteristic sub-group is either simple or is generated by n holohedrically isomorphous and independent simple groups, every operation of any one of which is permutable with every operation of any other.

When the simple groups are of prime order p, the group thus generated is an Abelian group of order p^n all of whose operations are of order p. If the simple groups are of composite order N, the group

* *Cf.* "Gruppentheoretische Studien," *Math. Ann.*, xx. If A and B are any operations of H and H_1 respectively, then $ABA^{-1}B^{-1}$ belongs to both and is therefore identity, so that A and B are permutable.

of order N^n generated by them has no self-conjugate operation except identity. In either case the type of groups arrived at is that which occurs at the last step but one when the chief composition-series (*Hauptreihe*) of any group is formed.

4. Suppose, now, that G is a group which has no characteristic sub-group, and that L is the group of isomorphisms of G. When the order of G is not the power of a prime, L will contain a self-conjugate sub-group which is holodedrically isomorphous with G, namely, the group of cogredient isomorphisms ; and this may without risk of confusion be denoted by G. If G is not a characteristic sub-group of L, let G' be a sub-group with which G is conjugate in the group of isomorphisms of L, so that G' is also a self-conjugate sub-group of L. Then, if G and G' have a common sub-group g, other than identity, g is a self-conjugate sub-group of L, and therefore a characteristic sub-group of G, contrary to supposition. Hence G and G' can have no common sub-group other than identity, and therefore, by the theorem of Herr Dyck already quoted, every operation of G' is permutable with every operation of G. But this again is inconsistent with G' being a sub-group of L, since every operation of L gives a distinct isomorphism of G. Finally, then, G is a characteristic sub-group of L.

If, now, L is susceptible of a contragredient isomorphism I, such an isomorphism must interchange the operations of G among themselves, and therefore as regards these operations it must effect the same change as a suitably chosen cogredient isomorphism J of L. Hence IJ^{-1} must be a contragredient isomorphism of L which leaves every operation of G unchanged. If, then, S is any operation of L which the isomorphism IJ^{-1} changes into S', while R represents in turn each operation of G, then IJ^{-1} changes

$$S^{-1}RS \quad \text{into} \quad S'^{-1}RS',$$

so that these operations of G are identical. Hence S and S' effect the same isomorphism of G, or, in other words, S and S' are identical. The isomorphism IJ^{-1} of L is therefore the identical isomorphism, and this is not consistent with the assumption that I is a contragredient isomorphism. Hence, finally, the group L admits of cogredient isomorphisms only.

Suppose next that G is an Abelian group of order p^n, which has no characteristic sub-group, and that L is the group of isomorphisms of G. Since G is Abelian, the cogredient isomorphisms reduce to the identical isomorphism only, and L will contain in this case no sub-group isomorphous with G. If, however,

$$S_1 (=1),\ S_2,\ \ldots\ S_N, \qquad (N = p^m),$$

are the operations of G, each operation of L can be expressed in the form of a substitution

$$\begin{pmatrix} S \\ S' \end{pmatrix},$$

while each operation S_x of G can be expressed in the form of a substitution

$$\begin{pmatrix} S \\ SS_x \end{pmatrix}.^*$$

If, when the operations of L and G are thus expressed, these groups are referred to as L_1 and G_1, then $\{L_1, G_1\}$ contains G_1 self-conjugately. For

$$\begin{pmatrix} S \\ S' \end{pmatrix}^{-1} \begin{pmatrix} S \\ SS_x \end{pmatrix} \begin{pmatrix} S \\ S' \end{pmatrix} = \begin{pmatrix} S' \\ S'S'_x \end{pmatrix} = \begin{pmatrix} S \\ SS'_x \end{pmatrix},$$

which shows not only that G_1 is contained self-conjugately in $\{L_1, G_1\}$, but also that the result of transforming the operations of G_1 by any operation of L_1 gives rise to that isomorphism of G which the corresponding operation of L actually represents.

The group $\{L_1, G_1\}$ is clearly the most general group that contains G_1 self-conjugately, so that no operation of the group which is not contained in G_1 is permutable with every operation of G_1. For, if Σ were any operation not contained in $\{L_1, G_1\}$ which transforms G_1 into itself, Σ would transform the operations of G_1 in the same way as some operation S of $\{L_1, G_1\}$, and therefore ΣS^{-1} would be permutable with every operation of G_1. The group $\{L_1, G_1\}$ thus arrived at is obviously identical with the general linear group defined by the congruences

$$\left.\begin{array}{l} x'_1 \equiv a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + \beta_1, \\ x'_2 \equiv a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + \beta_2, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ x'_n \equiv a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + \beta_n, \end{array}\right\} \ \text{mod. } p,$$

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* *Cf.* Herr W. Dyck, "Gruppentheoretische Studien II.," *Math. Ann.*, XXII., p. 85, and Herr G. Frobenius, "Ueber endliche Gruppen," *Berliner Sitzungsberichte*, 1895, p. 184.

where
$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix} \not\equiv 0, \qquad \text{mod. } p.$$

It may now be shown, by precisely the same reasoning as in the previous case, that G_1 is a characteristic sub-group of $\{L_1, G_1\}$. Every isomorphism of $\{L_1, G_1\}$ therefore transforms G_1 into itself. But the group contains just p^m operations of order 2, each of which transforms every operation of G_1 into its own inverse. Hence every isomorphism of the group must transform these p^m operations among themselves. Since $\{L_1, G_1\}$ can itself be expressed as a transitive group of degree p^m, it follows, exactly as in the first part of Section II., that its group of isomorphisms can also be so expressed. If now there are contragredient isomorphisms, the group of isomorphisms must contain operations which are permutable with every operation of G_1. But, when expressed as a transitive group of degree p^m, the only operations permutable with every operation of G_1 are its own operations; and there can therefore be no contragredient isomorphisms. It is assumed here that p is an odd prime, but when p is 2 there is no difficulty in showing that the same result holds by considering the operations of L_1 of order 2^m-1. Hence, finally,

If G is a group which has no characteristic sub-group, and if K is the group of greatest order that contains G self-conjugately, while at the same time no operation of K not contained in G is permutable with every operation of G, then the group K is capable only of cogredient isomorphisms and it contains no self-conjugate operation except identity.

To a group which only allows of cogredient isomorphisms and contains no self-conjugate operation except identity, Herr Hölder has given the name *vollkommene Gruppe* (*loc. cit.*, p. 325). Such a group I propose to call a *complete group.* The importance of complete groups in the theory of the composition of a group depends on a theorem due to Herr Hölder (*loc. cit.*), which may be stated as follows:— If a complete group H is contained self-conjugately in a group G, then G is the direct product of H and some other group K; *i.e.*, G is generated by H and K where every operation of H is permutable with every operation of K, while H and K have no common operation except identity.

I may point out here that, if G is *any* Abelian group and L the

[redacted]

group of isomorphisms of G, a group $\{L_1, G_1\}$ may be formed, exactly as above, which contains G_1 self-conjugately, and at the same time contains no operation permutable with every operation of G_1 except the operations of G_1 itself. Such a group will not, however, in general be a complete group, nor will G_1 in general be a characteristic sub-group of it.

III.

In a paper on "A Class of Simple Groups defined by Congruences," in Vol. xxv. of the Society's *Proceedings*, I have investigated some of the properties of certain simple groups of orders $2^n(2^{2n}-1)$ and $\frac{1}{2}p^n(p^{2n}-1)$, p being an odd prime. These include as a particular case ($n=1$) the groups of the modular equations the groups of isomorphisms of which have been determined in the memoir by Herr Hölder referred to above. I propose here to determine these groups of isomorphisms whatever n may be.

It is shown in my paper (Vol. xxv., p. 119) that the simple group H of order $\frac{1}{2}p^n(p^{2n}-1)$ contains p^n+1 conjugate sub-groups of order p^n, and therefore, by the first result in Section II., the group of isomorphisms L of H can be expressed as a transitive group of degree p^n+1. Now H is contained self-conjugately in a group G of order $p^n(p^{2n}-1)$ which can be expressed as a triply-transitive group of degree p^n+1, containing no operation permutable with every operation of H. Hence L must either be identical with or must contain G, and its order must be $p^n(p^{2n}-1)\,m$. If S is any operation contained in L and not in H, it is always possible, since H is doubly transitive, to find an operation S' of H such that SS' keeps any two given symbols fixed.

Suppose, then, that SS' keeps the symbols a and b fixed. The sub-group of H which keeps a fixed consists of an Abelian sub-group P of order p^n, all the operations of which are of order p, and p^n cyclical sub-groups of order $\frac{1}{2}(p^n-1)$ each keeping a and one other symbol fixed. One of the latter, say Q, will keep a and b fixed, and it consists of all the operations of H which keep a and b fixed. Hence SS' must transform the sub-groups P and Q into themselves, and, since every operation of P displaces all the symbols except a, SS' cannot be permutable with every operation of P. A superior limit to $2m$ is therefore obtained by finding the greatest sub-group of the group of isomorphisms of P within which a cyclical sub-group of order $\frac{1}{2}(p^n-1)$ is self-conjugate.

It is shown* in my former paper (pp. 138, 139) that the group of isomorphisms of P is identical with the linear group

$$\left.\begin{aligned} x_1' &\equiv a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ x_2' &\equiv a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ x_n' &\equiv a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n, \end{aligned}\right\} \quad \text{mod. } p.$$

Now the multipliers of an operation of order $\frac{1}{2}(p^n-1)$ of the linear group are of the form

$$\lambda^2,\ \lambda^{2p},\ \ldots\ \lambda^{2p^{n-1}},$$

where λ is a primitive root of the congruence

$$\lambda^{p^n-1}-1 \equiv 0, \qquad \text{mod. } p.$$

By bringing the operation to its canonical form, it may be shown immediately that it is only permutable with the operations which have the same fixed (imaginary) symbols as itself, and these form a cyclical sub-group of order p^n-1, generated by an operation whose multipliers are

$$\lambda,\ \lambda^p,\ \ldots\ \lambda^{p^{n-1}}.$$

Moreover, the multipliers of the operations of order $\frac{1}{2}(p^n-1)$ being all different, I have shown (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 64) that the operation is conjugate with every operation that has the same multipliers. If, then, Σ is an operation of the linear group of order $\frac{1}{2}(p^n-1)$, Σ is permutable only in a cyclical sub-group of order p^n-1, while

$$\Sigma,\ \Sigma^p,\ \Sigma^{p^2},\ \ldots\ \Sigma^{p^{n-1}}$$

have the same multipliers and are therefore conjugate operations. Hence the sub-group of order $\frac{1}{2}(p^n-1)$ generated by Σ must be contained self-conjugately in a sub-group of order $(p^n-1)\,n$, and is contained self-conjugately in no greater sub-group. Finally, then, m cannot exceed n, and the order of L cannot exceed $p^n\,(p^{2n}-1)\,n$.

That the order of L actually has this value may be verified by direct calculation. The group H is generated (Vol. XXV., p. 117) by the operations

$$z' \equiv \frac{-1}{z}, \quad z' \equiv z+1, \quad z' \equiv \lambda^2 z,$$

* *Cf.* also Dr. Cole, *Bulletin of the New York Mathematical Society*, 2nd Series, II., 2.

and the group G by

$$z' \equiv \frac{-1}{z}, \quad z' \equiv z+1, \quad z' \equiv \lambda z,$$

where λ is a primitive root of

$$\lambda^{p^n-1} - 1 \equiv 0, \qquad \text{mod. } p.$$

Now the operation J, $\qquad z' \equiv z^p,$

is of order n, and it transforms the groups H and G into themselves. This may be shown directly by transforming the generating operations. Moreover, it is easy to see that no operation, of the form

$$z' = \frac{\alpha z^{p^r} + \beta}{\gamma z^{p^r} + \delta},$$

where α, β, γ, δ are powers of λ, which results by combining a power of J with any operation of G, is permutable with every operation H; and therefore every operation of $\{J, G\}$, or L, whose order is $p^n(p^{2n}-1)\,n$ represents a distinct isomorphism of H.

The operations $\qquad z' \equiv \lambda z \quad \text{and} \quad z' \equiv z^p$

being represented by S and J, SJ and JS are given by

$$z' \equiv \lambda^p z^p \quad \text{and} \quad z' \equiv \lambda z^p,$$

so that

$$JS = SJS^{1-p},$$

and S^{1-p} is an operation of H. Also the operation SJ^r cannot belong to H, unless r is a multiple of n. Hence the factor group $\dfrac{L}{H}$ is the direct product of two cyclical groups of orders 2 and n. In particular, if n is odd, the factor group $\dfrac{L}{H}$ is a cyclical group of order $2n$.

The simple group of order $2^n(2^{2n}-1)$ (Vol. xxv., p. 118) contains 2^n+1 conjugate sub-groups of order 2^n, and can be expressed as a triply-transitive group of order 2^n+1. Its isomorphisms may be determined exactly as in the case just considered, with the exception that the simple group is here the analogue of the group G of the previous case. The result is to show that the group of isomorphisms L is a group of order $2^n(2^{2n}-1)\,n$, and the factor group $\dfrac{L}{H}$ of order n is cyclical.

Among the simple groups of order $\frac{1}{2}p^n(p^{2n}-1)$ here dealt with the particular case $p^n = 3^2$ gives the alternating group of degree 6. Herr Hölder in his memoir already quoted determines the group of isomorphisms of the alternating group for all degrees, and he finds that, as compared with all others, the alternating degree of degree 6 behaves in an exceptional manner and requires special treatment. There is no reason to regard the alternating groups of different degrees as characterized by common group properties, in the same sense that the groups of the modular equations for prime transformations are; though, no doubt, they have some common properties which make it convenient to deal with them as a class. When considered as one of a class with which it is connected by common group properties, it is here seen that the alternating group of degree 6 does not behave in an exceptional manner.

The Division of the Lemniscate. By G. B. Mathews.

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Since Abel's discovery* of the analogy which exists between the problem of the equisection of the lemniscate and the division of the circumference of a circle into equal parts, the lemniscate problem has attracted a good deal of attention: in particular, reference should be made to Schwering's important papers in *Crelle's Journal*, Vols. CVII., CX. But these researches are mainly analytical, and there is a certain interest in trying to develop the geometrical theory proper, so as to give the actual results for the section of the real period in a form suitable for geometrical construction. To do this in some of the simpler cases is the object of the present paper: the elements for the construction are found by analysis, and this is, of course, a kind of imperfection; but it is possible that the possession of explicit real formulæ which can be at once translated into geometry may lead to a purely geometrical method, at least when the section by rule and compass is possible.

* Gauss was undoubtedly familiar with the theory: see *Disq. Arith.*, Art. 335, and *Werke*, III., p. 404 and following, with the editorial note, p. 496.

In order to follow out this idea consistently, we avoid the theory of complex multiplication, and take the modulus to be $\frac{1}{\sqrt{2}}$, so that the elliptic functions to be employed are those defined by

$$\operatorname{sn}^{-1} x = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}},$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}},$$

and the problem is to find the elliptic functions of the arguments $\frac{2rK}{n}$, where r, n are real integers.

If we take the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

the arc measured from the vertex is

$$s = a\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta}} = \frac{a}{\sqrt{2}}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}},$$

where $$\sin\phi = \sqrt{2}\sin\theta.$$

If we put $$\frac{s\sqrt{2}}{a} = u, \quad \phi = \operatorname{am} u,$$

then $$\sin\theta = \frac{1}{\sqrt{2}}\operatorname{sn} u, \quad \cos\theta = \operatorname{dn} u,$$

and the whole length of the lemniscate is

$$\frac{4Ka}{\sqrt{2}} = L,$$

say. Hence, if $s = \frac{L}{n}$, where n is a real integer, $u = \frac{4K}{n}$; it is convenient, however, to consider $2K$, not $4K$, as a period, so that the problem of n-section is to be understood as primarily the question of finding the elliptic functions of $\frac{2K}{n}$.

Following Halphen (*Fonc. Ell.*, I., p. 10; II., p. 386), let

$$U = x^2+y^2-R^2,$$

$$V = (x-\delta)^2+y^2-r^2,$$

so that $U=0$, $V=0$ represent two circles. Then the discriminant of $kU-V$ is

$$-R^2k^3+(2R^2+r^2-\delta^2)\,k^2-(R^2+2r^2-\delta^2)\,k+r^2,$$

and hence, in Halphen's notation,

$$k_1=\frac{2R^2+r^2-\delta^2}{R^2},$$

$$k_2=\frac{R^2+2r^2-\delta^2}{R^2},$$

$$k_3=\frac{r^2}{R^2}.$$

The condition that the elliptic functions associated with the pair of circles may be to modulus $\dfrac{1}{\sqrt{2}}$ is found to be

$$r^2=R^2-6R\delta+\delta^2;$$

and hence, putting $$\frac{\delta}{R}=c,$$

we have
$$k_1=3\,(1-2c),$$
$$k_2=3-12c+c^2,$$
$$k_3=1-6c+c^2,$$

and Halphen's invariants X, Y, calculated by the corrected formula (*l.c.*, II., p. 649),

$$X=\frac{(4k_1k_3-k_2^2)^3}{2^6k_3^4},\quad Y=\frac{4k_1k_2k_3-k_2^3-8k_3^2}{2^3k_3^2},$$

are in the present case

$$X=\frac{(3-24c+6c^2-c^4)^3}{2^8\,(1-6c+c^2)^4},$$

$$Y=\frac{1-12c+5c^2-5c^4+12c^5-c^6}{2^3\,(1-6c+c^2)^2}.$$

It is convenient to change the parameter to d, where

$$d=\frac{1-c}{1+c}=\frac{R-\delta}{R+\delta};$$

the new values are

$$X = -\frac{(1+2d-4d^3-2d^4)^3}{2^4(1+d)^4(1-2d^2)^4},$$

$$Y = -\frac{d(1-2d^4)}{2(1+d)^2(1-2d^2)^2}.$$

If now we solve the equations

$$X = 0, \quad Y = 0, \quad X - Y = 0, \ \ldots$$

(see Halphen, II., p. 377), the corresponding values of d are those of

$$\operatorname{dn} \frac{2(r+2si)K}{n},$$

for $n = 3, 4, 5$, &c.; the positive real integers r, s assume, independently of each other, the values $0, 1, 2 \ldots n-1$ with the following limitations:—

If n is odd, the combination $(0, 0)$ is excluded;

If n is even, the combinations $(0, 0)$, $(0, \frac{1}{2}n)$, $(\frac{1}{2}n, 0)$ are to be excluded.

Further, it is to be remembered that the combinations (r, s), $(n-r, n-s)$ lead to the same value of d, and this only occurs once in the equation we have to consider. Moreover, when n is even, (r, s) coincides with $(n-r, n-s)$ for $r = s = \frac{n}{2}$.

Therefore the degree of the equation in d is

$$\tfrac{1}{2}(n^2-1) \quad \text{if } n \text{ is odd,}$$

$$\tfrac{1}{2}n^2-3 \quad \text{if } n \text{ is even.}$$

If n is composite, the d-equation contains irrelevant factors arising from the divisors of n, but by taking the cases in their natural order these can be eliminated as they arise. Let the final equation, cleared of these factors, be

$$F(d) = 0.$$

Then all the equations $F(d) = 0$ are irreducible in the ordinary domain of rationality, that is, in the domain of ordinary rational numbers; but they are all Abelian, and therefore are algebraically solvable. The questions that have to be answered relate to the groups of the equations, and the arithmetical nature of the irrational quantities which the roots necessarily involve.

First, as to the groups. In the complex theory all the roots are arranged in an Abelian cycle, and this is further sub-divided into minor groups. At present we are concerned with the real roots, and it is clear that these form an Abelian cycle of their own, when $n=p$, an odd prime. If g is a primitive root of p, the quantities we want to find are

$$\operatorname{dn}\frac{2K}{p},\quad \operatorname{dn}\frac{4K}{p},\quad \dots\quad \operatorname{dn}\frac{2(p-1)K}{2p},$$

and these are identical, save as to order, with

$$\operatorname{dn}\frac{2K}{p},\quad \operatorname{dn}\frac{2gK}{p},\quad \operatorname{dn}\frac{2g^2K}{p},\quad \dots\quad \operatorname{dn}\frac{2g^{\frac{1}{2}(p-3)}K}{p},$$

and, if we now write

$$p' = \tfrac{1}{2}(p-1),$$

$$d_s = \operatorname{dn}\frac{2g^sK}{p},$$

we have

$$d_1 = \vartheta(d_0),$$
$$d_2 = \vartheta(d_1),$$
$$\dots\quad \dots\quad \dots$$
$$d_{p'-1} = \vartheta(d_{p'-2}),$$
$$d_0 = d_{p'} = \vartheta(d_{p'-1}),$$

where ϑ denotes a rational function, which can be obtained by the multiplication theorem. Of course, in practice, the smallest value of g is the most convenient.

All this, and much besides, is illustrated by the worked out examples which follow.

I. $n=3$.

The equation is $X=0$, which may be written in the form

$$2(d^2+d-\tfrac{1}{2})^2 = \tfrac{3}{2};$$

whence

$$(2d+1)^2 = 3\pm 2\sqrt{3},$$

$$d = \tfrac{1}{2}\left\{-1\pm\sqrt{3\pm 2\sqrt{3}}\right\}.$$

The real roots are

$$d_0 = \operatorname{dn}\frac{2K}{3} = \tfrac{1}{2}\left\{-1+\sqrt{3+2\sqrt{3}}\right\}$$

$$= \tfrac{1}{2}\left\{-1+\sqrt[4]{3}\,\frac{\sqrt{3}+1}{\sqrt{2}}\right\} = \cdot 77123 = \cos 39^\circ 32' 8'',$$

[redacted]

and $$d_0' = \operatorname{dn}\frac{4iK}{3} = \tfrac{1}{2}\{-1-\sqrt{3+2\sqrt{3}}\}$$
$$= \tfrac{1}{2}\left\{-1-\sqrt[4]{3}\,\frac{\sqrt{3}+1}{\sqrt{2}}\right\}.$$

The other roots are given by
$$d = \tfrac{1}{2}\left\{-1 \pm i\sqrt[4]{3}\,\frac{\sqrt{3}-1}{\sqrt{2}}\right\},$$
and correspond to the arguments
$$\frac{(2+4i)\,K}{3},\quad \frac{(4+4i)\,K}{3}.$$

II. $n = 4$.

The equation is $Y = 0$, or
$$d\,(1-2d^4) = 0;$$
whence $$d = 0,\quad \pm\frac{1}{\sqrt[4]{2}},\quad \pm\frac{i}{\sqrt[4]{2}},$$
corresponding to the arguments
$$(1+i)\,K,\quad \frac{K}{2},\quad \frac{K}{2}+2iK,\quad \frac{(3+2i)\,K}{2},\quad \frac{(1+2i)\,K}{2},$$
respectively.

If $$\cos\phi = \frac{1}{\sqrt[4]{2}},$$
$$\phi = 32^\circ\,45'\,54'',$$
and ϕ may of course, as in the last case, be constructed geometrically.

III. $n = 5$.

This is the first case of special interest, and requires discussion in detail.

The equation is $$Y - X = 0,$$
or $$(1+2d-4d^3-2d^4)^3 - 8d\,(1+d)^2\,(1-2d^2)^2\,(1-2d^4) = 0,$$
which is of the twelfth degree. The two real roots we have to discover are
$$d_1 = \operatorname{dn}\frac{2K}{5},\quad d_2 = \operatorname{dn}\frac{4K}{5}.$$

Now, since $$\text{dn}\,(2K-u) = \text{dn}\,u,$$

and $$\frac{8K}{5} = 2K - \frac{2K}{5},$$

it follows that d_1, d_2 form an Abelian cycle of two terms; and, by applying the duplication formula, we find that

$$d_2 = -\frac{1-2d_1^2+2d_1^4}{1-4d_1^2+2d_1^4} = \vartheta\,(d_1), \text{ say,}$$

and, similarly, $d_1 = \vartheta\,(d_2)$.

Now, let x_1, x_2 be the roots of a quadratic equation

$$x^2+ax+b = 0,$$

and suppose $$x_1 = \vartheta\,(x_2), \quad x_2 = \vartheta\,(x_1),$$

ϑ being as above defined. Then, since

$$x_1 = \frac{b}{x_2} \quad \text{and} \quad x_2 = \frac{b}{x_1},$$

it follows that x_1, x_2 both satisfy

$$\frac{b}{x} = \vartheta\,(x) = -\frac{1-2x^2+2x^4}{1-4x^2+2x^4}.$$

Hence the polynomial

$$2x^5+2bx^4-2x^3-4bx^2+x+b$$

must be exactly divisible by x^2+ax+b. This leads to the conditions that

$$(a^2+1)(a^2+a-1)^2 = 0 \quad\text{.........................(i.),}$$

and that b is a common root of

$$2\,(a+1)\,b^2-2\,(a^2-1)\,b-(a-1) = 0 \quad\text{...............(ii.),}$$
$$2b^2-2\,(a^2+2a-2)\,b+(2a^2-2a+1) = 0 \quad\text{.........(iii.).}$$

The solutions are as follows—

$$a = i, \qquad b = -\frac{1-i}{2},$$

$$a = -i, \qquad b = -\frac{1+i}{2},$$

$$a = \frac{-1+\sqrt{5}}{2}, \quad 2b^2+(3-\sqrt{5})\,b+(\sqrt{5}-2) = 0,$$

$$a = \frac{-1-\sqrt{5}}{2}, \quad 2b^2+(3+\sqrt{5})\,b-(\sqrt{5}+2) = 0,$$

[redacted]

and both roots of each quadratic are available, because for these two values of a the equations (ii.) and (iii.) are identical.

The solution of $2b^2+(3-\sqrt{5})\,b+(\sqrt{5}-2)=0$

is
$$b=\frac{-3+\sqrt{5}\pm\sqrt{30-14\sqrt{5}}}{4}$$
$$=\frac{-3+\sqrt{5}\pm i\sqrt[4]{5}\,(3-\sqrt{5})}{4},$$

and that of the other quadratic is
$$b=\frac{-3-\sqrt{5}\pm\sqrt[4]{5}\,(3+\sqrt{5})}{4},$$

so that altogether there are six quadratics in x, namely,

$$2x^2+2ix-(1-i)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$2x^2-2ix-(1+i)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$4x^2+2\,(\sqrt{5}+1)\,x-3+\sqrt{5}\pm i\,(3-\sqrt{5})\,\sqrt[4]{5}=0 \;\ldots\ldots (3,\,4),$$
$$4x^2-2\,(\sqrt{5}+1)\,x-3-\sqrt{5}\pm\,(3+\sqrt{5})\,\sqrt[4]{5}=0 \;\ldots\ldots (5,\,6).$$

These, then, are the six quadratic factors of the original equation in x.

The only quadratics which can have real roots are the last pair. Choosing
$$4x^2-2\,(\sqrt{5}+1)\,x-3-\sqrt{5}+(3+\sqrt{5})\,\sqrt[4]{5}=0,$$

we have
$$(4x-\sqrt{5}-1)^2=6\,(3+\sqrt{5})-4\,(3+\sqrt{5})\,\sqrt[4]{5}$$
$$=(\sqrt{5}+1)^2\left\{3-2\sqrt[4]{5}\right\};$$

and therefore
$$x=\frac{\sqrt{5}+1}{4}\left\{1\pm\sqrt{3-2\sqrt[4]{5}}\right\}.$$

Both the roots are real positive proper fractions, and therefore correspond to the arguments $\frac{2K}{5}$, $\frac{4K}{5}$ respectively.

It may be observed that
$$\frac{\sqrt{5}+1}{4}=\cos\frac{\pi}{5},$$

and that $\sqrt{3-2\sqrt[4]{5}}$ is an algebraical unit.

It is easy to see how the values of x may be constructed geometrically.

The numerical values are

$$x_1 = \cdot 8870484 = \cos 27^\circ\, 29'\, 43'',$$
$$x_2 = \cdot 7309856 = \cos 43^\circ\, 1'\, 52''.$$

For the sake of comparison we will consider the quinquisection problem in the light of the complex theory. Here the divisor is, in the first instance, not 5 but $1+2i$, and the solution depends upon any one of the equations

$$\operatorname{sn}(1+2i)\,u = 0,$$
$$\operatorname{dn}(1+2i)\,u = \pm 1,$$
$$\operatorname{cn}(1+2i)\,u = \pm 1.$$

Now, if we make use of the formulæ

$$\operatorname{sn} iu = \frac{i \operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{cn} iu = \frac{1}{\operatorname{cn} u}, \quad \operatorname{dn} iu = \frac{\operatorname{dn} u}{\operatorname{cn} u},$$

the addition theorem gives

$$\operatorname{sn}(1+2i)\,u = \frac{2\left(\operatorname{sn} u \operatorname{dn} 2u + i \operatorname{cn} u \operatorname{dn} u \operatorname{sn} 2u \operatorname{cn} 2u\right)}{2 \operatorname{cn}^2 2u + \operatorname{sn}^2 u \operatorname{sn}^2 2u}$$
$$= \frac{P(\operatorname{sn} u)}{Q(\operatorname{sn} u)},$$

where $P(x) = -x\left\{(1-2i)\,x^8 - 2\,(1-7i)\,x^6 + (1-32i)\,x^4 + 4\,(1+7i)\,x^2 - 4\,(1+2i)\right\},$

$$Q(x) = 5x^8 - 20x^6 + 28x^4 - 16x^2 + 4.$$

We know from the general theory that $P(x)$ and $Q(x)$ must have a common factor of the fourth degree; this is found to be

$$(1-2i)\,x^4 - 2\,(1-2i)\,x^2 - 2i;$$

therefore

$$\operatorname{sn}(1+2i)\,u = -\frac{\operatorname{sn} u\left\{\operatorname{sn}^4 u - 2\,(2-i)\operatorname{sn}^2 u - 2i\,(1+2i)\right\}}{(1+2i)\operatorname{sn}^4 u - 2\,(1+2i)\operatorname{sn}^2 u + 2i}$$
$$= -\frac{\operatorname{sn} u\left\{\operatorname{sn}^4 u + 2i\,(1+2i)\operatorname{sn}^2 u - 2i\,(1+2i)\right\}}{(1+2i)\operatorname{sn}^4 u - 2\,(1+2i)\operatorname{sn}^2 u + 2i}.$$

Equating this to zero, and supposing that sn u does not vanish,

$$\text{sn}^2 u = -i(1+2i) \pm \sqrt{-(1+2i)^2 + 2i(1+2i)}$$

$$= -i(1+2i) \pm i\sqrt{1+2i},$$

$$\text{sn}\, u = \pm\sqrt{-i(1+2i) \pm i\sqrt{1+2i}}.$$

Since i is a primitive root of $1+2i$, the four values of sn u are those for which

$$u \equiv \pm\frac{2K}{1+2i}, \quad \pm\frac{2iK}{1+2i},$$

$$\equiv \pm\frac{2(1-2i)K}{5}, \quad \pm\frac{2(2+i)K}{5}.$$

The four values of sn u may be plotted off by rule and compass in the plane of the complex variable; and this is, in a sense, the proper geometrical solution of the quinquisection problem.

To find cn u and dn u, we have

$$\text{cn}^2 u = 1 - \text{sn}^2 u$$

$$= -1 + i \mp i\sqrt{1+2i},$$

$$\text{dn}^2 u = 1 - \tfrac{1}{2}\text{sn}^2 u$$

$$= \frac{i}{2}\{1 \pm \sqrt{1+2i}\} = \tfrac{1}{4}\{i \pm \sqrt{1+2i}\}^2.$$

If we like, we may reduce sn u, cn u, dn u to the form $\alpha + \beta i$; the results are

$$\text{sn}\, u = \pm\left[\sqrt{\tfrac{1}{2}\left\{\sqrt[4]{5}\left(\frac{\sqrt{5}+1}{2} + \sqrt{\frac{\sqrt{5}-1}{2}}\right) \pm \left(1+\sqrt{\frac{\sqrt{5}+1}{2}}\right)\right\}}\right.$$

$$\left. + \sqrt{\tfrac{1}{2}\left\{\sqrt[4]{5}\left(\quad\right) \mp \left(1+\sqrt{\frac{\sqrt{5}+1}{2}}\right)\right\}}\, i\right],$$

$$\text{cn}\, u = \pm\tfrac{1}{2}\left[\left\{1 \pm \sqrt{\frac{\sqrt{5}+1}{2}} \mp \sqrt{\frac{\sqrt{5}-1}{2}}\right\}\right.$$

$$\left. + \left\{1 \pm \sqrt{\frac{\sqrt{5}+1}{2}} \pm \sqrt{\frac{\sqrt{5}-1}{2}}\right\} i\right],$$

$$\text{dn}\, u = \pm\tfrac{1}{2}\left\{i \pm \left(\sqrt{\frac{\sqrt{5}+1}{2}} + i\sqrt{\frac{\sqrt{5}-1}{2}}\right)\right\}$$

$$= \pm\tfrac{1}{2}\left\{\sqrt{\frac{\sqrt{5}+1}{2}} + \left(\sqrt{\frac{\sqrt{5}-1}{2}} \pm 1\right) i\right\}.$$

The values of cn u, dn u may be neatly expressed in the trigonometrical forms

$$\operatorname{cn} u = \pm\tfrac{1}{2}\left[\left(1 \pm \frac{4}{\sqrt[4]{5}}\sin\frac{\pi}{10}\cos\frac{3\pi}{10}\right)+\left(1 \pm \frac{4}{\sqrt[4]{5}}\cos\frac{\pi}{10}\sin\frac{3\pi}{10}\right)i\right],$$

$$\operatorname{dn} u = \pm\tfrac{1}{2}\left[\frac{2}{\sqrt[4]{5}}\sin\frac{2\pi}{5}+\left(\frac{2}{\sqrt[4]{5}}\sin\frac{\pi}{5}\pm 1\right)i\right];$$

there does not appear to be any correspondingly simple expression for sn u.

The most elegant method of all, from the point of view of the complex theory, is to introduce the complex fifth roots of unity. If we put

$$\omega = e^{2\pi i/5},$$

$$\xi = \omega + i\omega^2 - i\omega^3 - \omega^4,$$

it is found that

$$\xi^2 = -(1+2i)\sqrt{5},$$

and we may put

$$\frac{i\xi}{\sqrt[4]{5}} = \sqrt{1+2i};$$

hence

$$\operatorname{dn} u = \pm\frac{i}{2}\left\{1 \pm \frac{\xi}{\sqrt[4]{5}}\right\},$$

$$\operatorname{cn} u = \pm\frac{1+i}{2}\left\{1 \mp \frac{i\xi}{\sqrt[4]{5}}\right\},$$

$$\operatorname{sn}^2 u = 2 - i \pm \frac{\xi}{\sqrt[4]{5}}.$$

The quartic of which dn u is a root is

$$2d^4 - 2id^2 + i = 0,$$

which breaks up into the two quadratics

$$2d^2 - 2id - (1+i) = 0,$$

$$2d^2 + 2id - (1+i) = 0.$$

Of these the first is the same as the second of the six quadratics obtained from the geometrical theory; the other is obtained from the first of the six by changing the sign of the middle term. This may be accounted for by the fact that, strictly speaking, we ought to write

$$\frac{R-\delta}{R+\delta} = \pm d,$$

thus obtaining twelve quadratics, in accordance with the analytical theory.

As to the arguments to be associated with the different values of d, it will be found that, if we write for convenience

$$\epsilon = \sqrt{\frac{\surd 5+1}{2}}, \quad \epsilon' = \sqrt{\frac{\surd 5-1}{2}},$$

$$d_1 = \tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\} = \frac{i}{2}\left\{1-\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\frac{(2-2i)K}{1+2i},$$

$$d_2 = -\tfrac{1}{2}\{\epsilon+(\epsilon'-1)i\} = \frac{i}{2}\left\{1+\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\frac{(4-4i)K}{1+2i},$$

$$d_3 = -\tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\} = -\frac{i}{2}\left\{1-\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\frac{2K}{1+2i},$$

$$d_4 = \tfrac{1}{2}\{\epsilon+(\epsilon'-1)i\} = -\frac{i}{2}\left\{1+\frac{\xi}{\sqrt[4]{5}}\right\} = \mathrm{dn}\frac{2iK}{1+2i}.$$

We are now in a position to deduce a solution of the real quinquisection. Putting

$$u = \frac{2K}{1+2i}, \qquad v = \frac{2K}{1-2i},$$

$$u+v = \frac{4K}{5}, \qquad u-v = -\frac{8iK}{5}.$$

Now, by the addition theorem,

$$\mathrm{dn}(u+v)+\mathrm{dn}(u-v) = \frac{2\,\mathrm{dn}\,u\,\mathrm{dn}\,v}{-1+2(\mathrm{dn}^2 u+\mathrm{dn}^2 v)-2\,\mathrm{dn}^2 u\,\mathrm{dn}^2 v},$$

$$1+\mathrm{dn}(u+v)\,\mathrm{dn}(u-v) = \frac{\mathrm{dn}^2 u+\mathrm{dn}^2 v}{-1+2(\mathrm{dn}^2 u+\mathrm{dn}^2 v)-2\,\mathrm{dn}^2 u\,\mathrm{dn}^2 v}.$$

In the present case

$$\mathrm{dn}\,u = -\tfrac{1}{2}\{\epsilon+(\epsilon'+1)i\},$$

$$\mathrm{dn}\,v = -\tfrac{1}{2}\{\epsilon-(\epsilon'+1)i\};$$

and therefore

$$\mathrm{dn}\,u\,\mathrm{dn}\,v = \tfrac{1}{4}\{\epsilon^2+\epsilon'^2+2\epsilon'+1\} = \tfrac{1}{4}\{\surd 5+1+2\epsilon'\},$$

$$\mathrm{dn}^2 u\,\mathrm{dn}^2 v = \tfrac{1}{4}(\surd 5+1)(1+\epsilon'),$$

$$\mathrm{dn}^2 u+\mathrm{dn}^2 v = -\epsilon'.$$

Putting in these values, and performing the necessary reductions, we find

$$\operatorname{dn}(u+v)+\operatorname{dn}(u-v)=\frac{\sqrt{5}+1}{2}-\frac{3+\sqrt{5}}{2}\epsilon',$$

$$\operatorname{dn}(u+v)\operatorname{dn}(u-v)=\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}+2}{2}\epsilon'.$$

Therefore $\operatorname{dn}\frac{4K}{5}$, $\operatorname{dn}\frac{8iK}{5}$ are the roots of

$$d^2-\left(\frac{\sqrt{5}+1}{2}-\frac{3+\sqrt{5}}{2}\epsilon'\right)d+\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}+2}{2}\epsilon'\right)=0.$$

Guided by previous results, we assume

$$d=\frac{\sqrt{5}+1}{4}t;$$

the equation then becomes

$$t^2-2(1-\epsilon)t+(\sqrt{5}-1-2\epsilon)=0,$$

and hence $\quad t=1-\epsilon\pm\sqrt[4]{5}\,.\,\epsilon'$

$$=1-\sqrt{\frac{\sqrt{5}+1}{2}}\pm\sqrt{\frac{5-\sqrt{5}}{2}}.$$

Since $\operatorname{dn}\frac{4K}{5}$ is positive,

$$\operatorname{dn}\frac{4K}{5}=\left(1-\sqrt{\frac{\sqrt{5}+1}{2}}+\sqrt{\frac{5-\sqrt{5}}{2}}\right)\frac{\sqrt{5}+1}{4},$$

$$\operatorname{dn}\frac{8iK}{5}=\left(1-\sqrt{\frac{\sqrt{5}+1}{2}}-\sqrt{\frac{5-\sqrt{5}}{2}}\right)\frac{\sqrt{5}+1}{4}=-\operatorname{dn}\frac{2iK}{5}.$$

The value previously obtained for $\operatorname{dn}\frac{4K}{5}$ was

$$\operatorname{dn}\frac{4K}{5}=\frac{\sqrt{5}+1}{4}\left(1-\sqrt{3-2\sqrt[4]{5}}\right);$$

it is easy to show that the two values coincide; thus, writing for the moment

$$\theta=\sqrt[4]{5},$$

we ought to have $\quad \theta\epsilon'-\epsilon=-\sqrt{3-2\theta}.$

The signs agree, and, on squaring,

$$\theta^2\epsilon'^2-2\theta+\epsilon^2=3-2\theta,$$

or

$$\theta^2\frac{\sqrt{5}-1}{2}=3-\frac{\sqrt{5}+1}{2}=\frac{5-\sqrt{5}}{2},$$

that is, $$\theta^2=\sqrt{5},$$

and the relation is verified.

In the same way,

$$\operatorname{dn}\frac{8iK}{5}=\frac{\sqrt{5}+1}{4}\left(1-\sqrt{3+2\sqrt[4]{5}}\right)=-\operatorname{dn}\frac{2iK}{5}.$$

It is now possible to construct a table of the values of $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$ for all the arguments

$$\frac{2(m+ni)K}{5};$$

but, on account of the trouble of reducing the expressions to their simplest forms, no attempt has been made to form a complete table. A few of the formulæ, however, are given. For convenience, we write

$$\epsilon=\sqrt{\frac{\sqrt{5}+1}{2}},\quad \epsilon'=\sqrt{\frac{\sqrt{5}-1}{2}},$$

$$\eta=\sqrt{3+2\sqrt[4]{5}},\quad \eta'=\sqrt{3-2\sqrt[4]{5}},$$

where observe that

$$\eta\eta'=\sqrt{5}-2,\quad \eta+\eta'=2\epsilon,\quad \eta-\eta'=2\epsilon'\sqrt[4]{5}.$$

so that we have $$\left.\begin{aligned}\eta&=\epsilon+\epsilon'\theta\\ \eta'&=\epsilon-\epsilon'\theta\end{aligned}\right\},\qquad \theta=\sqrt[4]{5}.$$

$$\operatorname{dn}\frac{2K}{5}=\tfrac{1}{2}\epsilon^2(1+\eta'),$$

$$\operatorname{dn}\frac{4K}{5}=\tfrac{1}{2}\epsilon^2(1-\eta'),$$

$$\operatorname{dn}\frac{2iK}{5}=\tfrac{1}{2}\epsilon^2(\eta-1),$$

$$\operatorname{dn}\frac{4iK}{5}=\tfrac{1}{2}\epsilon^2(\eta+1);$$

$$\operatorname{cn}\frac{2K}{5}=\frac{1+\eta'}{\eta-1}=\frac{\epsilon^2+\epsilon-1}{\theta+1},$$

$$\operatorname{cn}\frac{4K}{5}=\frac{1-\eta'}{\eta+1}=-\frac{\epsilon^2-\epsilon-1}{\theta+1},$$

$$\operatorname{cn}\frac{2iK}{5}=\frac{\eta-1}{1+\eta'}=-\frac{\epsilon^2-\epsilon-1}{\theta-1},$$

$$\operatorname{cn}\frac{4iK}{5}=\frac{\eta+1}{1-\eta'}=\frac{\epsilon^2+\epsilon-1}{\theta-1};$$

$$\operatorname{sn}\frac{2K}{5}=\frac{2\sqrt{\theta-\epsilon}}{\eta-1},$$

$$\operatorname{sn}\frac{4K}{5}=\frac{2\sqrt{\theta+\epsilon}}{\eta+1},$$

$$\operatorname{sn}\frac{2iK}{5}=\frac{2i\sqrt{\theta-\epsilon}}{1+\eta'},$$

$$\operatorname{sn}\frac{4iK}{5}=\frac{2i\sqrt{\theta-\epsilon}}{1-\eta'}.$$

It is interesting to verify these results by comparing them with those of Gauss and Schwering. The function which Gauss calls sin lemn v, and Schwering sin am v, is the inverse of the function

$$v=\int_0^x\frac{dx}{\sqrt{(1-x^4)}}=(\sin\operatorname{lemn})^{-1}x\,;$$

we shall write this $\quad \operatorname{sl} v = x,$

and put $\quad \operatorname{cl} v=\sqrt{1-x^2},\quad \operatorname{dl} v=\sqrt{1+x^2},$

$$\int_0^1\frac{dx}{\sqrt{1-x^4}}=\varpi=\frac{K}{\sqrt{2}}.$$

Then, if the arguments u, v relate to the same point on the lemniscate,

$$u=\sqrt{2}\,(\varpi-v),$$

$$\operatorname{sn} u=\operatorname{cl} v,$$

$$\operatorname{cn} u=\operatorname{sl} v,$$

$$\operatorname{dn} u=\frac{1}{\sqrt{2}}\operatorname{dl} v\,;$$

and when
$$u = \frac{2mK}{n},$$
$$v = \frac{(n-2m)\varpi}{n}.$$

Schwering gives a numerical result which in this notation is
$$\operatorname{sl}\frac{\varpi}{5} = \cdot 2620824 = \operatorname{cn}\frac{4K}{5},$$
and hence we find
$$\operatorname{dn}\frac{4K}{5} = \sqrt{\tfrac{1}{2}\left(1+\operatorname{cn}^2\frac{4K}{5}\right)} = \cdot 7310017.$$
The value found above (p. 375) was
$$\cdot 7309856,$$
so that there is a discrepancy of about ·0000161 which must be accounted for by errors of calculation, as the agreement is too close for any mistake in the argument of the function.

To take an example of a different kind, Gauss gives a number of quinquisection formulæ (*Werke*, III., p. 421) of which the first is, in our notation, equivalent to
$$\left\{\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5}\right\}^2 = 14\sqrt{5}-30 = \sqrt{5}\,(3-\sqrt{5})^2;$$
whence
$$\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5} = \theta\,(3-\sqrt{5}) = 2\theta\epsilon^{-1}.$$

To verify this, write
$$\operatorname{dn}\frac{2K}{5} = d_1, \quad \operatorname{dn}\frac{4K}{5} = d_2,$$
$$\operatorname{cn}\frac{2K}{5} = c_1, \quad \operatorname{cn}\frac{4K}{5} = c_2;$$
then, observing that
$$\operatorname{cn}^2 u = \frac{\operatorname{dn} 2u + \operatorname{cn} 2u}{1+\operatorname{dn} 2u},$$
we find
$$\operatorname{cn}^2\frac{K}{5} + \operatorname{cn}^2\frac{3K}{5} = \frac{d_1+c_1}{1+d_1} + \frac{d_2-c_2}{1+d_2}$$
$$= \frac{d_1+d_2+2d_1d_2+c_1-c_2+c_1d_2-c_2d_1}{1+d_1+d_2+d_1d_2}.$$

From the short table given above

$$d_1+d_2=\epsilon^2,\quad d_1d_2=\tfrac{1}{4}\epsilon^4(1-\eta^2)=\tfrac{1}{2}\epsilon^4(\theta-1),$$

$$c_1-c_2=\frac{2(\epsilon^2-1)}{\theta+1},\quad c_1d_2-c_2d_1=\frac{\epsilon^2(1-\eta^2)}{\eta^2-1}=\frac{\epsilon^2(\theta-1)}{\theta+1}.$$

Therefore the numerator of the fraction is

$$\epsilon^2+\epsilon^4(\theta-1)+\frac{2(\epsilon^2-1)+\epsilon^2(\theta-1)}{\theta+1}$$

$$=\frac{1}{\theta+1}\left[\epsilon^2+\epsilon^4(\sqrt{5}-1)+2\epsilon^2-2-\epsilon^2+2\epsilon^2\theta\right]$$

$$=\frac{1}{\theta+1}(4\epsilon^2-2+2\epsilon^2\theta)=2\epsilon^2+\frac{2\epsilon'^2}{\theta+1}=2\epsilon^2+\theta-1$$

$$=\sqrt{5}+\theta=\theta(\theta+1).$$

In a similar way, it may be shown that the denominator is equal to $\frac{1}{2}\epsilon^4(1+\theta)$; hence

$$\text{cn}^2\frac{K}{5}+\text{cn}^2\frac{3K}{5}=\frac{2\theta}{\epsilon^4}=2\theta\epsilon'^4,$$

which completes the verification.

It is found in a similar way that

$$\text{cn}^2\frac{K}{5}-\text{cn}^2\frac{3K}{5}=2\epsilon'^5;$$

whence

$$\left(\text{cn}^2\frac{K}{5}-\text{cn}^2\frac{3K}{5}\right)^2=4\epsilon'^{10}=10\sqrt{5}-22,$$

which is Gauss's second formula.

By combining these results we obtain

$$\text{cn}^2\frac{K}{5}=\epsilon'^4(\theta+\epsilon'),\quad \text{cn}^2\frac{3K}{5}=\epsilon'^4(\theta-\epsilon'),$$

so that

$$\text{cn}\,\frac{K}{5}=\frac{\sqrt{5}-1}{2}\sqrt{\sqrt[4]{5}+\sqrt{\frac{\sqrt{5}-1}{2}}}=\epsilon'^2\sqrt{\theta+\epsilon'},$$

$$\text{cn}\,\frac{3K}{5}=\frac{\sqrt{5}-1}{2}\sqrt{\sqrt[4]{5}-\sqrt{\frac{\sqrt{5}-1}{2}}}=\epsilon'^2\sqrt{\theta-\epsilon'}.$$

Thursday, May 14th, 1896.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

The following gentlemen were admitted into the Society:—Mr. F. W. Dyson, Royal Observatory, Greenwich; Mr. G. H. J. Hurst, Eton College; and Mr. F. W. Russell, University College School.

Mr. Baker spoke upon "The Bitangents of a Plane Quartic Curve and the Straight Lines of a Cubic Surface."

A paper by Prof. E. W. Brown, "On the Application of the Principal Function to the Solution of Delaunay's Canonical System of Equations," was, in the absence of the author, taken as read.

Short impromptu communications were made by the President, Prof. Hill, Col. Cunningham, Mr. Hammond, and Mr. Tucker.

The following presents had been received since the April meeting:

"Proceedings of the Royal Society," Vol. LIX., No. 356.

"Journal of the Institute of Actuaries," Vol. XXXII., No. 181, Pt. 5, April, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 7; April, 1896.

"Encyclopädie der Mathematischen Wissenschaften," 2 Probeartikele, 8vo; Leipzig, 1896. iii.C *6a*, "Flächer die Herordnung," von W. F. Meyer. ii.A *7b*, "Potential-theorie (Theorie der Laplace-Poisson'schen Differential Gleichung)," von H. Burkhardt und W. F. Meyer.

"Bulletin de la Société Mathématique de France," Tome XXIV., Nos. 2, 3; Paris, 1896.

"Bulletin des Sciences Mathématiques," Tome XX., Avril, 1896; Paris.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," I; 1896.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XX., St. 4; Leipzig, 1896.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXV., Heft 1; Berlin, 1896.

"Proceedings of the Physical Society of London," Vol. XIV., Pt. 5, May, 1896.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. X., No. 2, 1896.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 7, Sem. 1; Roma, 1896.

"Educational Times," May, 1896.

"Journal für die reine und angewandte Mathematik," Bd. CXVI., Heft 2; Berlin, 1896.

"Annali di Matematica, Milano," Tome XXIV., Fasc. 2; 1896.

"Indian Engineering," Vol. XIX., Nos. 14, 15, 16, April 4th to April 18th, 1896.

On the Application of the Principal Function to the Solution of Delaunay's Canonical System of Equations. By Ernest W. Brown. Read May 14th, 1896. Received April 28th, 1896.

(i.) The system of canonical equations adopted by Delaunay is

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \qquad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \qquad \frac{dH}{dt} = \frac{\partial R}{\partial h};$$

$$\frac{dl}{dt} = -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}.$$

In order to solve these equations, he puts

$$R = -B - A\cos\theta + R_1,$$

where $$\theta = il + i'g + i''h + i'''n't + q.$$

Here $-A\cos\theta$ is any periodic term of the disturbing function R, and $-B$ is the non-periodic portion of R; B, A are functions of L, G, H only, i, i', i'', i''' are positive or negative integers, n' is the solar mean motion, and q is a constant depending on the solar epoch and perigee. The equations are solved by neglecting R_1, and Delaunay then inquires what variable values the six arbitrary constants, introduced into the solution, are to have when R_1 is not neglected. These arbitraries are so chosen that the new equations are canonical. In order to satisfy this condition, three only of the six arbitraries may be chosen at will, the manner in which the other three arbitraries enter into the solution being then determinate. Delaunay adopts the constant terms in the expressions for l, g, h as three of the arbitraries, and the equations are afterwards transformed so that the new variables are the non-periodic, instead of the constant, parts of l, g, h.

The method adopted by Delaunay,* in order to arrive at the new system of equations, is one of direct transformation. M. Tisserand† has shown that, by the use of the principal function, Jacobi's dynamical method very materially shortens the process. The transformations may be also made by means of the calculus of variations, and this plan has been used by Radau‡ and myself.§ All these methods, however, are long and require several transformations, as well as the proof of a lemma which does not seem otherwise of value, and they labour under the further disadvantage of being unsymmetrical, while both the old and the new systems of equations are symmetrical.

The use of the principal function, S, appears to offer the most direct means of obtaining the required system of arbitraries, and the method given below is based on its properties. The chief difference from M. Tisserand's method lies in the form obtained for the principal function. He finds a solution of the partial differential equation for S in which the variables are L, G, H, t, and of the arbitrary constants one refers especially to L, G, H, while the other two refer to l, g, h. I find a solution in which the variables are l, g, h, t, while all the three arbitrary constants refer to L, G, H. The gain in simplicity and brevity will be immediately evident. All the essential properties of the solution are easily obtained, and one or two new ones, which do not appear to have been noticed hitherto, are added.

(ii.) Let $$L = i\Theta\,;$$

then, substituting for R and neglecting R_1, we obtain on integration

$$G = (G)+i'\Theta, \quad H = (H)+i''\Theta,$$

where (G), (H) are arbitrary constants. Substituting, we find easily

$$\frac{d\theta}{dt} = \frac{dA}{d\Theta}\cos\theta + \frac{dB}{d\Theta} + i'''n',$$

$$\frac{d\Theta}{dt} = A\sin\theta,$$

in which A, B are supposed to have been expressed in terms of (G),

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* *Théorie de la Lune*, Vol. I., Chap. III.
† *Journal de Liouville*, 1868 ; *Mécanique Céleste*, Chap. XI.
‡ *Bulletin Astronomique*, Vol. IX., p. 336.
§ *Treatise on the Lunar Theory*, Chap. IX.

(H), and the variable Θ. These equations may be solved by continued approximation, giving

$$\left.\begin{array}{l} \theta = \theta_0 (t+c) + \Sigma \theta_j \sin j\theta_0 (t+c), \\ \Theta = \Theta_0 + \Sigma \Theta_j \cos j\theta_0 (t+c); \\ \text{whence} \quad \Theta = \Theta_0' + \Sigma j \psi_j \cos j\theta \end{array}\right\} (j = 1 \dots \infty),$$

in which c and Θ_0 or Θ_0' may be taken as the arbitraries. The last expression does not contain c explicitly, since θ_0, θ_j, Θ_j, ψ_j are functions of (G), (H), and Θ_0 (or Θ_0') only. The equations also admit of the integral

$$A \cos \theta + B + i''' n' \Theta = C,$$

where C is a function of the same three arbitrary constants.

Putting $L' = i\Theta_0'$, $G' = i'\Theta_0' + (G)$, $H' = i''\Theta_0' + (H)$,

and considering L', G', H' as the arbitrary constants, we have

$$\left.\begin{array}{r} L = L' + i\ \Sigma j \psi_j \cos j\theta \\ G = G' + i'\ \Sigma j \psi_j \cos j\theta \\ H = H' + i'' \Sigma j \psi_j \cos j\theta \\ -B - A \cos \theta = \psi_0 + i''' n' \Sigma j \psi_j \cos j\theta \end{array}\right\} \dots\dots\dots\dots\dots\dots (1),$$

ψ_0 being a function of L', G', H' only.

(iii.) Write $$R' = A \cos \theta + B:$$

the canonical equations become

$$\frac{dL}{dt} = -\frac{\partial R'}{\partial l}, \quad \frac{dG}{dt} = -\frac{\partial R'}{\partial g}, \quad \frac{dH}{dt} = -\frac{\partial R'}{\partial h};$$

$$\frac{dl}{dt} = \frac{\partial R'}{\partial L}, \quad \frac{dg}{dt} = \frac{\partial R'}{\partial G}, \quad \frac{dh}{dt} = \frac{\partial R'}{\partial h}.$$

The principal function, S, satisfies the partial differential equation

$$\frac{\partial S}{\partial t} + R' = 0,$$

where R' is supposed to be expressed in terms of l, g, h, t, $\frac{\partial S}{\partial l}$, $\frac{\partial S}{\partial g}$, $\frac{\partial S}{\partial h}$ by means of the equations

$$L = \frac{\partial S}{\partial l}, \quad G = \frac{\partial S}{\partial g}, \quad H = \frac{\partial S}{\partial h}.$$

The preceding values for L, G, H give

$$\frac{\partial S}{\partial l} = L' + i\Sigma j\psi_j \cos j\theta,$$

$$\frac{\partial S}{\partial g} = G' + i'\Sigma j\psi_j \cos j\theta,$$

$$\frac{\partial S}{\partial h} = H' + i''\Sigma j\psi_j \cos j\theta,$$

$$\frac{\partial S}{\partial t} = \psi_0 + i'''n'\Sigma j\psi_j \cos j\theta.$$

A solution of the partial differential equation for S, involving the variables l, g, h, t, and the three arbitrary constants L', G', H', is therefore

$$S = L'l + G'g + H'h + t\psi_0 + \Sigma\psi_j \sin j\theta,$$

where $$\theta = il + i'g + i''h + i'''n't + q,$$

as before. The well-known results of Jacobi give, as the other three integrals, the equations

$$\left.\begin{aligned} (l) &= \frac{\partial S}{\partial L'} = l + t\frac{\partial \psi_0}{\partial L'} + \Sigma\frac{\partial \psi_j}{\partial L'}\sin j\theta \\ (g) &= \frac{\partial S}{\partial G'} = g + t\frac{\partial \psi_0}{\partial G'} + \Sigma\frac{\partial \psi_j}{\partial G'}\sin j\theta \\ (h) &= \frac{\partial S}{\partial H'} = h + t\frac{\partial \psi_0}{\partial H'} + \Sigma\frac{\partial \psi_j}{\partial H'}\sin j\theta \end{aligned}\right\} \quad \text{..................(2)},$$

where (l), (g), (h) are three new arbitrary constants.

The six constants L', G', H', (l), (g), (h) form a canonical system, and, if we put

$$R = -R' + R_1,$$

the equations satisfied by the arbitraries, supposed variable in order that R_1 may be included, are

$$\frac{dL'}{dt} = \frac{\partial R_1}{\partial (l)}, \quad \frac{dG'}{dt} = \frac{\partial R_1}{\partial (g)}, \quad \frac{dH'}{dt} = \frac{\partial R_1}{\partial (h)};$$

$$\frac{d(l)}{dt} = -\frac{\partial R_1}{\partial L'}, \quad \frac{d(g)}{dt} = -\frac{\partial R_1}{\partial G'}, \quad \frac{d(h)}{dt} = -\frac{\partial R_1}{\partial H'}.$$

Change the variables (l), (g), (h) to l', g', h', where

$$l' = (l) - t\frac{\partial\psi_0}{\partial L'}, \quad g' = \ldots, \quad h' = \ldots.$$

A simple and well-known transformation shows that the equations satisfied by the new variables are

$$\frac{dL'}{dt} = \frac{\partial R_2}{\partial l'}, \ \ldots, \ \ldots; \quad \frac{dl'}{dt} = -\frac{\partial R_2}{\partial L'}, \ \ldots, \ \ldots;$$

where $$R_2 = R_1 + \psi_0.$$

(iv.) It is only necessary now to show that this is the final canonical system obtained by Delaunay. Let

$$\theta' = il' + i'g' + i''h' + i'''n't + q.$$

By reversion of series, we obtain from (2)

$$l = (l) - t\frac{\partial\psi_0}{\partial L'} + \Sigma l_j \sin j\theta',$$

$$g = (g) - t\frac{\partial\psi_0}{\partial G'} + \Sigma g_j \sin j\theta',$$

$$h = (h) - t\frac{\partial\psi_0}{\partial H'} + \Sigma h_j \sin j\theta'.$$

Hence l', g', h' are the non-periodic parts of l, g, h. Also, from these equations combined with the value of θ previously obtained,

$$\theta = \theta' + \Sigma\theta_j \sin j\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\theta' = \theta_0(t+c), \quad \theta_j = il_j + i'g_j + i''h_j;$$

and the equations (1) become

$$\left.\begin{array}{c} L = L' + i\Theta, \quad G = G' + i'\Theta, \quad H = H' + i''\Theta \\ \Theta = \Theta_0 + \Sigma\Theta_j \cos j\theta' \end{array}\right\} \ldots\ldots\ldots(4),$$

where (the second line above)

the coefficients l_j, g_j, h_j, θ_j, Θ_j, Θ_0 being functions of L', G', H' only.

Since L', G', H' are the non-periodic parts of L, G, H when expressed in terms of θ, we have

$$\pi L' = \int_0^\pi L\, d\theta.$$

[redacted]

The equations (3), (4) give, since θ, θ', take the values 0, π together,

$$L' = \frac{1}{\pi}\int_0^{\pi}(L'+i\Theta_0+i\Sigma\Theta_j\cos j\theta')(1+\Sigma j\theta_j\cos j\theta')\,d\theta'$$

$$= L'+i\Theta_0+\tfrac{1}{2}i\phi,$$

where $$\phi = \theta_1\Theta_1+2\theta_2\Theta_2+3\theta_3\Theta_3+\dots.$$

Hence, if L_0, G_0, H_0 be the constant parts of L, G, H when expressed in terms of the time, the new variables are, by (4),

$$L_0+\tfrac{1}{2}i\phi,\quad G_0+\tfrac{1}{2}i'\phi,\quad H_0+\tfrac{1}{2}i''\phi.$$

Again, $$R_2 = R_1+\psi_0$$

$$= R_1-B-A\cos\theta-i'''n'\Sigma j\psi_j\cos j\theta$$

$$= R-i'''n'(L-L')/i;$$

and $$\psi_0 = -C+i'''n'\Theta-i'''n'(L-L')/i$$

$$= -C+i'''n'\left(\frac{1}{i}L_0+\tfrac{1}{2}\phi\right).$$

Finally, it is evident that, if A, the coefficient of the periodic term considered, be neglected, the values of L, G, H, l, g, h reduce to L', G', H', l', g', h'. This completes the relation of the solution to that of Delaunay.

We have thus seen (1) that the six new variables are the non-periodic parts of the old variables when expressed in terms of θ; (2) that, when all the variables are expressed in periodic series, the coefficients of the periodic terms can all be deduced directly from those of the expression for Θ; (3) that the secular parts of l, g, h are the derivatives with respect to L', G', H' of the negative of the constant term to be added to the disturbing function. The last result shows that, if l_0, g_0, h_0 are the secular motions of l, g, h, the expression

$$l_0\,dL'+g_0\,dG'+h_0\,dH'$$

is a perfect differential. This result is a case of a similar theorem obtained by Prof. Newcomb in the general problem of n bodies.*

* *Smithsonian Contributions*, Vol. XXI., "The General Integrals of Planetary Motion," § 5.

Thursday, June 11th, 1896.

Major P. A. MACMAHON, R.A., F.R.S., President, in the Chair.

The President announced that the Council had awarded the De Morgan Memorial Medal to Mr. Samuel Roberts. Mr. Roberts, who was present, cordially thanked the Council. The Medal will be presented, and the grounds of the award stated, at the Annual Meeting in November.

The President then informed the members present that the Council had adopted an address to Lord Kelvin on the occasion of the Jubilee of his Professoriate, and that it had been signed by the President and Secretaries, and stamped with the Seal of the Society, and that he (the President) had been requested to present it in the name of the Society. The text of the address, which had been illuminated and bound in a volume, was then read by the President, and ordered, on the motion of the Treasurer, to be entered in the minutes.

To the Right Hon. Lord Kelvin, D.C.L., LL.D., Past President of the Royal Society.

My Lord,—

It is at once the duty and the pleasure of a Society associated with that special branch of science of which you have made so brilliant applications to add its warmest congratulations to the many you will receive upon the occasion of the Jubilee of your Professoriate in the University of Glasgow.

The London Mathematical Society recalls with pride the fact that from the earliest days of the Society you have been a member. It recognizes that for more than half a century your work in Mathematical Physics has had a foremost place in sustaining the reputation of this country in the science which it represents.

The Council and members, represented at your Jubilee celebration by their President, tender you heartfelt wishes for your future happiness.

We remain, my Lord,

Your obedient servants,

Percy A. MacMahon, Major, R.A., F.R.S., *President.*
Robert Tucker, M.A. } *Hon. Secs.*
A. E. H. Love, M.A., F.R.S. }

London, 11, vi. 96.

Mr. Alfred Edward Western, B.A., Scholar of Trinity College, Cambridge, was elected a member of the Society.

The following communications were made:—

Waves in Canals: Mr. H. M. Macdonald (communicated by Mr. Love).

On the *a*, *b*, *c* Form of the Binary Quintic: Mr. J. Hammond.

Construction for the Four Normals to a Central Conic drawn through a Given Point: Prof. G. B. Mathews.

On a Two-fold Generalization of Stieltjes' Theorem: Dr. H. Taber.

These last two papers were taken as read.

Mr. Samuel Roberts explained the abstract of "Notes on Magic Squares," by the Rev. A. H. Frost (communicated by his brother, Dr. P. Frost).

The following presents to the Library were received:—

Williamson, B.—"An Elementary Treatise on the Integral Calculus," 8vo, 7th edition; London, 1896. From the author.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xx., St. 5; Leipzig, 1896.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. x., No. 3; May, 1896.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 2; Lent Term, 1896.

Darboux, G.—"Leçons sur la Théorie Générale des Surfaces et les Applications Géométriques du Calcul Infinitésimal," Partie 4 (second fascicule), 8vo; Paris, 1896.

"Monatshefte für Mathematik und Physik," Jahrgang VII., Hefte 4, 5, 6; Wien, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 8, May, 1896; New York.

"Proceedings of the Physical Society of London," Vol. XIV., Pt. 6; June, 1896.

"Nyt Tidsskrift for Matematik," A. Aargang 7, Nr. 2, B. Aargang 7, Nr. 1; Copenhagen, 1896.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXX., Liv. 1; Harlem, 1896.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. II., Fasc. 4, April, 1896; Napoli.

"Bulletin de la Société Mathématique de France," Tome XXIV., No. 4; Paris, 1896.

"Bulletin des Sciences Mathématiques," Tome XX., Mai, 1896; Paris.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, 1896, Heft 1; Geschäftliche Mittheilungen, 1896, Heft 1.

"Annales de la Faculté des Sciences de Marseille," Tome V., Fasc. 4; Tome VI., Fasc. 1, 2, 3; Paris, 1896.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. V., Fasc. 8, 9, 10; Roma, 1896.

"Educational Times," June, 1896.

"Indian Engineering," Vol. xix., Nos. 17–20, April 25th to May 16th, 1896.

"Memorie della Regia Accademia di Scienze, Lettere ed Arti in Modena," Serie 2, Vol. xi., 1895.

"Annals of Mathematics," Vol. x., Nos. 2–3; Jan.–March, 1896.

On the a, b, c Form of the Binary Quintic. By J. Hammond, M.A.

Received June 2nd, 1896. Read June 11th, 1896.

"Lo now it appeareth easie enough."

Recorde's *Arithmetick.**

1. If ω is an imaginary cube root of unity, and λ a disposable constant which may have any value we please except zero, then, writing

$$\left.\begin{aligned} X &= \lambda\,(\omega x+\omega^2 y) \\ Y &= \lambda\,(\omega^2 x+\omega y) \\ Z &= \lambda\,(x+y) \end{aligned}\right\} \qquad (1),$$

we have $$X+Y+Z=0,$$

and
$$\begin{aligned} AX^5+BY^5+CZ^5 = {} & x^5\lambda^5\,(A\omega^2+B\omega+C) \\ & +5x^4y\lambda^5\,(A+B+C) \\ & +10x^3y^2\lambda^5\,(A\omega+B\omega^2+C) \\ & +10x^2y^3\lambda^5\,(A\omega^2+B\omega+C) \\ & +5xy^4\lambda^5\,(A+B+C) \\ & +y^5\lambda^5\,(A\omega+B\omega^2+C), \end{aligned}$$

i.e., $$AX^5+BY^5+CZ^5 = (a, b, c, a, b, c)(x, y)^5 \qquad (2),$$

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* Robert Recorde wrote the first treatise on Algebra in the English language, which was published in 1557 under the title of "The Whetstone of Witte, which is the seconde parte of Arithmetike: containing the Extraction of Rootes; The Cossike Practise, with the Rule of Equation; and the Workes of Surde Nombers." See Hutton's *Tracts*, 3 vols., 8vo, 1812, Tract 33. He died in 1558, but his *Arithmetick* continued in use for more than a century, and went through many editions, of which the last known to De Morgan was that of Edward Hatton, 1699. (See *Companion to the Almanac*, 1837, or *Arithmetical Books*, 1847.)

where
$$a = \lambda^5 (A\omega^2 + B\omega + C),$$
$$b = \lambda^5 (A + B + C),$$
$$c = \lambda^5 (A\omega + B\omega^2 + C).$$

The form (2) to which, by aid of Sylvester's canonical form, the general quintic has now been brought may be called its *a, b, c* form, or we may speak of it as the quintic (a, b, c); and we shall use k, a', b', c', respectively, to denote the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ and its three minors, which may be called *the determinant of the quintic* (a, b, c) *and its three minors.*

Thus
$$k = 3abc - a^3 - b^3 - c^3 \quad\text{.......................(3),}$$

and
$$\left.\begin{aligned} a' &= bc - a^2 \\ b' &= ca - b^2 \\ c' &= ab - c^2 \end{aligned}\right\} \quad\text{..............................(4).}$$

It follows immediately that
$$\left.\begin{aligned} b'c' - a'^2 &= ka \\ c'a' - b'^2 &= kb \\ a'b' - c'^2 &= kc \end{aligned}\right\} \quad\text{...........................(5),}$$

i.e., the quintics (a, b, c) and (a', b', c') are so related that the coefficients of each of them are proportional to the corresponding minor determinants of the other. This relation may be briefly expressed by saying that these two quintics are *conjugate quintics.*

The determinant of (a, b, c) being k, that of (a', b', c') is k^2, and it will be seen hereafter that the canonizant of (a, b, c) is $k(x^3 + y^3)$, and its skew invariant $k^4(a'^3 - c'^3)$. Hence the canonizant of (a', b', c') is $k^2(x^3 + y^3)$, and its skew invariant $k^{11}(a^3 - c^3)$; for (5) shows that a', c' change into ka, kc respectively, whilst k^4 changes into k^8.

But
$$(a'^3 - c'^3) = (bc - a^2)^3 - (ab - c^2)^3 = k(a^3 - c^3),$$

so that, disregarding the power of k which enters as a factor, the skew invariants are the same for both quintics.

The constant λ, which is still at our disposal, may be so chosen as to make $k = 1$; and then the conjugate pair of quintics (a, b, c) and (a', b', c') have the same canonizant (x^3+y^3) and the same skew invariant (a^3-c^3), the determinant of each quintic is equal to unity, and the coefficients of either of them are equal to the corresponding minor determinants of the other.

2. The simplest quadratic covariant (2, 2) of the quintic, and its canonizant (3, 3), may, as is well known, be derived from the invariants of the quartic

$$ae-4bd+3c^2,$$

$$ace+2bcd-ad^2-b^2e-c^3,$$

by changing a, b, c, &c., into

$$ax+by, \quad bx+cy, \quad cx+dy, \text{ \&c.}$$

Thus, for the quintic (a, b, c), these two covariants are

$$-3\,(ax+by)(bx+cy)+3\,(cx+ay)^2,$$

and $3\,(ax+by)(bx+cy)(cx+ay)-(ax+by)^3-(bx+cy)^3-(cx+ay)^3$.

Or, dividing the quadratic covariant by -3, and simplifying by the use of equations (3) and (4), we obtain

$$(2.2) \qquad c'x^2-b'xy+a'y^2,$$

$$(3.3) \qquad k\,(x^3+y^3),$$

where k is the determinant of the quintic (a, b, c), and a', b', c' are the coefficients of its conjugate.

Referring now to the list of concomitants of a cubic and quadratic given on p. 348 of Elliott's *Algebra of Quantics*, it will be seen that, in consequence of the identical relation

$$3a'b'c'-a'^3-b'^3-c'^3 = k^2,$$

which subsists between the coefficients of (2.2) and (3.3) above, the five invariants of a *general* cubic and quadratic reduce to four only, when we substitute the coefficients of *this* cubic and quadratic. In fact No. 6 on the list becomes

$$I_{34} \equiv k^4,$$

and No. 14, $\quad I_{32} \equiv k^2\,(c'^3-3a'b'c'+b'^3+c'^3) \equiv -k^4,$

when we make the required substitutions. This is as it should be, since the quintic has only four invariants, viz., those obtained from Nos. 2, 13, 6, 15 of Elliott's list by substituting the coefficients of

[redacted]

(2.2) and (3.3) in them (*i.e.*, by changing a, b, c, a', c' into c', $-\frac{1}{2}b'$, a', k, k respectively). Thus we have

(4.0)	$4a'c'-b'^2$	(4 times No. 2),
(8.0)	k^2b'	(twice No. 13),
(12.0)	k^4	(No. 6),
(18.0)	$k^4(a'^3-c'^3)$	(No. 15).

In like manner we obtain the four linear covariants

(5.1)	$k(a'x+c'y)$	(No. 9),
(7.1)	$k\{-(a'b'+2c'^2)x+(b'c'+2a'^2)y\}$	(twice No. 10),
(11.1)	$k^3(a'x-c'y)$	(No. 11),
(13.1)	$k^3(c'^2x+a'^2y)$	(Elliott's L'_{13});

the three quadratic covariants

(2.2)	$c'x^2-b'xy+a'y^2$	(No. 1),
(6.2)	k^2xy	(No. 4),
(8.2)	$k^3(c'x^2-a'y^2)$	(No. 8);

and the three cubic covariants

(3.3)	$k(x^3+y^3)$	(No. 3),
(5.3)	$k(b'x^3-2a'x^2y+2c'xy^2-b'y^3)$	(twice No. 7),
(9.3)	$k^3(x^3-y^3)$	(No. 5).

3. The remaining covariants of the quintic (9 in number) are all of them of orders superior to 3. Five of these are accounted for by taking the quintic itself, and the four Jacobians of it and (5.1), (2.2), (6.2), (3.3) respectively. Writing down their values, we have

$$(1.5)\qquad ax^5+5bx^4y+10cx^3y^2+10ax^2y^3+5bxy^4+cy^5,$$

$$(6.4)\qquad k\{(ac'-ba')(x^4+4xy^3)+(bc'-ca')(4x^3y+y^4)+6(cc'-aa')x^2y^2\},$$

$$(3.5)\qquad \begin{vmatrix} ax^4+4bx^3y+6cx^2y^2+4axy^3+by^4 & 2c'x-b'y \\ bx^4+4cx^3y+6ax^2y^2+4bxy^3+cy^4 & -b'x+2a'y \end{vmatrix},$$

$$(7.5)\qquad k^2(ax^5+3bx^4y+2cx^3y^2-2ax^2y^3-3bxy^4-cy^5),$$

$$(4.6)\qquad k(bx^6+4cx^5y+5ax^4y^2-5cx^2y^4-4axy^5-by^6).$$

The Hessian (2.6) of the quintic (1.5), the Jacobian of (2.6) and (1.5), that of (2.6) and (3.3), and the result of operating with (2.2) on (2.6), complete the list of covariants. Forming the Hessian of the quintic (a, b, c), and remembering that equations (4) give $ac-b^2 = b'$, &c., we have

(2.6) $$b'x^6-3a'x^5y+6c'x^4y^2-7b'x^3y^3+6a'x^2y^4-3c'xy^5+b'y^6.$$

The Jacobian of (2.6) and (1.5) is

$$\begin{aligned}
&(aa'+2bb')\,x^9 && -(2bb'+cc')\,y^9\\
&-(4ac'+ba'-8cb')\,x^8y && +(4ca'+bc'-8ab')\,xy^8\\
&+(19ab'-8bc'-14ca')\,x^7y^2 && -(19cb'-8ba'-14ac')\,x^2y^7\\
&-(34aa'-29bb'-8cc')\,x^6y^3 && +(34cc'-29bb'-8aa')\,x^3y^6\\
&+(37ac'-47ba'+16cb')\,x^5y^4 && -(37ca'-47bc'+16ab')\,x^4y^5,
\end{aligned}$$

which, by using the identical relations

$$ab'+bc'+ca' = 0,$$

$$ba'+cb'+ac' = 0,$$

may be made to assume the somewhat simpler shape

(3.9) $$\begin{aligned}
&(aa'+2bb')\,x^9 && -(cc'+2bb')\,y^9\\
&-3\,(ac'-3cb')\,x^8y && +3\,(ca'-3ab')\,xy^8\\
&+3\,(9ab'-2ca')\,x^7y^2 && -3\,(9cb'-2ac')\,x^2y^7\\
&-(34aa'-29bb'-8cc')\,x^6y^3 && +(34cc'-29bb'-8aa')\,x^3y^6\\
&+21\,(ac'-3ba')\,x^5y^4 && -21\,(ca'-3bc')\,x^4y^5.
\end{aligned}$$

The Jacobian of (2.6) and (3.3) is

(5.7) $$k(a'x^7-4c'x^6y+9b'x^5y^2-13a'x^4y^3+13c'x^3y^4-9b'x^2y^5+4a'xy^6-c'y^7).$$

Operating with (2.2), *i.e.*, with $a'\partial_x^2+b'\partial_x\partial_y+c'\partial_y^2$, on (2.6), and dividing the result by 3, we obtain

$$(5a'b'+4c'^2)\,x^4+(2b'c'-20a'^2)\,x^3y+(48c'a'-21b'^2)\,x^2y^2$$
$$+(2a'b'-20c'^2)\,xy^3+(5b'c'+4a'^2)\,y^4.$$

This may be simplified a little by adding the square of (2.2), and dividing the sum by 5, which gives

(4.4) $$(a'b'+c'^2)\,x^4-4a'^2x^3y+(10c'a'-4b'^2)\,x^2y^2-4c'^2xy^3+(b'c'+a'^2)\,y^4,$$

or we may subtract 9 times the square of (2.2), and, after dividing by 5, use equations (5) to simplify, which yields the alternative form

$$k\,(cx^4+4ax^3y+6bx^2y^2+4cxy^3+ay^4).$$

4. Observing that 17 of the 23 concomitants of the quintic (a, b, c) are expressed naturally in terms of k, a', b', c', it seems right, for the sake of uniformity, to give similar expressions for the remaining 6. And it will only involve the repetition of three forms to give all the covariants whose orders surpass 3, in this shape; which is done in the following list,

$$(4.4)\quad (a'b'+c'^2)\,x^4-4a'^2x^3y+(10c'a'-4b'^2)\,x^2y^2-4c'^2xy^3$$
$$+(b'c'+a'^2)\,y^4+\mu\,(c'x^2-b'xy+a'y^2)^2,$$

$$(6.4)\quad (a'b'^2+b'c'^2-2c'a'^2)(x^4+4xy^3)+6\,(a'^3-c'^3)\,x^2y^2$$
$$+(2a'c'^2-b'a'^2-c'b'^2)(4x^3y+y^4),$$

$$k\,(1.5)\quad (b'c'-a'^2)(x^5+10x^2y^3)+5\,(c'a'-b'^2)(x^4y+xy^4)$$
$$+(a'b'-c'^2)(10x^3y^2+y^5),$$

$$k\,(3.5)\quad (a'^2b'+b'^2c'-2c'^2a')(x^5+10x^2y^3)$$
$$-(9a'b'c'+2a'^3-3b'^3-8c'^3)\,x^4y$$
$$+(9a'b'c'-8a'^3-3b'^3+2c'^3)\,xy^4$$
$$-(c'^2b'+b'^2a'-2a'^2c')(10x^3y^2+y^5),$$

$$(7.5)\quad k\,\{(b'c'-a'^2)(x^5-2x^2y^3)+3\,(c'a'-b'^2)(x^4y-xy^4)$$
$$+(a'b'-c'^2)(2x^3y^2-y^5)\},$$

$$(2.6)\quad b'\,(x^6-7x^3y^3+y^6)-3a'\,(x^5y-2x^2y^4)-3c'\,(xy^5-2x^4y^2),$$

$$(4.6)\quad (a'c'-b'^2)(x^6-y^6)+(a'b'-c'^2)(4x^5y-5x^2y^4)$$
$$+(b'c'-a'^2)(5x^4y^2-4xy^5),$$

$$(5.7)\quad k\,\{a'\,(x^7-13x^4y^3+4xy^6)-c'\,(y^7-13x^3y^4+4x^6y)$$
$$+9b'\,(x^5y^2-x^2y^5)\},$$

$$k\,(3.9)\quad (3a'b'c'-a'^3-2b'^3)\,x^9 \qquad -(3a'b'c'-2b'^3-c'^3)\,y^9$$
$$+3\,(3a'b'^2-4b'c'^2+c'a'^2)\,x^8y \qquad -3\,(a'c'^2-4b'a'^2+3c'b'^2)\,xy^8$$
$$+3\,(2a'c'^2-11b'a'^2+9c'b'^2)\,x^7y^2 \qquad -3\,(9a'b'^2-11b'c'^2+2c'a'^2)\,x^2y^7$$
$$+(3a'b'c'+34a'^3-29b'^3-8c'^3)\,x^6y^3-(3a'b'c'-8a'^3-29b'^3+34c'^3)\,x^3y^6$$
$$+21\,(3a'b'^2+b'c'^2-4c'a'^2)\,x^5y^4 \qquad -21\,(3c'b'^2+b'a'^2-4a'c'^2)\,x^4y^5,$$

in which all the covariants, except (1.5), (3.5), (3.9), are rational *integral* functions of k, a', b', c', and the variables; and these three forms become so when multiplied by k.

5. If in the 23 concomitants of the quintic (a, b, c) we change k, a', b', c' into k^2, ka, kb, kc, respectively, we obtain those of the conjugate quintic; and, if in these we change k, a, b, c into k^2, a', b', c', respectively, the original 23 forms are restored, each multiplied by a power of k.

Thus, from (5.1), the simplest linear covariant of the quintic, whose value (Art. 2) is $k(a'x+c'y)$, we obtain the corresponding covariant of the conjugate quintic, viz., $k^3(ax+cy)$: and from this we get $k^6(a'x+c'y)$, which is the original form (5.1) multiplied by k^5.

And so in general, if the form (p, q), whose degree is p, acquires the factor k^μ in consequence of the successive performance of both substitutions, then, since each of them doubles the degree of the form, $k^\mu(p, q)$ must be of the degree $4p$, *i.e.*,

$$3\mu+p = 4p,$$

so that

$$\mu = p.$$

The invariants of the conjugate quintic are

$$k^2(4ac-b^2), \quad k^5 b, \quad k^8, \quad k^{11}(a^3-c^3);$$

but the second and third of these are expressible as rational integral functions of the invariants of the original quintic, and the last of them is (see Art. 1) merely the original skew invariant multiplied by a power of k, so that the only fresh form is $k^2(4ac-b^2)$. In fact,

$$k^5 b = k^4(a'c'-b'^2) = \tfrac{1}{4}(4.0)(12.0)-\tfrac{3}{4}(8.0)^2,$$

$$k^8 = (12.0)^2,$$

$$k^{11}(a^3-c^3) = k^{10}(a'^3-c'^3) = k^6(18.0).$$

Similarly, we may reject all such covariants of the conjugate quintic as are rationally and integrally expressible in terms of k, and the 23 concomitants of the original quintic; but it is best to postpone the entire question of reducibility until we know more of the concomitants of a pair of conjugate quintics.

6. Those which remain to be calculated belong to both quintics conjointly, but not to either of them separately. They are sufficiently numerous to form the subject of a separate communication, but a few specimens of them are given in this article.

From the two quadratic covariants, viz.,

$$c'x^2 - b'xy + a'y^2$$

and
$$k\,(cx^2 - bxy + ay^2),$$

one of which belongs to the original quintic, and the other to its conjugate, we obtain the joint invariant

$$k\,(2ac' - bb' + 2ca'),$$

and the Jacobian

$$k\left\{(bc' - b'c)\,x^2 + 2\,(ca' - c'a)\,xy + (ab' - a'b)\,y^2\right\},$$

which are two of the forms in question.

If in this Jacobian we give a', b', c' their values in terms of a, b, c, it assumes the shape

$$k\,(bc + ca + ab)(b - c,\ c - a,\ a - b)(x,\ y)^2,$$

and, if we give ka, kb, kc their values in terms of a', b', c', it becomes

$$(b'c' + c'a' + a'b')(c' - b',\ a' - c',\ b' - a')(x,\ y)^2.$$

By operating with either of the conjugate quintics on the other, we obtain the well-known lineo-linear invariant

$$ac' - 5bb' + 10ca' - 10ac' + 5bb' - ca' = 9\,(ca' - c'a).$$

The resultant of two conjugate quintics is merely the fifth power of their lineo-linear invariant.

For, since
$$(ab + bc + ca) - a\,(a + b + c) = a',$$

with similar expressions for b', c', we have identically

$$(a',\ b',\ c') = (ab + bc + ca)(x + y)^5 - (a + b + c)(a,\ b,\ c),$$

where $(a,\ b,\ c)$ and $(a',\ b',\ c')$ denote the conjugate quintics $ax^5 + \&c.$, $a'x^5 + \&c.$

Now suppose

$$(a,\ b,\ c) = (xy_1 - yx_1)(xy_2 - yx_2)\ \ldots\ (xy_5 - yx_5),$$

and let Q_1, Q_2, &c., denote the results of substituting the roots of $(a,\ b,\ c)$ in $(a',\ b',\ c')$, so that

$$Q_1 = (ab + bc + ca)(x_1 + y_1)^5,$$

with similar expressions for Q_2, Q_3, Q_4, Q_5.

Hence the resultant is

$$
\begin{aligned}
Q_1 Q_2 Q_3 Q_4 Q_5 &= (ab+bc+ca)^5 (x_1+y_1)^5 \dots (x_5+y_5)^5 \\
&= (ab+bc+ca)^5 (x_1 x_2 x_3 x_4 x_5 + \dots + y_1 y_2 y_3 y_4 y_5)^5 \\
&= (ab+bc+ca)^5 (-c+5b-10a+10c-5b+a)^5 \\
&= 9^5 (c-a)^5 (ab+bc+ca)^5 \\
&= 9^5 (ca'-c'a)^5,
\end{aligned}
$$

which is exactly the fifth power of the lineo-linear invariant.

7. In conclusion, the case in which the invariant (8.0) vanishes will be considered, and the formulæ of Art. 2 will be used to prove that, in this case, the quintic can be brought by a linear transformation into the form

$$z^5 + 10z^3 + 45z = \text{const.},$$

which occurs in the theory of the solution of the quintic.* When $(8.0) = k^2 b'$ vanishes, we must have

$$b' = 0,$$

since k cannot vanish.

Thus the quintic, multiplied by k (see Art. 4), becomes

$$-a'^2 (x^5 + 10x^2y^3) + 5c'a' (x^4y + xy^4) - c'^2 (10x^3y^2 + y^5).$$

Multiplying this by $a'^3 c'^3$, and writing

$$\left.\begin{aligned} a'x &= \xi \\ c'y &= \eta \end{aligned}\right\},$$

we obtain $c'^3(-\xi^5 + 5\xi^4\eta - 10\xi^3\eta^2) + a'^3(-10\xi^2\eta^3 + 5\xi\eta^4 - \eta^5)$.

Now $(18.0) = k^4 (a'^3 - c'^3)$,

and $(12.0) = k^4$,

so that, if $(18.0) = k^6 I$,

we have $a'^3 - c'^3 = k^2 I$,

where I is an absolute invariant.

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* See Weber, *Elliptische Functionen und Algebraische Zahlen* (8vo, 1891), p. 319; and Kiepert, "Auflösung der Gleichung 5ten Grades," *Crelle's Journal*, Bd. LXXXVII.

And, writing $b' = 0$ in the determinant of the conjugate quintic, we find

$$-a'^3 - c'^3 = k^3.$$

Hence a'^3, c'^3 are proportional to $I-1$, $-I-1$, and, when these values are substituted in the quintic, we obtain

$$I(\xi-\eta)^5 + (\xi^5 - 5\xi^4\eta + 10\xi^3\eta^2 + 10\xi^2\eta^3 - 5\xi\eta^4 + \eta^5).$$

This, equated to zero and simplified by writing

$$\xi + \eta = u,$$

$$\xi - \eta = v,$$

gives
$$8Iv^5 + 3u^5 - 10u^3v^2 + 15uv^4 = 0,$$

which takes the form

$$z^5 + 10z^3 + 45z = -24I\sqrt{-3} = \text{const.},*$$

when we write
$$z = \frac{u}{v}\sqrt{-3}.$$

It should be noticed that

$$(5.1) = k\ (a'x + c'y) = ku,$$

$$(11.1) = k^3(a'x + c'y) = k^3v,$$

so that z is the absolute covariant $\frac{(5.1)}{(11.1)}\sqrt{(12.0)}$ multiplied by a numerical constant.

---

* Comparing $-24I\sqrt{-3}$ with the constant term of the equation given by Weber (*loc. cit.*), we have

$$-24I\sqrt{-3} = \gamma_3 = \frac{8(2+\kappa^2\kappa'^2)(\kappa'^2-\kappa^2)}{\kappa^2\kappa'^2},$$

so that κ^2 may be found, as a function of the absolute invariant I, by solving a cubic equation.

The Transformation and Division of Elliptic Functions.

By A. G. GREENHILL.

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Two important papers, with the title "The Transformation of Elliptic Functions," have been communicated by Professor Felix Klein to the London Mathematical Society, and published in Volumes IX and XI of the *Proceedings.*

Professor Klein has greatly honoured our Society by choosing its *Proceedings* for the publication of such fundamental ideas in his theory—ideas which have subsequently received their fullest development in Klein and Fricke's *Modulfunctionen.*

As the present communication is of the nature of a note or commentary on these two papers of Professor Klein, in Vols. IX and XI, the same title has been adopted; and the object of the present paper is to show how to express the various parameters employed by Klein, Kiepert, Fricke, and others, for a given transformation, explicitly in terms of a *single* parameter.

Thence it is easy to construct numerical cases required in the applications of elliptic functions; two such cases have been worked out in the sequel, and the chief results stated.

Starting with the various modular equations given by Kiepert and Klein, the object of the present paper is to express their parameters in terms of another, in such a manner that it is possible to write down the various division values (*Theil-werthe*) of the second as well as of the first stage (*Stufe*), the parameters of Klein and Kiepert being symmetric functions of these division values; and the method is illustrated at length in its application to the simplest cases.

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1. In the second paper, in Vol. XI, a certain quantity is introduced which Klein denotes by z_a; expressed in the Jacobian notation by the Eta-function, this z_a is given by a division value (*Theil-werthe*), in the form

$$(-1)^a z_a = q^{a^2/n} H(2aK'i, q^n), \qquad (1)$$

for a transformation of odd order n; and Klein shows (*Math. Ann.*, XIV, XV, and XVII) that the roots of the modular equation, for instance, of order 5, 7, or 11, can be expressed in terms of z_a.

Professor Klein's Note in Vol. XI is a mere statement of the principal results he had then arrived at; the theory is fully developed in Vol. II, Part V, of the *Modulfunctionen* ("*M. F.*"), and it is there shown that these z_a functions satisfy a number of biquadratic relations, which can be derived from the well-known four-part theta-function formula

$$\begin{aligned} &\theta(v+w)\,\theta(v-w)\,\theta(t+u)\,\theta(t-u) \\ +&\theta(w+u)\,\theta(w-u)\,\theta(t+v)\,\theta(t-v) \\ +&\theta(u+v)\,\theta(u-v)\,\theta(t+w)\,\theta(t-w) = 0 \end{aligned} \qquad (2)$$

(Brioschi, *Annali di Mathematica*, XII, 1883; XXI, 1893; XXI, 1894; *Rendiconti della R. A. dei Lincei*, 1893).

2. But in working with Halphen's γ_n function, defined in his *Fonctions Elliptiques*, I, p. 102, I have found that the relation connecting Halphen's γ with Klein's z, for a transformation of odd order n, can be written

$$(-1)^a z_a = f^3 x^{-\frac{1}{2}} \lambda^{-a^2/n} \gamma_a, \qquad (3)$$

where f is the function employed by Kiepert, defined in his article "Ueber Theilung und Transformation der Elliptischen Functionen," *Math. Ann.*, XXVI, p. 369, this f being connected with the τ employed by Klein by the relation

$$\tau^{n-1} = \Delta^{n-1} f^{24}, \qquad (4)$$

where Δ denotes the discriminant, so that

$$\Delta = g_2^3 - 27 g_3^2. \qquad (5)$$

We find also that

$$(-1)^a z_a = f^3 e^{-2(a/n)^2 \eta\omega} \sigma \frac{2a\omega}{n} = f^3 \tau\left(\frac{2a\omega}{n}\right) \qquad (6)$$

in Kiepert's notation (*Math. Ann.*, XXXIII, p. 7), a different τ function from that employed by Klein, but the reciprocal of Halphen's f function (*F. E.*, III, p. 216).

As for λ, it is a quantity defined by the relations

$$\lambda = \frac{\gamma_{\frac{1}{2}(n+1)}}{\gamma_{\frac{1}{2}(n-1)}}, \quad \lambda^3 = \frac{\gamma_{\frac{1}{2}(n+3)}}{\gamma_{\frac{1}{2}(n-3)}}, \quad \dots,$$

and generally
$$\lambda^{2p-1} = \frac{\gamma_{\frac{1}{2}(n+2p-1)}}{\gamma_{\frac{1}{2}(n-2p+1)}}, \tag{7}$$

with
$$p = 1, 2, 3, \dots \tfrac{1}{2}(n-1);$$

and now the relation
$$\gamma_n = 0 \tag{8}$$

is satisfied.

3. Klein's biquadratic relation (9), *M. F.*, II, p. 314, derived from the four-part theta-function formula (2), is now seen to be the equivalent of Halphen's formula (*Fonctions Elliptiques*, I, p. 102)

$$\gamma_{m+n}\gamma_{m-n} = \gamma_{m+1}\gamma_{m-1}\gamma_n^2 - \gamma_{n+1}\gamma_{n-1}\gamma_m^2, \tag{9}$$

or
$$\psi_{m+n}\psi_{m-n} = \psi_{m+1}\psi_{m-1}\psi_n^2 - \psi_{n+1}\psi_{n-1}\psi_m^2, \tag{10}$$

derivable at once from the defining relation

$$\wp nu - \wp mu = \frac{\psi_{m+n}\psi_{m-n}}{\psi_m^2\psi_n^2}$$

$$= x^4 \frac{\gamma_{m+n}\gamma_{m-n}}{\gamma_m^2\gamma_n^2}, \tag{11}$$

and the identity

$$\wp nu - \wp mu = (\wp u - \wp mu) - (\wp u - \wp nu), \tag{12}$$

or of the relation (Halphen, *F. E.*, I, p. 104)

$$\gamma_{m+n}\gamma_{m-n}\gamma_{p+q}\gamma_{p-q} + \gamma_{n+p}\gamma_{n-p}\gamma_{m+q}\gamma_{m-q} + \gamma_{p+m}\gamma_{p-m}\gamma_{n+q}\gamma_{n-q} = 0; \tag{13}$$

of which (9) is a particular case, obtained by putting

$$p = 1 \quad \text{and} \quad q = 0.$$

4. The relation (8) can be treated as the equation of a curve, connecting Halphen's x and y as coordinates; and when x and y can be expressed, rationally or irrationally, as functions of a parameter c, the quantities γ_a and z_a and the roots of the modular equation can be expressed as functions of this parameter.

Moreover, the separate division values, of which Klein and Kiepert's parameters are symmetric functions, can be disentangled from each other and written down.

[redacted]

5. The formulas in the *Modulfunctionen* by which relation (3) can be established are, in the first place, equation (8), *M. F.*, II, p. 282,

$$\frac{(-1)^{a+1}}{\sigma_{a,0}^2} = \wp'_{a,0} \prod_{\lambda=1}^{\frac12(n-1)} (\wp_{a,0} - \wp_{\lambda,0}), \tag{14}$$

and equation (1), p. 281,

$$(-1)^a z_a = \sqrt[8]{\frac{\Delta}{\Delta^n}}\, \sigma_{a/n,0}\left(\omega_1, \frac{\omega_2}{n}\right); \tag{15}$$

or, introducing Kiepert's f function, defined in the *Math. Ann.*, XXVI, p. 388, by

$$f^8 = \sqrt[8]{\frac{\Delta}{\Delta^n}} = \frac{(-1)^{\frac12(n-1)}}{\prod\limits_{\lambda=1}^{\frac12(n-1)} \wp' \dfrac{2a\varpi}{n}}; \tag{16}$$

then (108) may be written

$$(-1)^a z_a = f^8 \sigma_{a/n,0}\left(\omega_1, \frac{\omega_2}{n}\right). \tag{17}$$

Changing then in formula (14) to the transformed modulus,

$$(-1)^a z_a^n = \frac{f^{3n}}{\sqrt{S_a}\,(s_a - s_1)(s_a - s_2) \dots * \dots (s_a - s_{\frac12(n-1)})}, \tag{18}$$

where, with the notation of the article on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, p. 200,

$$\wp_{a,0} - \wp_{\lambda,0} = s_a - s_\lambda = -\frac{\psi_{a+\lambda}\psi_{a-\lambda}}{\psi_a^2 \psi_\lambda^2} = -x^{\frac12}\frac{\gamma_{a+\lambda}\gamma_{a-\lambda}}{\gamma_a^2\gamma_\lambda^2}, \tag{19}$$

$$\wp'_{a,0} = \sqrt{S_a} = -\frac{\psi_{2a}}{\psi_a^4} = -x\frac{\gamma_{2a}}{\gamma_a^4}. \tag{20}$$

Thence, from these relations,

$$(-1)^a z_a^n = \frac{\gamma_a^4 \cdot \gamma_a^2\gamma_1^2 \cdot \gamma_a^2\gamma_2^2 \cdots \gamma_a^2\gamma_{\frac12(n-1)}^2}{x^{\frac12 n}\gamma_{2a} \cdot \gamma_{a+1}\gamma_{a-1} \cdot \gamma_{a+2}\gamma_{a-2} \cdots \gamma_{\frac12(n-1)+a}\gamma_{\frac12(n-1)-a}}. \tag{21}$$

The numerator N of this expression (21) is given by

$$\begin{aligned} N &= \gamma_a^{n+1}(\gamma_1\gamma_2\gamma_3 \cdots \gamma_{a-1})^2 (\gamma_{a+1}\gamma_{a+2} \cdots \gamma_{\frac12(n-1)})^2 \\ &= \gamma_a^{n-1}(\gamma_1\gamma_2 \cdots \gamma_{\frac12(n-1)})^2, \end{aligned} \tag{22}$$

while the denominator D is given by

$$\begin{aligned} D = x^{\frac12 n}&\gamma_{2a}\gamma_{a-1}\gamma_{a-2}\gamma_{a-3} \cdots \gamma_3\gamma_2\gamma_1 \cdot \gamma_1\gamma_2\gamma_3 \cdots \gamma_{a-1}\gamma_a\gamma_{a+1} \\ &\cdots \gamma_{\frac12(n-1)-a} \cdot \gamma_{a+1}\gamma_{a+2} \cdots \gamma_{2a-3}\gamma_{2a-2}\gamma_{2a-1}\gamma_{2a+1}\gamma_{2a+2} \cdots \gamma_{\frac12(n-1)+a} \\ &\gamma_{\frac12(n-1)-a+1}\gamma_{\frac12(n-1)-a+2} \cdots \gamma_{\frac12(n-1)+a-1}\gamma_{\frac12(n-1)+a} \\ = x^{\frac12 n}&(\gamma_1\gamma_2\gamma_3 \cdots \gamma_{a-1})^2 \gamma_a (\gamma_{a+1}\gamma_{a+2} \cdots \gamma_{\frac12(n-1)-a})^2, \end{aligned} \tag{23}$$

and this, from relations (7) and (8), reduces to

$$D = x^{\frac{1}{2}n}(\gamma_1\gamma_2 \dots \gamma_{a-1})^2 \gamma_a (\gamma_{a+1}\gamma_{a+2} \dots \gamma_{\frac{1}{2}(n-1)-a})^2$$

$$\lambda^{a^2}(\gamma_{\frac{1}{2}(n-1)-a+1}\gamma_{\frac{1}{2}(n-1)-a+2} \dots \gamma_{\frac{1}{2}(n-1)})^2$$

$$= x^{\frac{1}{2}n}\lambda^{a^2}\gamma_a^{-1}(\gamma_1\gamma_2 \dots \gamma_{\frac{1}{2}(n-1)})^2; \tag{24}$$

so that, finally,

$$(-1)^a z_a^n = \frac{f^{3n}\gamma_a^n}{x^{\frac{1}{2}n}\lambda^{a^2}}, \tag{25}$$

as in equation (3); and thence Klein's function z_a can be expressed in terms of a single parameter c, when x and y, and therefore γ_a and λ, are given as functions of the parameter c, satisfying Halphen's relation

$$\gamma_n = 0. \tag{26}$$

6. Using the notation σ_a and τ_a to denote $\sigma\left(\frac{2a\omega}{n}\right)$ and $\tau\left(\frac{2a\omega}{n}\right)$, and putting (Halphen, *F. E.*, I, pp. 102 and 198),

$$\gamma_a = \psi_a x^{-\frac{1}{2}(a^2-1)}, \tag{27}$$

$$\psi_a = \frac{\sigma_a}{(\sigma_1)^{a^2}} = \frac{\tau_a}{(\tau_1)^{a^2}}; \tag{28}$$

then (7) may be written

$$\lambda^{2p-1} = \frac{\psi_{\frac{1}{2}(n+2p-1)}}{\psi_{\frac{1}{2}(n-2p+1)}} x^{-\frac{1}{2}n(2p-1)} \tag{29}$$

$$= \frac{\sigma(n+2p-1)\frac{\omega}{n}}{\sigma(n-2p+1)\frac{\omega}{n}} \sigma_1^{-n(2p-1)} x^{-\frac{1}{2}n(2p-1)}$$

$$= e^{(2p-1)(2\eta\omega)/n} \sigma_1^{-(2p-1)n} x^{-\frac{1}{2}(2p-1)n}, \tag{30}$$

in consequence of equation (9), *F. E.*, I, p. 170; so that

$$\lambda = e^{(2\eta\omega)/n}\sigma_1^{-n}x^{-\frac{1}{2}n}, \tag{31}$$

or, in Kiepert's notation,

$$\lambda = \left[\frac{1}{x^{\frac{1}{2}}\tau\left(\frac{2\omega}{n}\right)}\right]^n. \tag{32}$$

Otherwise, writing v for $\dfrac{2\omega}{n}$,

$$\frac{\sigma(av)}{(\sigma v)^{a^2}} = \psi_a v = x^{\frac{1}{2}(a^2-1)}\gamma_a, \tag{33}$$

$$\frac{\sigma(n-a)v}{(\sigma v)^{(n-a)^2}} = x^{\frac{1}{2}(n-a)^2-\frac{1}{2}}\gamma_{n-a}, \tag{34}$$

and
$$\sigma(n-a)v = \sigma(2\omega-av)$$
$$= e^{2\eta(\omega-av)}\sigma(av), \tag{35}$$

so that
$$e^{2\eta(\omega-av)} = \frac{\sigma(n-a)v}{nav}$$
$$= (\sigma v)^{(n-a)^2-a^2}x^{\frac{1}{2}(n-a)^2+\frac{1}{2}a^2}\frac{\gamma_{n-a}}{\gamma_a}, \tag{36}$$

or
$$e^{(2\eta\omega)/n(n-2a)} = (\sigma v)^{n(n-2a)}x^{\frac{1}{2}n(n-2a)}\lambda^{n-2a}, \tag{37}$$

$$e^{-(2\eta\omega)/n}(\sigma v)^n = x^{-\frac{1}{2}n}\lambda^{-1}, \tag{38}$$

$$e^{-(2\eta\omega)/n^2}\sigma\frac{2\omega}{n} = x^{-\frac{1}{2}}\lambda^{-1/n}; \tag{39}$$

and Kiepert's function

$$\tau\frac{2a\omega}{n} = e^{-2(a/n)^2\eta\omega}\sigma\frac{2a\omega}{n} = (e^{-(2\eta\omega)/n^2}\sigma v)^{a^2}\psi_a = x^{-\frac{1}{2}a^2}\lambda^{-a^2/n}x^{\frac{1}{2}(a^2-1)}\gamma_a$$
$$= x^{-\frac{1}{2}}\lambda^{-a^2/n}\gamma_a. \tag{40}$$

7. A number of cases of this relation (8) or (26), for the simplest values of n, have been worked out in my paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, and by means of these results the roots of the modular equations given by Prof. Klein in *Proc. Lond. Math. Soc.*, Vol. IX, p. 123, can now be expressed in terms of a single parameter.

In his paper "Elliptische Functionen und Gleichungen fünften Grades," *Math. Ann.*, XIV, Professor Klein continues his investigations, and expresses the roots of the modular equations of order $n = 2, 3, 4$, and 5, in terms of a single parameter.

The degree of the modular equation of prime order n being $n+1$, and the roots being denoted by

$$\tau_\infty,\ \tau_0,\ \tau_1,\ \tau_2,\ \ldots\ \tau_{n-1},$$

then for $n = 2$ (*Math. Ann.*, XIV, p. 153), comparing

$$J = \frac{(4\tau - 1)^3}{27\tau}, \tag{41}$$

and

$$J = \frac{4(1-\sigma+\sigma^2)^3}{27\sigma^2(1-\sigma)^2}, \tag{42}$$

the roots, τ_∞, τ_0, τ_1, for a given value of J, are expressed by Klein in terms of σ, the *anharmonic ratio* (*Doppelverhältniss*), or squared Legendrian modulus κ^2, by

$$\left.\begin{aligned} \tau_\infty &= -\frac{(1-\sigma)^2}{4\sigma} \\ \tau_0 &= -\frac{\sigma^2}{4(1-\sigma)} \\ \tau_1 &= \frac{1}{4\sigma(1-\sigma)} \end{aligned}\right\}. \tag{43}$$

For $n = 3$, Klein expresses the roots of the modular equation in terms of $x_1 : x_2$, which ratio he calls the *tetrahedron irrationality*, by means of the relation

$$J = \frac{(\tau-1)(9\tau-1)^3}{-64\tau} = -64\frac{(x_1^4 - x_1 x_2^3)^3}{(8x_1^3 x_2 + x_2^4)^3}, \tag{44}$$

and he finds

$$\tau_\infty = \frac{x_2^4}{8x_1^3 x_2 + x_2^4},$$

$$\tau_r = \frac{1}{9}\,\frac{(2\epsilon^r x_1 + x_2)^4}{8x_1^3 x_2 + x_2^4}, \tag{45}$$

where

$$\epsilon = e^{\frac{2}{3}\pi i}, \quad r = 0, 1, 2. \tag{46}$$

For $n = 4$, the modular equation

$$J = \frac{(\tau^2 + 14\tau + 1)^3}{108\tau(\tau-1)^4} = \frac{(\eta^8 + 14\eta^4 + 1)^3}{108\eta^4(\eta^4-1)^4} \tag{47}$$

gives the six values of τ as

$$\eta^4, \quad \frac{1}{\eta^4}, \quad \left(\frac{1\pm\eta}{1\mp\eta}\right)^4, \quad \left(\frac{1\pm\eta i}{1\mp\eta i}\right)^4, \tag{48}$$

in terms of η, called the *octahedron irrationality;* and Legendre's modulus

$$\kappa = \eta^2, \quad \tau = \kappa^2 = \eta^4. \tag{49}$$

8. The next important extension is to the case of

$$n = 5,$$

where
$$J = \frac{(r^2-10r+5)^3}{-1728r}, \tag{50}$$

and Klein expresses the six roots of this equation in r in terms of the ratio
$$\eta = \eta_1 : \eta_2, \tag{51}$$

which he calls the *ikosahedron irrationality*, by means of the *ikosahedron form f*, defined by
$$f = \eta_1\eta_2(\eta_1^{10}+11\eta_1^5\eta_2^5-\eta_2^{10}), \tag{52}$$

and the relations
$$\tau_\infty = \frac{125\eta_1^6\eta_2^6}{f}, \tag{53}$$

$$\tau_\rho = \frac{(\epsilon^{-\rho}\eta_1^2+\eta_1\eta_2-\epsilon^\rho\eta_2^2)^6}{f}, \tag{54}$$

where
$$\epsilon = e^{\frac{2}{5}\pi i}, \quad \rho = 0, 1, 2, 3, 4. \tag{55}$$

So also for the roots of the quintic resolvent

$$J : J-1 : 1 = (r-3)^3(r^2-11r+64) : r(r^2-10r+45)^2 : -1728, \tag{56}$$

or, putting $r = x^2$, in Brioschi's form,

$$x^5-10x^3+45x+\frac{216g_3}{\surd(-\Delta)} = 0. \tag{57}$$

The values $r = 3, 11, 19$ make $K'/K = \surd 3, \surd(11), \surd(19)$; and lead to interesting numerical results; thus, when $r = 19$,

$$\eta - \frac{1}{\eta} = \tfrac{1}{2}\left[-1+i\surd(19)\right] = -\tau_\infty^{\frac{1}{5}}.$$

(*Lectures on the Ikosahedron*, p. 60, by Felix Klein, translated by G. G. Morrice, 1888; *Math. Ann.*, XIV, p. 417; *Modulfunctionen*, I, p. 649.)

9. Professor Klein passes on to the solution of the modular equation of order

$$n = 7,$$

in the *Math. Ann.*, XIV, p. 428, and XV, p. 251; also in the *Modulfunctionen*, I, p. 692.

The modular equation (*Proc. Lond. Math. Soc.*, Vol. IX, p. 124) is

$$J = \frac{(\tau^2+13\tau+49)(\tau^2+5\tau+1)^3}{1728\tau}, \tag{58}$$

and Klein shows how the eight roots of this equation may be given in terms of three parameters

$$\lambda, \ \mu, \ \nu, \quad \text{or} \quad z_1, \ z_2, \ z_4,$$

connected by the biquadratic relation

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda = 0, \tag{59}$$

or

$$z_1^3 z_2 + z_2^3 z_4 + z_4^3 z_1 = 0. \tag{60}$$

In the *Modulfunctionen*, I, p. 701, we must take

$$\lambda, \ \mu, \ \nu = z_4, \ z_2, \ z_1; \tag{61}$$

and the biquadratic relation is

$$z_4^3 z_2 + z_2^3 z_1 + z_1^3 z_4 = 0. \tag{62}$$

10. But it was shown in the *Proc. Lond. Math. Soc.*, Vol. XXV, p. 223, that these quantities can be expressed in terms of a single parameter z, there defined, by means of the relations

$$\left.\begin{aligned} \lambda &= z_1 = -z^{\frac{4}{7}}(z-1)^{\frac{2}{7}} \\ \mu &= z_2 = -z^{\frac{2}{7}}(z-1)^{\frac{1}{7}} \\ \nu &= z_4 = \quad z^{\frac{1}{7}}(z-1)^{\frac{4}{7}} \end{aligned}\right\}, \tag{63}$$

satisfying the relations (59) and (60) above.

Halphen's x and y are given in terms of the parameter z by the relations

$$x = z(1-z)^2, \quad y = z(1-z), \tag{64}$$

and thus the relation

$$\gamma_7 = (y-x)\,x - y^3 = 0 \tag{65}$$

is satisfied; also

$$z = \frac{\wp\frac{6}{7}\omega - \wp\frac{4}{7}\omega}{\wp\frac{6}{7}\omega - \wp\frac{2}{7}\omega}. \tag{66}$$

The roots of the modular equation (58) are then shown to be expressible in terms of z by the relations

$$\begin{aligned} \tau_\infty &= -\frac{49z(z-1)}{z^3+5z^2-8z+1} \\ &= -\frac{49}{z+5+\dfrac{1}{1-z}+\dfrac{z-1}{z}}, \end{aligned} \tag{67}$$

$$\tau_r = -\frac{(1+\epsilon^{-r}a_1+\epsilon^{-2r}a_2+\epsilon^{-4r}a_4)^4}{z+5+\frac{1}{1-z}+\frac{z-1}{z}}, \tag{68}$$

where $$\epsilon = e^{\frac{2}{7}\pi i}, \quad r = 0, 1, 2, 3, 4, 5, 6, \tag{69}$$

and $$a_1 = \frac{z_1}{z_4} = -z^{\frac{3}{7}}(z-1)^{-\frac{2}{7}}, \tag{70}$$

$$a_2 = \frac{z_4}{z_2} = -z^{-\frac{1}{7}}(z-1)^{\frac{3}{7}}, \tag{71}$$

$$a_4 = \frac{z_2}{z_1} = z^{-\frac{2}{7}}(z-1)^{-\frac{1}{7}}. \tag{72}$$

Thus the irrationality

$$z^{\frac{2}{7}}(z-1)^{\frac{1}{7}} = \lambda^{\frac{3}{7}}, \tag{73}$$

where $$\lambda = \frac{\gamma_4}{\gamma_8}, \quad \lambda^3 = \frac{\gamma_8}{\gamma_2} = y-x = z^2(1-z), \tag{74}$$

plays the same part here as the ikosahedron irrationality η in equations (53) and (54).

Also
$$\left.\begin{aligned}
\frac{a_2}{a_1^2} &= -\frac{z-1}{z} = \frac{\wp\frac{4}{7}\omega-\wp\frac{2}{7}\omega}{\wp\frac{8}{7}\omega-\wp\frac{4}{7}\omega}\\
\frac{a_4}{a_2^2} &= \frac{1}{z-1} = -\frac{\wp\frac{8}{7}\omega-\wp\frac{2}{7}\omega}{\wp\frac{4}{7}\omega-\omega\frac{2}{7}\omega}\\
\frac{a_1}{a_4^2} &= -z = -\frac{\wp\frac{8}{7}\omega-\wp\frac{4}{7}\omega}{\wp\frac{8}{7}\omega-\wp\frac{2}{7}\omega}
\end{aligned}\right\}. \tag{75}$$

This parameter z is seen to be intimately bound up with Gierster's *Hauptmodul* M ("Ueber Classenanzahl-relationen," *Math. Ann.*, XVII, p. 81); for

$$M = \frac{\lambda^3\mu}{\nu^8} = -\frac{z}{z-1}, \tag{76}$$

and, as the values of τ are unaltered by the group of substitutions

$$z, \quad \frac{1}{1-z}, \quad \frac{z-1}{z}, \tag{77}$$

Gierster's M is equivalent to minus the reciprocal of our z.

Put
$$z = -x - \frac{1}{x};$$

then
$$\tau = \frac{1-8z+5z^2+z^3}{z\,(1-z)}$$

$$= \frac{x^6-5x^5-5x^4-11x^3-5x^2-5x+1}{x\,(x^2+1)(x^2+x+1)}, \tag{78}$$

$$\tau+6 = \frac{x^6+x^5+x^4+x^3+x^2+x+1}{x\,(x^2+1)(x^2+x+1)}$$

$$= \frac{x^7-1}{x\,(x^2+1)(x^3-1)}. \tag{79}$$

11. A straightforward algebraical verification of equation (58) by the roots given by (67) and (68) would be very formidable; but meanwhile Mr. T. I. Dewar has performed the verification for the special numerical case corresponding to

$$z = 2,$$

and he finds that this makes

$$\tau_\infty = -\frac{98}{13},$$

and
$$1728J = -8478{\cdot}4438756456985. \tag{80}$$

Also, considering only the real seventh roots,

$$\left.\begin{aligned} \alpha_1 &= -2^{\frac{3}{7}} &&= -1{\cdot}34590019263234 \\ \alpha_2 &= -2^{-\frac{1}{7}} &&= -0{\cdot}90572366426391 \\ \alpha_4 &= 2^{-\frac{2}{7}} &&= +0{\cdot}82033535600764 \end{aligned}\right\}, \tag{81}$$

and these make
$$\tau_0 = -0{\cdot}0053230207 54, \tag{82}$$

and then
$$1728J = -8478{\cdot}443876. \tag{83}$$

A similar numerical calculation makes

$$\tau_1 = -3{\cdot}555254192736 - 2{\cdot}712985482813i,$$

and
$$1728J = \frac{30143{\cdot}023137291471 + 23001{\cdot}895151881873i}{\tau_1}$$

$$= -8478{\cdot}44387579 - 0{\cdot}000000001688045i, \tag{84}$$

and these three values of J are sufficiently close to serve as an arithmetical verification.

Another numerical case can be constructed by taking

$$z = -2\cos\tfrac{2}{7}\pi, \quad \tau = -6.$$

So also we may verify that the roots of Galois' resolvent of the seventh degree,

$$x^7 + \frac{7+7i\sqrt{7}}{2}\Delta x^4 - \frac{35-7i\sqrt{7}}{2}\Delta^2 x - 12g_2\Delta^2 = 0, \tag{85}$$

are given by

$$x_r = \epsilon^r z_1^2 + \epsilon^{2r} z_2^2 + \epsilon^{4r} z_4^2 - \frac{1+i\sqrt{7}}{2}\left(\epsilon^{-r} z_1 z_4 + \epsilon^{-2r} z_2 z_1 + \epsilon^{-4r} z_4 z_2\right),$$

$$r = 1, 2, 3, 4, 5, 6, 7 \tag{86}$$

(Klein, *Math. Ann.*, XIV, pp. 426, 458; *M. F.*, I, p. 754).

12. The highest prime number for which Klein's modular equation is rational in τ is

$$n = 13;$$

and now (*Proc. Lond. Math. Soc.*, Vol. IX, p. 126)

$$J = \frac{(\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3}{1728\tau}, \tag{87}$$

and J', the transformed absolute invariant, is the same function of τ', where

$$\tau\tau' = 13. \tag{88}$$

It was shown, in the article "Pseudo-Elliptic Integrals," §50, that the relation

$$\gamma_{13} = 0 \tag{89}$$

can be satisfied when x and y are rational functions of a parameter c and of $\sqrt{C}$, where

$$\begin{aligned} C &= 1+4c+6c^2+2c^3+c^4+2c^5+c^6 \\ &= (1+2c-c^2-c^3)^2+4c^2(1+c)^2, \end{aligned} \tag{90}$$

so that the curve of equation (89) has a deficiency 2.

13. Then γ_a can also be expressed as a rational function of c and $\sqrt{C}$, and now we are able, from relation (3) and from the relation

$$\frac{A_a}{A_0} = \frac{z_{2a}}{z_a} \tag{91}$$

(Klein, "Ueber gewisse Theilwerthe der Θ-Function," *Math. Ann.*, XVII, p. 569), to express the fourteen roots of equation (87), the modular equation of the thirteenth order, in terms of c and $\sqrt{C}$, in the form

$$\tau_\infty = -\frac{13}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}}, \tag{92}$$

$$\tau_r = -\frac{\left(1+\sum\limits_{a=1}^{a=6}\dfrac{A_a}{A_0}\,\epsilon^{36a^2r}\right)^2}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}}, \tag{93}$$

where
$$\epsilon = e^{\frac{2}{13}\pi i}, \quad r = 0,\ 1,\ 2,\ \ldots\ 12, \tag{94}$$

the A's being expressions such that A^{13} is a rational function of c and $\sqrt{C}$, and A being taken as the real thirteenth root, the various imaginary roots being obtained by appropriate factors of powers of $e^{\frac{2}{13}\pi i}$ (*Proc. Lond. Math. Soc.*, Vol. xxv, p. 256).

14. In the general case, from (3) and (91),

$$\frac{A_a}{A_0} = (-1)^a\lambda^{-(3a^2)/n}\frac{\gamma_{2a}}{\gamma_a} = \mu^{a^2}\frac{\gamma_{2a}}{\gamma_a}, \tag{95}$$

on putting
$$\lambda^{-3} = -\mu^n, \tag{96}$$

or
$$\left(\frac{A_a}{A_0}\right)^n = (-1)^a\lambda^{-3a^2}\left(\frac{\gamma_{2a}}{\gamma_a}\right)^n. \tag{97}$$

15. In the special case of $n = 13$ it was shown (*Proc. Lond. Math. Soc.*, Vol. xxv, pp. 252–255) that the relation (89) is reduced to

$$p^2-(1-c^2-c^3)\,p-c\,(1+c)^2 = 0, \tag{98}$$

by means of the substitutions

$$x = y\,(1-z), \quad y = z-\frac{z^2}{p}, \quad z = c\,(p-1), \tag{99}$$

and then
$$\tau = \frac{1-c-4c^2-c^3}{c\,(1+c)} = -4-c+\frac{1}{1+c}+\frac{1+c}{c}, \tag{100}$$

also
$$\left.\begin{aligned} \lambda &= \frac{\gamma_7}{\gamma_6} = -\frac{yz}{cx^4} \\ \lambda^3 &= \frac{\gamma_8}{\gamma_5} = -y^2z^3\,\frac{1+c}{cp} \end{aligned}\right\}, \tag{101}$$

and
$$p = \frac{\wp 2v-\wp v}{\wp 2v-\wp 5v}, \quad c = \frac{\wp 5v-\wp v}{\wp 3v-\wp v}\,\frac{\wp 3v-\wp 2v}{\wp 2v-\wp 5v}, \quad v = \frac{2\omega}{13}. \tag{102}$$

Denoting $\frac{A_a}{A_0}$ by a_a, then we find

$$a_1^{13} = -\frac{1}{\lambda^8} = \frac{P+Q\sqrt{C}}{2c^7(1+c)^5},$$

$$P = 1+4+11+29+70+86+69+84+100+68+35+27+19+7+c^{14},$$

$$Q = 1+4+12+10+14+32+29+14+13+13+6+c^{11},$$

$$P^2-Q^2C = -4c(1+c)^{10};$$

$$a_2^{13} = \frac{y^{13}}{\lambda^{13}} = \frac{R+S\sqrt{C}}{2c^3(1+c)^7},$$

$$R = 1+7+19+19-9-26+7+26-8-17+8+6-4-1+c^{14},$$

$$S = 1+5+8-1-12-1+12-1-7+2+2-c^{11},$$

$$R^2-S^2C = 4c^{17}(1+c);$$

R and S being obtained from P and Q by writing $-\frac{1+c}{c}$ for c;

$$a_3^{13} = -\frac{1}{\lambda^{27}}\left(\frac{yz^2}{p}\right)^{13} = \frac{R-S\sqrt{C}}{2c^3(1+c)^7},$$

$$a_4^{13} = \frac{z^{13}}{\lambda^9} = -\frac{T+U\sqrt{C}}{2c^5(1+c)^2},$$

$$T = 1+15+100+388+965+1604+1825+1482+960+581+334+155+50+10+c^{14},$$

$$U = 1+13+73+230+443+537+416+216+83+27+7+c^{11},$$

$$T^2-U^2C = 4c^{10}(1+c)^{17};$$

obtained from P and Q by writing $-\frac{1}{1+c}$ for c;

$$a_5^{13} = \frac{\lambda^8}{c^{13}} = \frac{P-Q\sqrt{C}}{2c^7(1+c)^5},$$

$$a_6^{13} = \lambda^9\left(\frac{1+c}{z}\right)^{13} = -\frac{T-U\sqrt{C}}{2c^5(1+c)^2}. \qquad (103)$$

Thence $$a_1a_5 = -\frac{1}{c}, \quad a_2a_3 = \frac{c}{1+c}, \quad a_4a_6 = 1+c; \qquad (104)$$

so that a change of $\sqrt{C}$ into $-\sqrt{C}$ interchanges a_1 and a_5, a_2 and a_3, a_4 and a_6, but leaves r unaltered.

So also it will be found that, while r is unaltered,

(i.) the change of c into $-\dfrac{1}{1+c}$ changes

a_1	into	a_4,
a_2	,,	a_3,
a_3	,,	a_1,
a_4	,,	a_2,
a_5	,,	a_6,
a_6	,,	a_5;

(ii.) the change of c into $-\dfrac{1+c}{c}$ changes

a_1	into	a_3,
a_2	,,	a_6,
a_3	,,	a_4,
a_4	,,	a_1,
a_5	,,	a_2,
a_6	,,	a_5.

16. Various other relations connecting the a's can be written down, thus:—

$$a_1^3 a_6 = -\frac{1+c}{z},$$

with similar expressions for

$$a_6^3 a_3, \quad a_3^3 a_5, \quad a_5^3 a_4, \quad a_4^3 a_2, \quad a_2^3 a_1; \tag{105}$$

$$a\ \ a_3 = -\frac{yz^2}{\lambda^3 p} = \frac{c}{y(1+c)},$$

with similar expressions for $a_3^4 a_4$ and $a_4^4 a_1$; (106)

$$a_2^4 a_6 = \frac{y^4 (1+c)}{\lambda^8 z},$$

with similar expressions for $a_6^4 a_5$ and $a_5^4 a_2$; (107)

$$a_1^3 a_3 = \frac{1}{\lambda^9}\left(\frac{yz^2}{p}\right)^4,$$

with similar expressions for $a_3^3 a_4^4$ and $a_4^3 a_1$; (108)

$$a_2^3 a_6^3 = \frac{y^3 (1+c)^4}{z^4},$$

with similar expressions for $a_6^3 a_5^4$ and $a_5^3 a_2^4$; (109)

$$\frac{a_1^5 a_2}{a_3} = \frac{p}{z^3}$$

with similar expressions for $\dfrac{a_2^5 a_4}{a_6}, \dfrac{a_3^5 a_6}{a_4}, \dfrac{a_4^5 a_5}{a_1}, \dfrac{a_5^5 a_3}{a_2}, \dfrac{a_6^5 a_1}{a_5}$; (110)

and so on.

These relations are important in showing that the irrationality of the thirteenth root may be taken once for all.

17. A direct algebraical verification of equation (87) by the roots given in (92) and (93) would be a task still more formidable than that required for the corresponding case of $n = 7$; but here again Mr. T. I. Dewar has performed the numerical verification for the special case obtained by taking in (90)

$$1+2c-c^2-c^3 = 0, \tag{111}$$

thus making

$$\sqrt{C} = 2c+2c^2. \tag{112}$$

Taking the positive value of $\sqrt{C}$, this makes

$$\begin{aligned} p &= c^3, \\ z &= -c\,(1-c^2) = \frac{c^3}{1+c}, \\ y &= \frac{c^3}{(1+c)^2}, \\ \lambda^8 &= \frac{c^7}{(1+c)^5}, \end{aligned} \tag{113}$$

and

$$\left.\begin{aligned} a_1^{13} &= \frac{(1+c)^5}{c^7} \\ a_2^{13} &= \frac{c^{11}}{(1+c)^6} \\ a_3^{13} &= \frac{c^3}{(1+c)^7} \\ a_4^{13} &= -c^5\,(1+c)^2 \\ a_5^{13} &= -\frac{1}{c^6\,(1+c)^5} \\ a_6^{13} &= -\frac{(1+c)^{11}}{c^5} \end{aligned}\right\}; \tag{114}$$

while the negative value of $\sqrt{C}$ would merely permute

$$a_1 \text{ and } a_5, \quad a_2 \text{ and } a_3, \quad a_4 \text{ and } a_6.$$

Writing equation (100) in the form

$$\tau = \frac{1+2c-c^2-c^3}{c(1+c)} - 3, \tag{115}$$

then the roots of (111) make

$$\tau = -3, \quad \tau_\infty = -\frac{13}{3}, \tag{116}$$

and this value of τ_∞, substituted in (87), makes

$$\begin{aligned} 1728 J_\infty &= -\frac{2^{12} \times 7 \times 17^3 \times 23^3}{3^{13}} \\ &= -1075008 \cdot 6252986. \end{aligned} \tag{117}$$

The three roots of (111) are

$$2 \cos \frac{2\pi}{7}, \quad 2 \cos \frac{4\pi}{7}, \quad 2 \cos \frac{6\pi}{7}; \tag{118}$$

and we can thus put

$$\begin{aligned} c &= 2 \cos \frac{2\pi}{7} \\ &= 1 \cdot 246979603717467, \end{aligned}$$

$$\left. \begin{aligned} -\frac{1}{1+c} &= 2 \cos \frac{4\pi}{7} \\ -\frac{1+c}{c} &= 2 \cos \frac{6\pi}{7} \end{aligned} \right\}. \tag{119}$$

Mr. Dewar now calculates

$$\left. \begin{aligned} a_1 &= 1 \cdot 212310485995257 \\ a_2 &= 0 \cdot 829535995876051 \\ a_3 &= 0 \cdot 668998253055627 \\ -a_4 &= 1 \cdot 232994543430555 \\ -a_5 &= 0 \cdot 661495338916045 \\ -a_6 &= 1 \cdot 822375951044485 \end{aligned} \right\}, \tag{120}$$

[redacted]

and thence
$$\tau_0 = -\tfrac{1}{3}(1+a_1+a_2+a_3+a_4+a_5+a_6)^2$$
$$= -\tfrac{1}{3}(-0{\cdot}006021232698252)^2$$
$$= -0{\cdot}000012084542226; \qquad (121)$$

$$1728J_0 = \frac{13}{\tau_0} + 746 + 15145\tau_0 + 124852\tau_0^2 + \dots$$
$$= -1075008{\cdot}6257122, \qquad (122)$$

a close agreement with the result of (117).

18. Mr. Dewar has gone on further to the calculation of the imaginary root τ_1, and the corresponding value of J; and he finds in a similar manner, from (93),

$$\tau_r = -\tfrac{1}{3}(1+e^{36r}a_1+e^{2^2\cdot 36r}a_2+\dots+e^{6^2\cdot 36r}a_6)^2,$$
$$\tau_1 = -\tfrac{1}{3}(1+e^{10}a_1+ea_2+e^{12}a_3+e^{4}a_4+e^{3}a_5+e^{9}a_6)^2$$
$$= -3{\cdot}521253401250 + 2{\cdot}86124793802i, \qquad (123)$$

$$\tau_1^2 = +4{\cdot}212485752989 - 20{\cdot}150358067145i, \qquad (124)$$

$$\tau_1^3 = +42{\cdot}821940684550 + 83{\cdot}007483054998i, \qquad (125)$$

$$\tau_1^4 = -388{\cdot}291894015211 - 169{\cdot}766192550990i, \qquad (126)$$

$$\tau_1^2 + 5\tau_1 + 13 = -0{\cdot}3943781253261 - 5{\cdot}844118377045i, \qquad (127)$$

$$\tau_1^4 + 7\tau_1^3 + 20\tau_1^2 + 19\tau_1 + 1$$
$$= -70{\cdot}1902408787331 + 62{\cdot}6427383134 76i, \qquad (128)$$

and substituting these values in (87),

$$1728J_1 = \frac{3785377{\cdot}7728290 - 3075866{\cdot}2102691i}{-3{\cdot}52125340125 + 2{\cdot}86124793802i}$$
$$= -1075008{\cdot}6257 - 0{\cdot}00036754i, \qquad (129)$$

a close agreement with (117) and (122).

Mr. Dewar employed in these calculations the new multiplying machine invented by Mr. Macfarlane Gray, which is capable of multiplying together two numbers of sixteen figures.

19. The special case of $n = 9$

may be considered at this stage; this is the case receiving particular attention in Joubert's memoir "Sur les équations qui se rencontrent dans la théorie de la transformation des fonctions elliptiques," Paris, 1876.

The relation $$\gamma_9 = 0,$$

or $$y^3(y-x-y^2)-(y-x)^3 = 0, \tag{130}$$

is satisfied by putting

$$x = p^2(1-p)(1-p+p^2), \quad y = p^2(1-p), \tag{131}$$

so that the curve (130) is unicursal; and now

$$\lambda = \frac{\gamma_5}{\gamma_4} = 1-\frac{x}{y} = p-p^2. \tag{132}$$

With these values,
$$\left.\begin{aligned} a_1^9 &= -\frac{1}{p^3(1-p)^3} \\ a_2^9 &= \frac{p^6}{(1-p)^3} \\ a_3^9 &= -1 \\ a_4^9 &= \frac{(1-p)^6}{p^3} \end{aligned}\right\}, \tag{133}$$

and $$a_1 a_2 a_3 a_4 = 1. \tag{134}$$

Changing p into $\dfrac{1}{1-p}$ changes

$$a_1 \text{ into } a_4, \quad a_4 \text{ into } a_2, \quad a_2 \text{ into } a_1;$$

and changing p into $-\dfrac{1-p}{p}$ changes

$$a_1 \text{ into } a_2, \quad a_2 \text{ into } a_4, \quad a_4 \text{ into } a_1;$$

also $$a_1+a_2+a_4 = 0. \tag{135}$$

As stated previously, we need only consider the real ninth roots of a^9.

[redacted]

20. It was shown (*Proc. Lond. Math. Soc.*, Vol. xxv, p. 233) that

$$-3\tau = \frac{1-6p+3p^2+p^3}{p(1-p)}; \qquad (136)$$

and therefore, to the complementary modulus,

$$\tau_\infty = \frac{3}{\tau} = \frac{-9p(1-p)}{1-6p+3p^2+p^3}, \qquad (137)$$

and now all the twelve roots of Gierster's modular equation

$$J = \frac{(\tau-1)^3\{9(\tau-1)^3+8\}^3}{-64\tau\{(\tau-1)^3+1\}} \qquad (138)$$

can be expressed in terms of the parameter p.

The τ employed here, distinguished as τ_9, is connected with the τ or τ_3 employed in the modular equation of the third order by the relation

$$(\tau_9-1)^3 = \tau_3-1,$$

or

$$\tau_9 = 1+\omega(\tau_3-1)^{\frac{1}{3}}, \quad \omega^3 = 1, \qquad (139)$$

and thence the twelve values of τ_9 can be inferred from the four values of τ_3.

If a denotes the tetrahedron irrationality $x_1 : x_2$ in § 7,

$$\tau_3 = \frac{1}{8a^3+1}, \quad \tau_3' = 8a^3+1,$$

$$\tau_9'-1 = (\tau_3'-1)^{\frac{1}{3}} = 2a, \quad \tau_9' = 2a+1. \qquad (140)$$

For instance, in the Transformation of the Nineteenth Order (Fricke, *Math. Ann.*, XL)

$$19\tau = -x, \quad 19\tau' = x',$$

$$x'^2 = 4x^3+(8x+19)^2,$$

$$\tau_{3,\infty} = \frac{1}{513}, \quad a = 4, \quad \tau_{3,\infty}' = 513;$$

$$\tau_{3,0} = \frac{27}{19}, \quad \tau_{3,0}' = \frac{19}{27},$$

$$\tau_{9,0}' = 1+\left(-\frac{8}{27}\right)^{\frac{1}{3}} = \tfrac{1}{3}, \; a = -\tfrac{1}{3}, \; \tau_{9,0} = 9.$$

With $$\sqrt[3]{J} = -\frac{2^4 . 7}{3 . 19},$$

$$a = 4, \quad \text{or} \quad -\tfrac{1}{3},$$

or $$19a^2 - 5a + 7 = 0, \quad a = \frac{5 + 13i\sqrt{3}}{38};$$

and $$r_9 = 9, \quad r_9' = \tfrac{1}{3}$$

give the same value of J, and make

$$M = 1, \quad M' = \tfrac{1}{9},$$

and thus correspond to a multiplication by 3.

So also, for $n = 27$, and $n = 81, 243, \ldots$ (*Math. Ann.*, XXXII, p. 67),

$$\eta - 3 = \xi_3 = \frac{1 - 6p + 3p + p^3}{p(1-p)}, \quad \xi_1 = \frac{p^3(1-p)^3}{1 - p + p^2},$$

$$\xi_5 = \frac{1 - 6p + 3p^2 + p^3}{(-p+p^2)(p-p^2)^{\frac{1}{2}}},$$

$$\xi_2 = \frac{1 + 3p - 6p^2 + p^3 - 3(1 - p + p^2)(p - p^2)^{\frac{1}{2}}}{1 - 6p + 3p^2 + p^3},$$

$$\xi_4 = \frac{(1 - p + p^2)(p - p^2)^{\frac{1}{2}}}{1 + 3p - 6p^2 + p^3 - 3(1 - p + p^2)(p - p^2)^{\frac{1}{2}}},$$

$$w = \frac{(1+p)(2-p)(1-2p)(p-p^2)^{\frac{1}{2}}}{(1 - p + p^2)^2}.$$

21. The expression of x, y, and γ_s as functions of a single parameter has been given in the paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. XXV, for the odd numbers

$$3, \ 5, \ 7, \ 9, \ 11, \ 13, \text{ and } 15.$$

As a verification, let us examine again the three simplest cases of

$$n = 3, \ 5, \text{ and } 7.$$

For $n = 3$, the single z function is Dedekind's $\eta(\omega)$ (Brioschi, *Annali di Mathematica*, XII, 1883).

For $$n = 5, \quad \text{and} \quad \gamma_5 = 0,$$

$$y = x,$$

and $$\lambda = \frac{\gamma_3}{\gamma_2} = x^{\frac{1}{2}}, \quad f^{-2} = x^2;$$

$$z_1 = -f^3 x^{-\frac{1}{5}} \lambda^{-\frac{1}{5}} = -x^{-\frac{33}{10}},$$

$$z_2 = \quad f^3 x^{-\frac{1}{5}} \lambda^{-\frac{3}{5}} = \quad x^{-\frac{7}{2}};$$

so that $$\frac{z_1}{z_2} = -\lambda^{\frac{2}{5}} = -x^{\frac{1}{5}}, \tag{141}$$

the ikosahedron irrationality (*Proc. L. M. S.*, XXV, p. 215).

[redacted]

For $$n = 7 \quad \text{and} \quad \gamma_7 = 0,$$

$$x = z(1-z)^2, \quad y = z(1-z),$$

$$\lambda^3 = \frac{\gamma_5}{\gamma_2} = y - x = z^2(1-z); \tag{142}$$

and
$$\left.\begin{aligned} \left(\frac{A_1}{A_0}\right)^7 &= \left(\frac{z_2}{z_1}\right)^7 = -\lambda^{-3} = -z^{-2}(1-z)^{-1} = a_4^7 \\ \left(\frac{A_2}{A_0}\right)^7 &= \left(\frac{z_4}{z_2}\right)^7 = \lambda^{-12}\gamma_4^7 = z^{-1}(1-z)^{-3} = a_2^7 \\ \left(\frac{A_4}{A_0}\right)^7 &= -\left(\frac{z_1}{z_4}\right)^7 = -\lambda^{15}\gamma_4^{-7} = -z^3(1-z)^{-2} = a_1^7 \end{aligned}\right\}, \tag{143}$$

employing the relations (70), (71), (72).

The case of $n = 13$ has already received a full discussion in §§ 12–18.

22. The case of $$n = 11$$

is important, as being the earliest number for which the relation

$$f(J, J') = 0, \tag{144}$$

connecting the absolute invariant J and its transformed value J', considered as the equation of a curve, is no longer unicursal, but has a deficiency

$$p = 1.$$

The equation connecting J with the parameter η employed by Kiepert (*Math. Ann.*, XXXIII, p. 97), or with Klein's parameter τ (*Modulfunctionen*, II, p. 440), is, when rationalized, a quadratic in J, and of the twelfth degree in η or τ; and these are connected by the relation

$$\eta + 8 = \frac{1}{\tau}. \tag{145}$$

The relation $$\gamma_{11} = 0 \tag{146}$$

is reduced to

$$z(1-z) = p^2(1-p),$$

or
$$2z = 1 + \surd\{4p^2(p-1)+1\}, \tag{147}$$

by the substitutions

$$x = y(1-z), \quad y = z\left(1 - \frac{z}{p}\right);$$

so that
$$\lambda = \frac{\gamma_6}{\gamma_5} = x^{\frac{1}{3}}\frac{y-x-y^2}{y-x} = x^{\frac{1}{3}}\frac{z}{p},$$
$$\lambda^3 = \frac{\gamma_7}{\gamma_4} = \frac{(y-x)\,x-y^3}{y} = -yz^2\frac{p-1}{p}, \tag{148}$$
and to agree with the notation employed in "Pseudo-Elliptic Integrals," p. 241, we must put
$$p = 1+c, \quad z = -q,$$
so that
$$\left.\begin{aligned} q\,(q+1) &= c\,(1+c)^2 \\ 2q+1 &= \sqrt{C} \\ C &= 4c\,(c+1)^2+1 \end{aligned}\right\}. \tag{149}$$

23. Kiepert's f is given by ("Pseudo-Elliptic Integrals," p. 243)
$$f^{-2} = \frac{x^{\frac{10}{3}}\lambda^{10}}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^2} = \frac{x^6 z^{10}}{y^4 z^2 p^{10}}, \tag{150}$$
and now, written in the order employed in the *M. F.*, II, p. 403,
$$\left.\begin{aligned} z_1 &= -f^3x^{-\frac{1}{3}}\lambda^{-\frac{1}{11}}\gamma_1 = -f^3x^{-\frac{1}{3}}\lambda^{-\frac{1}{11}} \\ z_3 &= -f^3x^{-\frac{1}{3}}\lambda^{-\frac{9}{11}}\gamma_3 = -f^3x^{-\frac{1}{3}}\lambda^{-\frac{9}{11}}x^{\frac{1}{3}} \\ z_9 &= -f^3x^{-\frac{1}{3}}\lambda^{-\frac{81}{11}}\gamma_9 = -f^3x^{-\frac{1}{3}}\lambda^{-\frac{4}{11}} \\ z_5 &= -f^3x^{-\frac{1}{3}}\lambda^{-\frac{25}{11}}\gamma_5 = -f^3x^{-\frac{1}{3}}\lambda^{-\frac{25}{11}}yz \\ z_4 &= f^3x^{-\frac{1}{3}}\lambda^{-\frac{16}{11}}\gamma_4 = f^3x^{-\frac{1}{3}}\lambda^{-\frac{16}{11}}y \end{aligned}\right\}. \tag{151}$$

Thence various relations ensue, which are independent of the eleventh root of λ; for instance,
$$\left.\begin{aligned} s_a - s_\lambda &= -f^6\,\frac{z_{a+\lambda}z_{a-\lambda}}{z_a^2 z_\lambda^2}, \\ \sqrt{S_a} &= -f^9\,\frac{z_{2a}}{z_a^4}; \end{aligned}\right\} \tag{152}$$
which is true for all values of n; and
$$\left.\begin{aligned} z_a^2 z_{3a} &= -\frac{f^9}{x\lambda^{a^2}}\gamma_a^2\gamma_{3a}, \\ \frac{z_{5a}}{z_a^3} &= \frac{x^{\frac{1}{3}}}{f^6\lambda^{2a^2}}\,\frac{\gamma_{5a}}{\gamma_a^3}, \quad \text{\&c.}; \end{aligned}\right\} \tag{153}$$
$$\left.\begin{aligned} z_a^3 z_{5a} + z_{9a}^3 z_{4a} + z_{3a}^3 z_a &= 0, \\ z_a^2 z_{5a} z_{9a} + z_a z_{3a} z_{4a}^2 - z_{3a} z_{5a} z_{9a}^2 &= 0; \end{aligned}\right\} \tag{154}$$
(*Math. Ann.*, XVII, p. 567; *M. F.*, II, p. 409; Brioschi, *Annali*, XXI).

Also the invariant of the third order (*Modulfunctionen*, II, p. 410)

$$\Phi(z_a) = z_1^2 z_3 + z_3^2 z_9 + z_9^2 z_5 + z_5^2 z_4 + z_4^2 z_1$$

$$= \frac{f^9}{x}\left(-\frac{x^3}{\lambda} - \frac{x^3}{\lambda^2} - \frac{yz}{\lambda^3} + \frac{y^3z^2}{\lambda^6} - \frac{y^3}{\lambda^3}\right)$$

$$= \Delta f^{12}, \tag{155}$$

after reduction; and the invariant $\Psi(z_a)$ on p. 411 can also be expressed as a rational function of x, y, z, p, f, and λ; and thence as a function of the parameter p or c; but

$$\Psi(z_a) = 0.$$

So also the roots of Galois' resolvent of the Eleventh degree can be written down in terms of c; this resolvent is

$$J : J-1 : 1$$

$$= \{\tau^2+3\tau+5-i\surd(11)\}\left\{\tau^3-\tau^2-\frac{3+3i\surd(11)}{2}\tau-\frac{7-i\surd(11)}{2}\right\}^3$$

$$: \left\{\tau^3-4\tau^2+\frac{7-5i\surd(11)}{2}\tau-4+6i\surd(11)\right\}$$

$$\times\left\{\tau^4+2\tau^3+\frac{3-3i\surd(11)}{2}\tau^2-[5+i\surd(11)]\,\tau-\frac{15+3i\surd(11)}{2}\right\}^2$$

$$: 1728; \tag{156}$$

or, putting $x^3 = \Delta\{\tau^2+3\tau+5-i\surd(11)\}$,

so that $\quad 12g_2 = x\left\{\tau^3-\tau^2-\dfrac{3+3i\surd(11)}{2}\tau-\dfrac{7-i\surd(11)}{2}\right\}$,

the elimination of τ leads to the resolvent in the form

$$x^{11}-22\Delta x^8+11\{9+2i\surd(11)\}\Delta^2x^5-11\,.\,12g_2\Delta^2x^4+88i\surd(11)\,\Delta^3x^2$$

$$+11\{3-i\surd(11)\}\,6g_2\Delta^3x-144g_2^2\Delta^3 = 0, \tag{157}$$

the roots of which are given by (*Modulfunctionen*, II, p. 428)

$$4x_r = \sum_{a=1}^{a=5} \epsilon^{ra^2}(z_a^2-z_{3a}z_{9a}) - \tfrac{1}{2}\{1-i\surd(11)\}\,\Sigma\epsilon^{-ra^2}z_{4a}z_{9a}. \tag{158}$$

24. Again, as before,

$$\left.\begin{aligned}
a_1^{11} &= \left(\frac{A_1}{A_0}\right)^{11} = \left(\frac{z_2}{z_1}\right)^{11} = -\left(\frac{z_9}{z_1}\right)^{11} = -\frac{1}{\lambda^3} \\
a_3^{11} &= \left(\frac{A_3}{A_0}\right)^{11} = \left(\frac{z_6}{z_3}\right)^{11} = -\left(\frac{z_5}{z_3}\right)^{11} = -\frac{y^{11}z^{11}}{\lambda^{16}x^{\frac{11}{2}}} \\
a_9^{11} &= \left(\frac{A_9}{A_0}\right)^{11} = \left(\frac{z_{18}}{z_9}\right)^{11} = -\left(\frac{z_4}{z_9}\right)^{11} = -\frac{y^{11}}{\lambda^{13}} \\
a_5^{11} &= \left(\frac{A_5}{A_0}\right)^{11} = \left(\frac{z_{10}}{z_5}\right)^{11} = -\left(\frac{z_1}{z_5}\right)^{11} = -\frac{\lambda^{24}}{y^{11}z^{11}} \\
a_4^{11} &= \left(\frac{A_4}{A_0}\right)^{11} = \left(\frac{z_8}{z_4}\right)^{11} = -\left(\frac{z_3}{z_4}\right)^{11} = +\frac{\lambda^7 x^{\frac{11}{2}}}{y^{11}}
\end{aligned}\right\} \qquad (159)$$

Thence various relations ensue, such as

$$\frac{a_3 a_5}{a_1} = -\frac{z_1^2}{z_3 z_9} = \frac{\lambda}{x^{\frac{1}{2}}} = \frac{z}{p},$$

with similar relations for $\frac{a_9 a_4}{a_3}$, $\frac{a_5 a_1}{a_9}$, $\frac{a_4 a_3}{a_5}$, $\frac{a_1 a_9}{a_4}$; (160)

$$a_1 a_4^2 = -\frac{\lambda x^{\frac{3}{2}}}{y^3} = -\frac{xz}{py^3} = -\frac{1-z}{p-z},$$

with similar relations for $a_3 a_1^2$, $a_9 a_3^2$, $a_5 a_9^2$, $a_4 a_5^2$; (161)

$$\frac{a_5}{a_1^3} = \frac{\lambda^3}{yz} = -z\frac{p-1}{p},$$

with similar relations for $\frac{a_4}{a_3^3}$, $\frac{a_1}{a_9^3}$, $\frac{a_3}{a_5^3}$, $\frac{a_9}{a_4^3}$. (162)

These, and other similar relations, show that one eleventh root will serve for the system; and thereby the appropriate power of $e^{\frac{2}{11}\pi i}$ is settled in the case of the imaginary roots of the modular equation (Klein, *Math. Ann.*, XVII, p. 567).

25. It was shown ("Pseudo-Elliptic Integrals," p. 245) that Klein's τ is connected with the c employed above by the relation

$$\frac{1-10\tau+\tau'}{2\tau^2} = H = \frac{1+4c+2c^2-5c^3-2c^4+c^5}{c^2(1+c)^2}, \tag{163}$$

so that

$$\eta+8 = \frac{1}{\tau} = \frac{10H+11+H'}{2(H-11)}$$

$$= \frac{c(1+c)(10+40c+31c^2-28c^3-9c^4+10c^5) + (2+8c+12c^2+9c^3-c^4-3c^5-c^6)\sqrt{C}}{2c(1+c)(1+4c-9c^2-27c^3-13c^4+c^5)}, \tag{164}$$

$$\tau = \frac{-10H-11+H'}{2H^2}$$

$$= \frac{-c^2(1+c)^2(10+40c+31c^2-28c^3-9c^4+10c^5) + c(1+c)(2+8c+12c^2+9c^3-c^4-3c^5-c^6)\sqrt{C}}{2(1+4c+2c^2-5c^3-2c^4+c^5)^2}, \tag{165}$$

where $$C = 1+4c(1+c)^2, \tag{166}$$

and $$H'^2 = 4H^2(H-11)+(10H+11)^2. \tag{167}$$

Also, we shall find that

$$\frac{d\tau}{\tau'} = \frac{dH}{H'} = \frac{dc}{\sqrt{C}} = \frac{2d\xi}{w}, \tag{168}$$

where $$\frac{1}{\tau} = \xi^2+4\xi+8+\frac{4}{\xi},$$

$$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16), \tag{169}$$

so that the above relation (163) is a quintic transformation of the elliptic functions obtained by putting $\tau_5 = 11$, or the ikosahedron irrationality $\eta = 1$, in the Transformation of the Fifth Order.

26. We can put $$c = \wp(u;\, J;\, \omega, \omega') - \tfrac{2}{3}, \tag{170}$$

where $$g_2 = \frac{4}{3},\quad g_3 = -\frac{19}{27},\quad 1728J = -\frac{2^{12}}{11}, \tag{171}$$

and then H and τ will be given by

$$H = \wp\left(u;\, J';\, \frac{\omega}{5},\, \omega'\right) - \frac{14}{3},\qquad J' = -\frac{2^6 . 31^3}{3^3 . 11^5}, \tag{172}$$

$$11\tau = -\wp\left(u-\frac{2\omega}{25}\right)+\frac{14}{3}. \tag{173}$$

Thus, if we put $$a = \wp\left(u - \frac{2\omega}{25}\right) - \frac{2}{3}, \tag{174}$$

we can write $$\tau = -\frac{1+4a+2a^2-5a^3-2a^4+a^5}{11a^2(1+a)^2}, \tag{175}$$

and, for any given value of u, τ, and H, the five roots of the quintic in c or a will correspond to the group of arguments

$$u, \quad u \pm \tfrac{2}{5}\omega, \quad u \pm \tfrac{4}{5}\omega. \tag{176}$$

27. Putting $$c = -\frac{x}{x^2+x+1} \tag{177}$$

makes $$H = \frac{x^{11}-1}{x^2(x^2+1)^2(x^3-1)}; \tag{178}$$

and therefore the roots of $$H = 0 \tag{179}$$
are given by

$$c_p = -\frac{1}{1+2\cos\frac{2p\pi}{11}}, \quad p = 1, 2, 3, 4, 5. \tag{180}$$

The roots of $$H - 11 = 0 \tag{181}$$
correspond to the duplication of the argument in c_p, so that, denoting them by b_p,

$$b_p = \frac{(c_p+1)(c_p^3 - c_p^2 - c_p - 1)}{C_p}; \tag{182}$$

and the substitution $$c = \frac{(b+1)(b^3-b^2-b-1)}{B} \tag{183}$$

reproduces the roots c_p in the order c_5, c_1, c_4, c_2, c_3. (184)

This has been verified numerically by Mr. T. I. Dewar; he finds

$$\left.\begin{array}{ll} c_1 = -0{\cdot}37278, & b_1 = -\ \ 1{\cdot}241098 \\ c_2 = -0{\cdot}54620, & b_2 = -\ \ 0{\cdot}754925 \\ c_3 = -1{\cdot}39788, & b_3 = +\ 14{\cdot}856874 \\ c_4 = +3{\cdot}22871, & b_4 = +\ \ 0{\cdot}346486 \\ c_5 = +1{\cdot}08815, & b_5 = -\ \ 0{\cdot}207337 \end{array}\right\}; \tag{185}$$

we can thus take the correspondence

$$c = b_3,\ b_4,\ b_5,\ b_4,\ 0,\ b_5,\ c_1,\ c_2,\ b_2,\ -1,\ b_1,\ c_3 \tag{186}$$
to the arguments

$$u = (2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24)\frac{\omega}{25}, \tag{187}$$

where $$c = \wp u - \tfrac{2}{3}. \tag{188}$$

28. The corresponding Transformation of the Twenty-fifth Order

$$P=\frac{1}{M^2}\,\frac{(c,\,1)^{25}}{c^2\,(c+1)^2\,(c^5-2c^4-5c^3+2c^2+4c+1)^2 \times (c^5-13c^4-27c^3-9c^2+4c+1)^2} \tag{189}$$

will be given by taking Gierster's (*Math. Ann.*, XIV, p. 543)

$$\tau_{25}=1,\quad M=\frac{1}{25}, \tag{190}$$

equivalent to two quintic transformations, with

$$\tau_5=11,\quad \tau_5'=\frac{125}{11},$$

$$\tau_5=\frac{1}{11},\quad \tau_5'=1375\,;$$

the first being that given in (163), and the second

$$P=H+\frac{14}{3}+\frac{11^4}{(H-11)^2}+\frac{12\,.\,11^2}{H-11}+\frac{11^3}{H^2}+\frac{10\,.\,11}{H},$$

with

$$P'^2=4P^3-G_2P-G_3,$$

$$G_2=\frac{4\times 29\times 809}{3},\quad G_3=\frac{61\times 471281}{27}. \tag{191}$$

It is curious that the special pseudo-elliptic integral considered by Abel (*Œuvres*, I, p. 142),

$$\int\frac{5x-1}{\sqrt{\{(x^2+1)^2-4x\}}}\,dx$$
$$=2\,\mathrm{ch}^{-1}\tfrac{1}{2}x\sqrt{(x^3+x^2+3x-1)}$$
$$=2\,\mathrm{sh}^{-1}\tfrac{1}{2}(x^2+x+2)\sqrt{(x-1)},$$

introduces elliptic functions of the same modulus and invariants, as is seen when we substitute

$$x-1=1/c.$$

Also, in § 8, Klein's ikosahedron irrationality $\eta=1$, and

$$\tau=11,\quad \tau_\infty=\frac{125}{11},\quad \tau_0=\frac{1}{11},\quad \tau=\frac{64}{11}.$$

29. To change from the argument u to the argument

$$v = u - \frac{2r\omega}{25}, \tag{192}$$

we must take

$$c = \frac{2ab(a+b)+8ab+2(a+b)+1-\sqrt{A}\sqrt{B}}{2(a-b)^2}, \tag{193}$$

$$\sqrt{C} = \frac{\{(3b^2+4b+1)a+b^3+4b^2+3b+1\}\sqrt{A} - \{(3a^2+4a+1)b+a^3+4a^2+3a+1\}\sqrt{B}}{(a-b)^3}, \tag{194}$$

where
$$b_r = \wp\frac{2r\omega}{25} - \frac{2}{3} = -\frac{1}{1+2\cos\frac{2r\pi}{11}}, \tag{195}$$

and then
$$\begin{aligned}\sqrt{B} &= \sqrt{\{4b(b+1)^2+1\}} \\ &= \frac{2b_r^4-8b_r^2-6b_r-1}{2b_r+1} \\ &= -2b_r^2 b_{2r} - 4b_r - 1;\end{aligned} \tag{196}$$

also
$$b_r = b, \quad b_{4r} = \frac{b^4}{(2b+1)(2b^2-2b-1)},$$

$$b_{5r} = -\frac{1}{b} - \frac{1}{b+1}, \quad \text{\&c.,} \tag{197}$$

and now r assumes the form in equation (175); and the division values (*Theilwerthe*)

$$\wp\,(1, 2, 3, 4, 5)\frac{2\omega_3}{11}$$

will presumably assume more symmetrical forms than those given in "Pseudo-Elliptic Integrals," p. 243.

30. The next odd number for which the deficiency of Kiepert's or Klein's modular equation is not zero is

$$n = 15;$$

and the division values in this case have been worked out, in terms of a single parameter c, on p. 260 of "Pseudo-Elliptic Integrals."

[redacted]

Equation (97) can now be written

$$a_a^5 = \left(\frac{A_a}{A_0}\right)^5 = (-1)^a \lambda^{-a^2} \left(\frac{\gamma_{2a}}{\gamma_a}\right)^5, \tag{198}$$

and $$\lambda = \frac{\gamma_8}{\gamma_7} = (c+1)\,y$$

$$= -\frac{c\,(c+1)^2}{2\,(c^2+3c+3)} \{(c^2+3c+3)(c^4-2c^3-c+1) + (c^4+2c^3-3c-1)\sqrt{C}\}, \tag{199}$$

$$C = (c^2-c-1)(c^2+3c+3). \tag{200}$$

so that

$$\left.\begin{aligned}
a_1^5 &= -\frac{1}{\lambda} = \frac{(c^2+3c+3)(c^4+0-2c^3-c+1)-(c^4+2c^3+0-3c-1)\sqrt{C}}{2c\,(c+1)^3} \\
a_2^5 &= \frac{1}{\lambda^4}\left(\frac{\gamma_4}{\gamma_2}\right)^5 = \frac{y}{(c+1)^4} \\
a_3^5 &= -\frac{1}{\lambda^9}\left(\frac{\gamma_6}{\gamma_3}\right)^5 = -\frac{\gamma_2}{\gamma_{12}}\left(\frac{\gamma_6}{\gamma_5}\right)^5 \\
a_4^5 &= \frac{1}{\lambda^{16}}\left(\frac{\gamma_8}{\gamma_4}\right)^5 = \frac{\gamma_7^5}{\gamma_{13}\gamma_4^5} \\
a_5^5 &= -\frac{1}{\lambda^{25}}\left(\frac{\gamma_{10}}{\gamma_5}\right)^5 = -1 \\
a_6^5 &= \frac{1}{\lambda^{36}}\left(\frac{\gamma_{12}}{\gamma_6}\right)^5 = \lambda^9\left(\frac{\gamma_2}{\gamma_6}\right)^5 \\
a_7^5 &= -\frac{1}{\lambda^{49}}\left(\frac{\gamma_{14}}{\gamma_7}\right)^5 = -\frac{\lambda^{16}}{\gamma_7^5}
\end{aligned}\right\} \tag{201}$$

and $$a_a a_{2a} = (-1)^a \lambda^{-a^2} \frac{\gamma_{4a}}{\gamma_{2a}}, \qquad a_a a_{3a} = \lambda^{-2a^2} \frac{\gamma_{2a}\gamma_{6a}}{\gamma_a \gamma_{3a}}, \text{ \&c.}$$

Thence
$$\left.\begin{aligned}
a_3 a_6 &= -1 \\
a_4 a_7 &= -\frac{\lambda}{y} = -c-1 = \frac{1}{a_1 a_2} \\
a_1 a_3 &= \frac{1}{\lambda^2}\frac{\gamma_6}{\gamma_3} = \frac{-c^2-c+1+\sqrt{C}}{2\,(c+1)} \\
a_2 a_4 &= \frac{\gamma_7}{\lambda^3} = \frac{c^2+c+1+\sqrt{C}}{2\,(c+1)}
\end{aligned}\right\}. \tag{202}$$

31. We can also express the ξ parameters employed by Kiepert for $n = 15$ (*Math. Ann.*, XXXII, p. 121) in terms of c.

Starting with Kiepert's relations, for any odd number n,

$$L(n) = Q^{n-1} f(n), \tag{203}$$

$$Q^{24} = \Delta, \tag{204}$$

$$f(n)^{-3} = (-1)^{\frac{1}{2}(n-1)} \prod_{a=1}^{\frac{1}{2}(n-1)} \wp' \frac{2a\omega}{n} \tag{205}$$

(*Math. Ann.*, XXVI, pp. 394, 427), then

$$\xi_1 = \frac{L(15)^8}{L(5)^8 L(3)^8} = \Delta \frac{f(15)^8}{f(5)^8 f(3)^8}$$

$$= \frac{\Delta}{R} = \frac{\Delta}{\wp' \frac{2\omega}{15} \wp' \frac{4\omega}{15} \wp' \frac{8\omega}{15} \wp' \frac{14\omega}{15}}, \tag{206}$$

$$\xi_2 = \frac{L(15)^2 L(3)^2}{L(5)^2} = \Delta \frac{f(15)^2 f(3)^2}{f(5)^2}, \tag{207}$$

so that

$$\xi_2^3 = \Delta^3 \frac{\wp'^2 \frac{2\omega}{5} \wp'^2 \frac{4\omega}{5}}{\wp'^2 \frac{2\omega}{15} \wp'^2 \frac{4\omega}{15} \wp'^2 \frac{6\omega}{15} \wp'^2 \frac{8\omega}{15} \wp'^2 \frac{10\omega}{15} \wp'^2 \frac{12\omega}{15} \wp'^2 \frac{14\omega}{15} \wp'^2 \frac{2\omega}{3}}$$

$$= \frac{\Delta^3}{S_1 S_2 S_4 S_5^2 S_7}$$

$$= \frac{\Delta^3 \gamma_1^8 \gamma_2^8 \gamma_4^8 \gamma_5^{16} \gamma_7^8}{x^{12} \gamma_2^2 \gamma_4^2 \gamma_8^2 \gamma_{10}^4 \gamma_{14}^2}$$

$$= \frac{\Delta^3 \gamma_4^6 \gamma_5^{12} \gamma_7^6}{x^{12} \lambda^{48}}, \tag{208}$$

$$\xi_2 = \frac{\Delta \gamma_4^2 \gamma_5^4 \gamma_7^2}{x^4 \lambda^{16}}, \tag{209}$$

while

$$\xi_1 = \frac{\Delta \gamma_1^4 \gamma_2^4 \gamma_4^4 \gamma_7^4}{x^4 \gamma_2 \gamma_4 \gamma_8 \gamma_{14}}$$

$$= \frac{\Delta \gamma_4^3 \gamma_7^3}{x^4 \lambda^{14}}, \tag{210}$$

[redacted]

so that
$$\xi_4 = \frac{\xi_1}{\xi_3} = \frac{\lambda^2 \gamma_4 \gamma_7}{\gamma_5^4}$$

$$= -\frac{(c+1)^2 (p-z)}{p^3}, \tag{211}$$

reducing to a rational expression ("Pseudo-Elliptic Integrals," p. 258)
$$\xi_4 = \frac{(c+1)^2}{c^3 (c^2+3c+3)}. \tag{212}$$

Making use of this value of ξ_4 in Kiepert's equation (448), we find
$$\xi_1 = \frac{-5c(c+1)^2(c^2+3c+3)+(c+1)(c+2)(2c^2+3c+3)\sqrt{C}}{2c(c^2+3c+3)(c^4+3c^3+4c^2+2c+1)}, \tag{213}$$

$$\xi_3 = \frac{\xi_1}{\xi_4} = \frac{-5c^2(c+1)(c^2+3c+3)+c(c+2)(2c^2+3c+3)\sqrt{C}}{2(c+1)(c^4+3c^3+4c^2+2c+1)}, \tag{214}$$

and so forth; thence the values of x, y, z in terms of c in Kiepert's equations (620) *Math. Ann.*, XXXVII, p. 390, can be inferred.

32. In the preceding cases of Transformation the order n has been taken as an odd number, and the resolution of the cubic
$$4p^3 - g_2 p - g_3 = 0, \tag{215}$$

or of the associated form, employed in "Pseudo-Elliptic Integrals,"
$$4s(s+x)^2 - \{(y+1)s+xy\}^2 = 0, \tag{216}$$

is not required; and the associated elliptic functions are of the *First Stage* (*Erster Stufe*).

But in most dynamical applications this resolution must be effected, and elliptic functions of the *Second Stage* (*Zweiter Stufe*) must be employed; so that we shall find it useful, for the purpose of mechanical problems, to follow Kiepert, and to determine the modular functions corresponding to an even order.

(Brioschi, *Annali di Matematica*, XXII, 1894.)

33. Referring to Kiepert's paper "Zur Transformation der Elliptischen Functionen," *Math. Ann.*, XXXII, p. 1, for an explanation of the notation, and for the meaning of his parameters denoted by ξ, the following table of results shows the expression of the ξ's in terms of a single parameter, as defined in "Pseudo-Elliptic Integrals":—

$n = 2$ (*Math. Ann.*, xxxii, p. 55).

$$\xi = L(2)^{24} = \Delta f(2)^{24} = -64\tau_2$$
$$= 16\frac{\kappa'^4}{\kappa^2},\quad 16\frac{\kappa^4}{\kappa'^2},\quad \text{or}\quad -\frac{16}{\kappa^2\kappa'^2}. \tag{217}$$

$n = 4$

(*Math. Ann.* xxxii, p. 55; and "Pseudo-Elliptic Integrals," p. 211).

$$\left.\begin{aligned} \xi_1 &= \frac{L(4)^{16}}{L(2)^{24}} = 1-16x = \tau_4 = \eta^4 \\ \xi_2 &= L(4)^8 = \frac{1-16x}{x} \\ \xi_3 &= \frac{L(4)^8}{L(2)^{24}} = \frac{f(4)^8}{f(2)^{24}} = x \end{aligned}\right\}. \tag{218}$$

$n = 6$ (p. 83; and p. 216).

$$\left.\begin{aligned} \xi_2 &= \frac{L(6)^4}{L(3)^8 L(2)^4} = \frac{f(6)^4}{f(3)^8 f(2)^4} = y = \frac{-c}{(2-c)(1-2c)} \\ \xi_3 &= \frac{L(6)^3}{L(3)^3 L(2)^9} = \frac{f(6)^3}{f(3)^3 f(2)^9} = \frac{y}{1-y} = \frac{-c}{2(1-c)^2} \\ \xi_1 &= \frac{1-9\xi_2}{1-\xi_2} = \frac{1-9y}{1-y} = \left(\frac{1+c}{1-c}\right)^2 \\ \xi_4 &= \frac{8\xi_1}{1-\xi_1} = \frac{1-9y}{y} = 2\frac{(1+c)^2}{-c} \\ \xi_5 &= \frac{8}{9-\xi_1} = 1-y = \frac{2(1-c)^2}{(2-c)(1-2c)} \\ \xi_6 &= \frac{8\xi_1}{9-\xi_1} = 1-9y = \frac{2(1+c)^2}{(2-c)(1-2c)} \end{aligned}\right\}; \tag{219}$$

and (*Math. Ann.*, xxxvii, p. 385)

$$\left.\begin{aligned} u &= \sqrt{\xi} + \frac{8}{\sqrt{\xi}}, \quad \text{where}\quad \xi = \frac{\xi_6}{\xi_3} \\ v &= \sqrt{\eta} + \frac{1}{\sqrt{\eta}}, \quad \text{where}\quad \eta = \frac{1}{\xi_3\xi_6} \\ w &= \sqrt{\zeta} + \frac{9}{\sqrt{\zeta}}, \quad \text{where}\quad \zeta = \frac{\xi_1}{\xi_2} \end{aligned}\right\}. \tag{220}$$

$n=8$ (p. 57; and p. 226).

$$\left.\begin{aligned}
\xi_6 &= \frac{f(4)^4}{f(2)^{12}} = \frac{z(1-z)}{(1-2z)^2} = \frac{c(1-c)(1-2c)}{(1-2c+2c^2)^2} \\
\xi_1 &= \frac{1-4\xi_6}{1+4\xi_6} = 1-8z+8z^2 = \left(\frac{1-4c+2c^2}{1-2c^2}\right)^2 \\
\xi_2 &= \frac{1}{1+4\xi_6} = (1-2z)^2 = \left(\frac{1-2c+2c^2}{1-2c^2}\right)^2 \\
\xi_3 &= \frac{\xi_6}{1+4\xi_6} = \frac{1}{2r_8} = z(1-z) = \frac{c(1-c)(1-2c)}{(1-2c+2c^2)^2} \\
\xi_4 &= \frac{1}{1-4\xi_6} = \frac{(1-2z)^2}{1-8z+8z^2} = \left(\frac{1-2c+2c^2}{1-4c+2c^2}\right)^2 \\
\xi_5 &= \frac{\xi_6}{1-4\xi_6} = \frac{z(1-z)}{1-8z+8z^2} = \frac{c(1-c)(1-2c)}{(1-4c+2c^2)^2}
\end{aligned}\right\}. \tag{221}$$

$n=10$ (p. 86; and p. 235).

$$\xi_2 = \frac{f(10)^2}{f(5)^4 f(2)^2} = \frac{-a}{1-a-a^2}, \tag{222}$$

because
$$\left.\begin{aligned}
f(10)^2 &= x^{-\frac{9}{5}}\lambda^{-\frac{1}{10}(1^2+2^2+\dots+9^2)}\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9 \\
f(2)^2 &= x^{-\frac{1}{5}}\lambda^{-\frac{1}{10}5^2}\gamma_5 \\
f(5)^2 &= x^{-\frac{4}{5}}\lambda^{-\frac{1}{10}(2^2+4^2+6^2+8^2)}\gamma_2\gamma_4\gamma_6\gamma_8
\end{aligned}\right\}; \tag{223}$$

and these values lead here, as elsewhere, to the result, employing the values of γ given on p. 204 of "Pseudo-Elliptic Integrals."

Thence
$$\left.\begin{aligned}
\xi_1 &= \frac{1-5\xi_2}{1-\xi_2} = \frac{1+4a-a^2}{1-a^2} \\
\xi_3 &= \frac{1-\xi_1}{4} = \frac{-a}{1-a^2} \\
\xi_4 &= \frac{4\xi_1}{1-\xi_1} = \frac{1+4a-a^2}{-a} \\
\xi_5 &= \frac{4}{5-\xi_1} = \frac{1-a^2}{1-a-a^2} \\
\xi_6 &= \frac{4\xi_1}{5-\xi_1} = \frac{1+4a-a^2}{1-a-a^2}
\end{aligned}\right\}. \tag{224}$$

Thence (*Math. Ann.*, XXXVII, p. 385)

$$\left.\begin{aligned} u,\ \bar{u} &= \sqrt{\xi} \pm \frac{4}{\sqrt{\xi}}, \quad \text{where} \quad \xi = \xi_4 \xi_5 \\ v,\ \bar{v} &= \sqrt{\eta} \pm \frac{1}{\sqrt{\eta}}, \quad \text{where} \quad \eta = \frac{1}{\xi_1 \xi_2} \\ w,\ \bar{w} &= \sqrt{\zeta} \pm \frac{5}{\sqrt{\zeta}}, \quad \text{where} \quad \zeta = \frac{\xi_1}{\xi_2} \end{aligned}\right\}, \qquad (225)$$

and putting

$$\begin{aligned} 1-b &= -\frac{1}{a}+1+a \\ &= \frac{1}{1-p_{10}} = \frac{\wp\frac{1}{5}\omega - \wp\omega}{\wp\frac{2}{5}\omega - \wp\omega}, \end{aligned} \qquad (226)$$

$$\left.\begin{aligned} u^2 = \bar{v}^2 &= \frac{(4+b^2)^2}{b(1-b)(4+b)} \\ v^2 = w^2 &= \frac{(4+2b-b^2)^2}{b(1-b)(4+b)} \\ \bar{u}^2 = \bar{w}^2 &= \frac{(4-8b-b^2)^2}{b(1-b)(4+b)} \end{aligned}\right\}. \qquad (227)$$

The eighteen values of Gierster's τ_{10} (*Math. Ann.*, XIV, p. 452) can now be exhibited as a group of substitutions, involving the parameter a.

Passing on here to the case of $n = 20$ (*Math. Ann.*, XXXII, p. 105),

$$\begin{aligned} w &= \sqrt{\{\xi_3(\xi_3+1)(4\xi_3^2+1)\}} \\ &= \frac{\sqrt{(4+b^2)}\,\sqrt{(1-b)}}{b^2} \\ &= \frac{(1+a^2)\sqrt{A}}{(1-a^2)^2}, \end{aligned}$$

$$A = -a+a^2+a^3;$$

$$\begin{aligned} \eta_1 &= \left\{\frac{\sqrt{(4+b^2)}-2\sqrt{(1-b)}}{b}\right\}^2 \\ &= \left\{\frac{1+a^2-2\sqrt{A}}{1-a^2}\right\}^2, \end{aligned}$$

$$\eta_6 = \frac{(1-a-a^2)(1-6a-a^2)-5(1+a^2)\sqrt{A}}{(1-a-a^2)(1+4a-a^2)},$$

and similarly the values of η_2, η_3, η_4, η_5 can be written down.

$n = 12$ (p. 103; and p. 248).

Beginning with ξ_2, as it does not introduce Δ, we find

and thence

$$\left.\begin{aligned} \xi_2 &= \frac{f(6)^2}{f(3)^4 f(2)^2} = \frac{-a}{1+a+a^2} \\ \xi_1 &= \frac{1-3\xi_2}{1+\xi_2} = \frac{1+4a+a^2}{1+a^2} \\ \xi_3 &= \frac{1+\xi_1}{3-\xi_1} = \left(\frac{1+a}{1-a}\right)^2 \\ \xi_4 &= \frac{1-\xi_1}{4} = \frac{-a}{1+a^2} \\ \xi_5 &= 2\,\frac{1+\xi_1}{1-\xi_1} = \frac{(1+a)^2}{-a} \\ \xi_6 &= \frac{1+\xi_1}{2} = \frac{(1+a)^2}{1+a^2} \\ \dots & \quad \dots \quad \dots \quad \dots \end{aligned}\right\}, \tag{228}$$

and

$$a = \frac{\wp\frac{1}{3}\omega - \wp\frac{5}{6}\omega}{\wp\frac{1}{3}\omega - \wp\frac{1}{6}\omega}, \tag{229}$$

and thence the twenty-four values of Gierster's τ_{12} in terms of a can be exhibited by a group of substitutions.

$n = 14$ (p. 87; and p. 257).

Beginning with ξ_4, we find

$$\xi_4^2 = \frac{L(14)^2}{L(7)^2 L(2)^{14}} = \frac{f(14)^2}{f(7)^2 f(2)^{14}}$$

$$= \frac{x^{-\frac{16}{3}} \lambda^{-\frac{151}{3}} \gamma_1\gamma_2\gamma_3 \cdots \gamma_{12}\gamma_{13}}{x^{-3}\lambda^{-26}\gamma_2\gamma_4\gamma_6\gamma_8\gamma_{10}\gamma_{12} \cdot x^{-\frac{7}{3}} \lambda^{-\frac{49}{3}} \gamma_7^7}$$

$$= \frac{\gamma_1\gamma_3\gamma_5\gamma_7\gamma_9\gamma_{11}\gamma_{13}}{\lambda^8 \gamma_7^7}$$

$$= \lambda^{16} \frac{\gamma_1^2\gamma_3^2\gamma_5^2}{\gamma_7^6}, \tag{230}$$

$$\xi_4 = \lambda^8 \frac{\gamma_1\gamma_3\gamma_5}{\gamma_7^3} = \frac{\gamma_{11}}{\gamma_3}\,\frac{\gamma_1\gamma_3\gamma_5}{\gamma_7^3} = \frac{\gamma_5\gamma_{11}}{\gamma_7^3}$$

$$= \frac{p^3 - c^3p + c + c^2}{(p-1)^3}. \tag{231}$$

Putting, as in p. 257 of "Pseudo-Elliptic Integrals,"

$$p = \frac{r+c}{r-1},$$

$$\xi_4 = \frac{r^2 - c^2 r + c + c^2}{1+c} = \frac{c(1+c)^2}{1+2c}, \tag{232}$$

a rational function of

$$c = \frac{\wp 5v - \wp v}{\wp 3v - \wp v}\,\frac{\wp 3v - \wp 2v}{\wp 2v - \wp 5v}, \quad v = \tfrac{1}{7}\omega. \tag{233}$$

With this value of ξ_4, Kiepert's equation (290), p. 90, for ξ_3 becomes

$$\frac{1+c-2c^2-c^3}{1+2c}\xi_3^2 + 7\frac{c(1+c)^2}{1+2c}\xi_3 - \frac{c(1+c)^2(1-6c-16c^2-8c^3)}{(1+2c)^2} = 0,$$

and thence

$$\xi_8 = \frac{-7c(1+c)^2(1+2c) + (1+c)(1+3c+4c^2)\sqrt{C}}{2(1+2c)(1+c-2c^2-c^3)}, \tag{234}$$

where $\quad C = c(1+2c)(4+5c+2c^2),$ (235)

$$\xi_2 = \frac{\xi_8}{\xi_4} = \frac{-7c(1+c)(1+2c) + (1+3c+4c^2)\sqrt{C}}{2c(1+c)(1+c-2c^2-c^3)}, \tag{236}$$

and $\quad \xi_1 = \dfrac{\xi_2^2}{\xi_4} = \dfrac{1+2c}{c(1+c)^2}\xi_2^2,$ (237)

$$w = \frac{8(1+c)(1+3c+4c^2)\sqrt{C}}{(1+2c)^3}. \tag{238}$$

Also $\quad \dfrac{1}{\xi_3} = \dfrac{7c(1+c)^2(1+2c) + (1+c)(1+3c+4c^2)\sqrt{C}}{2c(1+c)(1-6c-16c^2-8c^3)};$ (239)

and thence (*Math. Ann.*, XXXVII, p. 385), for $n = 14$,

$$v^2 - 2 = \eta + \frac{1}{\eta} = \frac{1}{\xi_3} + \xi_3 = y$$

$$= \frac{7(1+2c+4c^2+22c^3+44c^4+32c^5+8c^6)}{2(1+c-2c^2-c^3)(1-6c-16c^2-8c^3)}$$

$$+ \frac{(1+3c+4c^2)(1+4c-4c^2-32c^3-48c^4-32c^5-8c^6)\sqrt{C}}{2c(1+c)(1+2c)(1+c-2c^2-c^3)(1-6c-16c^2-8c^3)}, \tag{240}$$

and (*Math. Ann.*, XXXVII, p. 386)

$$x = \xi + \frac{8}{\xi} = \xi_2 + \frac{8}{\xi_2}, \tag{241}$$

where $$\frac{1}{\xi_2} = \frac{\xi_4}{\xi_3} = \frac{7c(1+c)^2(1+2c)+(1+c)(1+3c+4c^2)\sqrt{C}}{2(1+2c)(1-6c-16c^2-8c^3)}, \qquad (242)$$

and thence we find that Kiepert's relation

$$x = y-7 = v^2-9 \qquad (243)$$

is satisfied.

$$n = 16 \quad \text{(p. 59; and p. 262).}$$

Converting Kiepert's expression for ξ_4 into one involving λ and γ,

$$\xi_4 = \frac{\dfrac{\gamma_8\gamma_4}{\gamma_2^2\gamma_6^2}\,\dfrac{\gamma_6\gamma_2}{\gamma_3^2\gamma_4^2}\,\dfrac{\gamma_{10}\gamma_2}{\gamma_6^2\gamma_4^2}}{\dfrac{\gamma_{12}\gamma_4}{\gamma_4^2\gamma_8^2}\,\dfrac{\gamma_8\gamma_6}{\gamma_1^2\gamma_7^2}\,\dfrac{\gamma_8\gamma_2}{\gamma_3^2\gamma_5^2}} = \frac{\gamma_{10}\gamma_8\gamma_7^2\gamma_5^2\gamma_3^2}{\gamma_{12}\gamma_6^4\gamma_4^2} = \frac{\gamma_8\gamma_7^2\gamma_5^2\gamma_3^2}{\lambda^4\gamma_6^3\gamma_4^3} \qquad (244)$$

and this reduces finally to

so that

$$\left.\begin{aligned}
\xi_4 &= \frac{a^2-1}{a^2-2a-1} \\
\xi_3 &= \frac{1-\xi_4}{2\xi_4} = \frac{-a}{a^2-1} \\
\xi_1 &= \frac{1-2\xi_3}{1+2\xi_3} = \frac{a^2+2a-1}{a^2-2a-1} \\
\xi_5 &= \frac{\xi_3}{1+2\xi_3} = \frac{-a}{a^2-2a-1} \\
\xi_7 &= 1-2\xi_3 = \frac{a^2+2a-1}{a^2-1} \\
\xi_8 &= \frac{1-2\xi_3}{\xi_3} = \frac{a^2+2a-1}{-a} \\
\xi_2 &= \frac{\sqrt{(1+4\xi_3^2)}}{1+2\xi_3} = \frac{a^2+1}{a^2-2a-1} \\
\xi_6 &= \frac{\xi_1}{\xi_2} = \frac{a^2+2a-1}{a^2+1} \\
\xi_9 &= \frac{\xi_4}{\xi_2} = \frac{a^2-1}{a^2+1} \\
\xi_{10} &= \frac{\xi_5}{\xi_2} = \frac{-a}{a^2+1}
\end{aligned}\right\} . \qquad (245)$$

Referring to "Pseudo-Elliptic Integrals," p. 263, and *Math. Ann.*, XIV, p. 542, we see that Gierster's

$$\tau_{16} = -\xi_8. \qquad (246)$$

$$n = 18 \qquad \text{(p. 126; and p. 265).}$$

To agree with the notation of the *Modulfunctionen*, I, p. 685, we must put Gierster's

$$\tau_{18} = -x-2, \tag{247}$$

$$\tau_{12} = \quad y+3, \tag{248}$$

and then (*Math. Ann.*, XIV, pp. 540, 541)

$$2\tau_6+9 = (\tau_{12}-3)^2 = y^2, \tag{249}$$

$$2\tau_6+8 = (\tau_{18}+2)^3 = x^3, \tag{250}$$

$$x^3 = -\frac{\wp\frac{2}{3}\omega-\wp\frac{1}{3}\omega}{\wp\omega-\wp\frac{2}{3}\omega}, \quad y^2 = \frac{\wp\omega-\wp\frac{1}{3}\omega}{\wp\omega-\wp\frac{2}{3}\omega} = \frac{(\wp\omega-\wp\frac{1}{3}\omega)^2}{(\wp\omega-\wp\omega')(\wp\omega-\wp\omega'')},$$

so that

$$x^3+y^2 = 1. \tag{251}$$

Changing the sign of the x on p. 269 of "Pseudo-Elliptic Integrals," we now find that the relation connecting this x with Kiepert's ξ_3 is

$$\xi_3 = -\frac{1}{x} = \frac{1}{\tau_{18}+2}$$

$$= \frac{-q^3-q^2+2q+1+\sqrt{Q}}{4q\,(q+1)}, \tag{252}$$

so that

$$\xi_5 = \frac{1-2\xi_3}{\xi_3} = \tau_{18}$$

$$= \frac{q^3-3q^2-6q-1+\sqrt{Q}}{2q\,(q+1)}, \tag{253}$$

$$\xi_1 = 1-2\xi_3$$

$$= \frac{q^3+3q^2+0-1-\sqrt{Q}}{2q\,(q+1)}, \tag{254}$$

$$\xi_2 = \frac{\xi_3}{1+\xi_3}$$

$$= \frac{q^3+3q^2+0-1-\sqrt{Q}}{2\,(q^3-3q-1)}, \tag{255}$$

$$\xi_4 = \frac{1-2\xi_3}{1+\xi_3}$$

$$= \frac{-q^3-9q^2-6q+1+3\sqrt{Q}}{2\,(q^3-3q-1)}, \tag{256}$$

$$\xi_6 = \frac{1}{1+\xi_3}$$

$$= \frac{q^3 - 3q^2 - 6q - 1 + \sqrt{Q}}{2(q^3 - 3q - 1)}, \tag{257}$$

so that

$$\frac{\xi_1}{\xi_3} = \frac{\xi_5}{\xi_6} = \frac{q^3 - 3q - 1}{q(q+1)}$$

$$= q - 1 - \frac{1}{q} - \frac{1}{q+1}. \tag{258}$$

$n = 22$ (p. 91; and p. 274).

34. Expressed in terms of our λ and γ, we find that Kiepert's three parameters ξ, ξ_1, ξ_2 are given by

$$\xi = \frac{L(22)^2 L(2)^2}{L(11)^2} = \Delta \frac{f(22)^2 f(2)^2}{f(11)^2}$$

$$= \Delta \frac{x^{-7} \lambda^{-\frac{293}{2}} \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_{20} \gamma_{21} \cdot x^{-\frac{1}{2}} \lambda^{-\frac{11}{2}} \gamma_{11}}{x^{-\frac{7}{2}} \lambda^{-70} \gamma_2 \gamma_4 \gamma_6 \cdots \gamma_{18} \gamma_{20}}$$

$$= \frac{\Delta}{x^4 \lambda^{86}} \gamma_1 \gamma_3 \gamma_5 \gamma_7 \gamma_9 \gamma_{11}^2 \gamma_{13} \gamma_{15} \gamma_{17} \gamma_{19} \gamma_{21}$$

$$= \frac{\Delta}{x^4 \lambda^{86}} (\gamma_1 \gamma_3 \gamma_5 \gamma_7 \gamma_9 \gamma_{11})^2, \tag{259}$$

$$\xi_1 = \frac{L(22)^4}{L(11)^8 L(2)^4} = \frac{f(22)^4}{f(11)^8 f(2)^4}$$

$$= \lambda^{10} \left(\frac{\gamma_1 \gamma_3 \gamma_5 \gamma_7 \gamma_9}{\gamma_2 \gamma_4 \gamma_6 \gamma_8 \gamma_{10}} \right)^4, \tag{260}$$

$$\xi_2 = \frac{L(22)^8}{L(11)^4 L(2)^8} = \Delta^5 \frac{f(22)^8}{f(11)^4 f(2)^8}$$

$$= \frac{\Delta^5}{x^{20} \lambda^{76}} \frac{(\gamma_1 \gamma_3 \gamma_5 \gamma_7 \gamma_9)^8}{(\gamma_2 \gamma_4 \gamma_6 \gamma_8 \gamma_{10})^2}, \tag{261}$$

and we have to express these quantities as functions of a single parameter c by means of the relations on p. 274 of "Pseudo-Elliptic Integrals."

35. Judging by analogy with preceding cases, the parameter ξ_1 seemed likely to assume the simplest form, because it did not involve the discriminant Δ; and now we shall find that, expressed in terms of the q and c of p. 274, "Pseudo-Elliptic Integrals,"

$$\xi_1 = \frac{c^3 q^4 (q-1-c-c^2)^3 (q+1+c)^6 (q-c^2)}{(1+c)(q-1)^5 (q+c)^3 (q-c-c^2)^9}, \tag{262}$$

where q is given as a function of c by the quartic equation

$$(q+c)(q-c-c^2)^2 \{cq^2-(1+2c+c^2+c^3)\, q+c^2+c^3\} \\ -cq\,(q-1-c-c^2)(q-c^2)(q+1+c)^2 = 0, \tag{263}$$

which can also be written, according to the calculations of Mr. St. Bodfan Griffiths, of University College, Bangor, in a form ready for solution,

$$\{2\,(1+c)^2 q^2-(2c+5c^2+4c^3+2c^4)\, q-c^4\,(1+c)\}^2 \\ = (4c+8c^2+4c^3+c^4)\,\{(1+2c)\, q-c^2\,(1+c)\}^2, \tag{264}$$

or $$c^2\,\{cq^2-(2+4c+c^2+c^3)\, q+2c^2\,(1+c)\}^2 \\ = (4c+8c^2+4c^3+c^4)\, q^2\,(q-c-c^2)^2. \tag{265}$$

Thus, from (263), we can also write

$$\xi_1 = \frac{c^2 q^3 (q+1+c)^3 (q-1-c-c^2)^2 \{cq^2-(1+2c+c^2+c^3)\, q+c^2+c^3\}}{(1+c)(q+c)(q-1)^5 (q-c-c^2)^7}, \tag{266}$$

and the elimination of q between this and (263) will lead to a quartic equation in ξ_1, which is discussed in the sequel.

36. To connect up these values of q and c, which may be distinguished when required by q_{22} and c_{22}, with the q_{11} and c_{11}, employed in the Transformation of the Eleventh Order in § 22, we notice that

$$q_{11} = -z_{11} = -\frac{\wp\frac{6\omega}{11}-\wp\frac{4\omega}{11}}{\wp\frac{6\omega}{11}-\wp\frac{2\omega}{11}}$$

$$= -\frac{\wp\frac{12\omega}{22}-\wp\frac{8\omega}{22}}{\wp\frac{12\omega}{22}-\wp\frac{4\omega}{22}}$$

$$= -\frac{\dfrac{\gamma_{10}\gamma_2}{\gamma_6^2\gamma_4^2}}{\dfrac{\gamma_8\gamma_4}{\gamma_6^2\gamma_2^2}}$$

$$= -\frac{\gamma_{10}}{\gamma_8\gamma_4^3}$$

$$= -\frac{(q+c)(q-c-c^2)}{(q-c^2)^2}, \qquad (267)$$

$$1+c_{11} = p_{11}$$

$$= \frac{\wp\dfrac{8\omega}{22} - \wp\dfrac{4\omega}{22}}{\wp\dfrac{20\omega}{22} - \wp\dfrac{4\omega}{22}}$$

$$= \frac{\gamma_{10}^2\gamma_6}{\gamma^{1\cdot}\gamma_8\gamma_4^2}$$

$$= \frac{(q+c)(q-c-c^2)^2}{(q+1+c)(q-c^2)^2}, \qquad (268)$$

so that
$$\frac{q_{11}}{1+c_{11}} = -\frac{z_{11}}{p_{11}} = -\frac{q+1+c}{q-c-c^2}, \qquad (269)$$

$$\frac{1+c_{11}+q_{11}}{1+c_{11}} = \frac{-(1+c)^2}{q-c-c^2}, \qquad (270)$$

and these are true in general when z_{11}, p_{11}, q_{22}, c_{22} are replaced by z_n, p_n, q_{2n}, c_{2n}.

37. In Kiepert's notation (*Math. Ann.*, XXXII, p. 96)

$$\frac{1}{\tau} = \eta + 8 = \xi^2 + 4\xi + 8 + \frac{4}{\xi},$$

$$\frac{\tau'}{\tau^2} = W = (\xi + 2 - 2\xi^{-2})\, w,$$

$$w^2 = (\xi^3 + 4\xi^2 + 8\xi + 4)(\xi^3 + 8\xi^2 + 16\xi + 16),$$

so that
$$-\frac{\tau'}{\tau^2}\frac{d\tau}{d\xi} = 2\,(\xi + 2 - 2\xi^{-2}),$$

and
$$\frac{d\tau}{\tau'} = -\frac{2d\xi}{w}. \qquad (271)$$

Again (*Math. Ann.*, XXXVII, p. 386), putting

$$u, \quad \bar{u} = \sqrt{\xi} \pm \frac{2}{\sqrt{\xi}}, \tag{272}$$

$$\frac{4}{\sqrt{\xi}} = u - \sqrt{(u^2-8)},$$

$$-\frac{2d\xi}{\xi^{\frac{3}{2}}} = du - \frac{u\,du}{\sqrt{(u^2-8)}},$$

while

$$\frac{w^2}{\xi^3} = u^6 - 4u^2 + 4, \tag{273}$$

so that, from (168),

$$\frac{dc}{\sqrt{C}} = \frac{d\tau}{\tau'} = \frac{du}{\sqrt{(u^6-4u^2+4)}} - \frac{u\,du}{\sqrt{\{(u^2-8)(u^6-4u^2+4)\}}}. \tag{274}$$

Put

$$u^2 = \frac{1}{a+1}; \tag{275}$$

then

$$\frac{du}{\sqrt{(u^6-4u^2+4)}} = \frac{-\frac{1}{2}da}{\sqrt{A}}, \tag{276}$$

$$A = 4a\,(a+1)^2 + 1. \tag{277}$$

Put

$$\bar{u}^2 = u^2 - 8 = \frac{11^2}{b-11}; \tag{278}$$

then

$$\frac{u\,du}{\sqrt{\{(u^2-8)(u^6-4u^2+4)\}}} = \frac{-\frac{1}{2}db}{\sqrt{B}}, \tag{279}$$

$$B = 4b^2\,(b-11) + (10b+11)^2, \tag{280}$$

the same function as H'^2 in (167), § 25.

Thus

$$\frac{dc}{\sqrt{C}} = \frac{-\frac{1}{2}da}{\sqrt{A}} + \frac{\frac{1}{2}db}{\sqrt{B}}, \tag{281}$$

and $\frac{db}{\sqrt{B}}$ can be reduced to the form of $\frac{dc}{\sqrt{C}}$

by means of the quintic transformation (163), so that the preceding relations conceal an elliptic function relation, the interpretation of which is given hereafter, in § 61.

38. In Kiepert's notation, distinguishing this new η by an accent,

$$2\sqrt{\eta'} = v+\bar{v}, \quad 2\sqrt{\zeta} = w+\bar{w},$$

so that

$$\frac{4}{\xi_1} = 4\sqrt{(\eta'\zeta)} = (v+\bar{v})(w+\bar{w}). \tag{282}$$

Also

$$2\sqrt[4]{\eta'} = \sqrt{(v+2)} + \sqrt{(v-2)},$$

$$2\sqrt[4]{\zeta} = \sqrt{(w+22)} + \sqrt{(w-22)},$$

$$\frac{4}{\sqrt{\xi_1}} = \{\sqrt{(v+2)} + \sqrt{(v-2)}\}\{\sqrt{(w+22)} + \sqrt{(w-22)}\}$$

$$= 2(\xi+3)\sqrt{(\xi^3+8\xi^2+16\xi+16)} + 2(\xi+5)\sqrt{(\xi^3+4\xi^2+8\xi+4)}. \tag{283}$$

These relations seem to show that u, v, w should be determined, as the simplest functions of a single parameter c.

$n = 26$ (*Math. Ann.*, XXXII, p. 98).

39. We notice here that Kiepert's ξ_1 is the same as Klein's τ_{13}, so that we can put

$$\xi_1 = \frac{1-c-4c^2-c^3}{c(1+c)}, \tag{284}$$

and thus ξ can be determined as a function of c by the solution of Kiepert's cubic equation (333).

To obtain the ξ's as explicit functions of a parameter, we should have to discuss Halphen's relation

$$\gamma_{26} = 0; \tag{285}$$

but this leads to difficulties not yet surmounted.

40. The connexion between Kiepert's ξ parameters and the function p_n employed by Abel in the expression of the square root of a quartic in the form of a continued fraction is remarkable (Abel, *Œuvres*, II, p. 157).

Expressed by Halphen's γ functions, we find that Abel's

$$q_m = -2x^3\frac{\gamma_m\gamma_{m+2}}{\gamma_{m+1}^2}, \tag{286}$$

and

$$p_0 = p = -4x; \tag{287}$$

thence

$$p_1 = \frac{2q_1}{p}, \tag{288}$$

and, since
$$p_m p_{m-1} = 2q_m, \tag{289}$$

$$p_{2a} = \frac{q_{2a}}{q_{2a-1}}\,\frac{q_{2a-2}}{q_{2a-3}}\dots\frac{q_2}{q_1}\,p = -4x\left(\frac{\gamma_2\gamma_4\gamma_6\dots\gamma_{2a}}{\gamma_1\gamma_3\gamma_5\dots\gamma_{2a-1}}\right)^4\frac{\gamma_{2a+2}}{\gamma_{2a+1}^3}, \tag{290}$$

$$p_{2a+1} = 2\,\frac{q_{2a+1}}{q_{2a}}\,\frac{q_{2a-1}}{q_{2a-2}}\dots\frac{q_3}{q_2}\,\frac{q_1}{p} = \left(\frac{\gamma_1\gamma_3\gamma_5\dots\gamma_{2a+1}}{\gamma_2\gamma_4\gamma_6\dots\gamma_{2a+2}}\right)^4\frac{\gamma_{2a+2}\gamma_{2a+3}}{x^2}. \tag{291}$$

Written in Halphen's notation (*Fonctions Elliptiques*, II, p. 582) Abel's continued fraction expression is

$$\sqrt{\{(x^2+ax+b)^2+px\}}$$
$$= (x^2+ax+b)+1 : \frac{2(x+g)}{p}+1 : \frac{2(x+g_1)}{p_1}+1 : \dots$$
$$\dots : \frac{2(x+g_m)}{p_m}+1 : \dots$$
$$= (x^2+ax+b)+p : 2(x+g)+pp_1 : 2(x+g_1)+p_1p_2 : \dots$$
$$: 2(x+g_{m-1})+p_{m-1}p_m : 2(x+g_m)+p_m p_{m+1} : \dots, \tag{292}$$

and we find
$$g_m = \tfrac{1}{2}\,\frac{\wp'(m+1)v-\wp'v}{\wp(m+1)v-\wp v}, \tag{293}$$

with
$$p_{m-1}p_m = 2q_m = 4\{\wp(m+1)v-\wp v\}; \tag{294}$$

so that the continued fraction is readily written down when it is periodic, and
$$\gamma_n = 0, \tag{295}$$

leading to a pseudo-elliptic integral.

41. But without having recourse to the transformations of the even orders, we can obtain the resolution of the cubic

$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2 \tag{296}$$

by means of Halphen's expressions for his x and y in terms of a and γ on p. 377, t. II, *Fonctions Elliptiques*,

$$x = -\frac{\{a^2-2a\gamma(2\gamma^2-3\gamma+2)+\gamma^4\}^2}{2^8a^2(a-1)^2\gamma^4(\gamma-1)^4}, \tag{297}$$

$$y = -\frac{(\gamma^2-a)(\gamma^2-2\gamma+a)(\gamma^2-2a\gamma+a)}{2^3a(a-1)\gamma^2(\gamma-1)^2}. \tag{298}$$

Now, if s_α, s_β, s_γ denote the roots of the cubic (1), we can put

$$s_\alpha = M^2(\gamma^2-2\gamma+\alpha)^2, \tag{299}$$

$$s_\beta = M^2(\gamma^2-2\alpha\gamma+\alpha)^2, \tag{300}$$

$$s_\gamma = M^2(\gamma^2-\alpha)^2, \tag{301}$$

where
$$M = \frac{\alpha^2-2\alpha\gamma(2\gamma^2-3\gamma+2)+\gamma^4}{2^4\alpha(\alpha-1)\gamma^2(\gamma-1)^2}. \tag{302}$$

42. But, if we put

$$\alpha = \frac{q^2}{p^2}\,\frac{1-p^2}{1-q^2}, \quad \gamma = \frac{1-p^2}{1-q^2}, \tag{303}$$

then
$$s_\alpha = N^2(1+p^2-q^2)^2, \tag{304}$$

$$s_\beta = N^2(1-p^2+q^2)^2, \tag{305}$$

$$s_\gamma = N^2(1-p^2-q^2)^2, \tag{306}$$

where
$$N = \frac{1-2(p^2+q^2)+(p^2-q^2)^2}{16p^2q^2}, \tag{307}$$

and then

$$\begin{aligned} x &= -\frac{\{1-2(p^2+q^2)+(p^2-q^2)^2\}^3}{2^8p^4q^4} \\ &= -\frac{\{(1+p+q)(1+p-q)(1-p+q)(1-p-q)\}^3}{2^8p^4q^4}, \end{aligned} \tag{308}$$

$$y = -\frac{(1+p^2-q^2)(1-p^2+q^2)(1-p^2-q^2)}{2^8p^2q^2}. \tag{309}$$

43. In the poristic problem of the polygon of n sides, inscribed in a circle of radius R and circumscribed to a circle of radius r, the centres being a distance c apart, we may put

$$\left.\begin{aligned} p &= \operatorname{cn}\frac{2K}{n} = \frac{r}{R-c} \\ q &= \operatorname{sn}\left(K-\frac{2K}{n}\right) = \frac{r}{R+c} \end{aligned}\right\}, \tag{310}$$

and
$$\left.\begin{aligned} \kappa^2 &= 1-\alpha = \frac{p^2-q^2}{p^2-p^2q^2} = \frac{4Rc}{(R+c)^2-r^2} \\ \gamma &= \operatorname{dn}^2\left(K-\frac{2K}{n}\right) = \left(\frac{R-c}{R+c}\right)^2 \end{aligned}\right\}. \tag{311}$$

44. Putting
$$\left.\begin{aligned}4\mu &= 1-(p+q)^2\\ 4\nu &= 1-(p-q)^2\end{aligned}\right\}, \qquad (312)$$

then
$$N = \frac{\mu\nu}{(\mu-\nu)^2}, \qquad (313)$$

$$x = -\frac{16\mu^3\nu^3}{(\mu-\nu)^4}, \qquad (314)$$

$$y = -\frac{(\mu+\nu)(\mu+\nu-4\mu\nu)}{(\mu-\nu)^2}. \qquad (315)$$

and then, putting
$$\mu+\nu = 2a, \quad \mu\nu = \beta,$$

$$(\mu-\nu)^2 = 4\,(a^2-\beta), \qquad (316)$$

then
$$x = -\frac{\beta^3}{(a^2-\beta)^2}, \qquad (317)$$

$$y = -\frac{a\,(a-2\beta)}{a^2-\beta}, \qquad (318)$$

$$N = \frac{\beta}{4\,(a^2-\beta)}, \qquad (319)$$

and
$$s_r = \frac{a^2\beta^2}{(a^2-\beta)^2}. \qquad (320)$$

45. We shall find it convenient to use the symbol s_r to denote the value of s corresponding to the aliquot part $2r\omega_\gamma/n$ of the period; it also simplifies the expressions to put

$$\beta = ma;$$

and now
$$x = -\frac{m^3a}{(a-m)^2}, \qquad (321)$$

$$y = -\frac{(1-2m)\,a}{a-m}, \qquad (322)$$

$$y+1 = \frac{(2a-1)\,m}{a-m}; \qquad (323)$$

and we find, after reduction

$$s_\gamma - s_1 = s_\gamma + x \quad = \frac{m^2 a}{a-m}, \tag{324}$$

$$s_\gamma - s_2 = s_\gamma \quad = \frac{m^2 a^2}{(a-m)^2}, \tag{325}$$

$$s_\gamma - s_3 = s_\gamma + x - y = \frac{(1-m)^2 a}{a-m}, \tag{326}$$

so that

$$\frac{s_\gamma - s_2}{s_\gamma - s_1} = \frac{a}{a-m}, \tag{327}$$

$$\frac{s_\gamma - s_3}{s_\gamma - s_1} = \left(\frac{1-m}{m}\right)^2, \tag{328}$$

$$s_\gamma - s_4 = \left\{\frac{m(1-2m)\,a - m^2(1-m)}{(1-2m)(a-m)}\right\}^2, \tag{329}$$

$$s_\gamma - s_5 = \frac{m^2 a}{a-m}\left\{\frac{(1-2m)\,a - m^2(1-m)}{(1-2m)\,a - m(1-m)^2}\right\}^2, \tag{330}$$

$$s_\gamma - s_6 = \frac{a^2}{(a-m)^2}\left\{\frac{(1-2m+2m^2)(1-2m)\,a - m(1-m)(1-3m+3m^2)}{2(1-m)(1-2m)\,a - m(1-m)^2}\right\}^2, \tag{331}$$

$$s_\gamma - s_7 = \frac{m^2 a}{a-m}$$

$$\times\left\{\frac{(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)\,a + m^2(1-m)^4}{(1-2m)^3 a^2 - m(1-m)(1-2m)(1-3m)\,a - m^4(1-m)^2}\right\}^2, \tag{332}$$

$$s_\gamma - s_8 = \frac{m^2}{(1-2m)^2(a-m)^2}$$

$$\times\left\{\frac{(1-2m)^3 a^3 - m(1-m)(1-2m)^2(1+2m-2m^2)\,a^2 + 4m^3(1-m)^3(1-2m)\,a - m^4(1-m)^4}{\{(1-2m)\,a - m(1-m)\}\times\{(1-2m)\,a - m(1-m)(1-2m+2m^2)\}}\right\} \tag{333}$$

Mr. G. H. Stuart is engaged on the calculation of the succeeding equations, and he has found

$$s_\gamma - s_9 = \frac{m^2 a}{a-m} - x^3\frac{\gamma_8\gamma_{10}}{\gamma_9^2}$$

$$= \frac{(1-m)^2 a}{a-m}\left\{\frac{Aa^3 - Ba^2 + Ca - D}{Pa^3 - Qa^2 + Ra - S}\right\}^2, \tag{334}$$

where
$$\left.\begin{aligned} A &= (1-2m)^3(1-2m+4m^2) \\ B &= 2m(1-m)(1-2m)^2(1-3m+5m^2) \\ C &= m^2(1-m)^2(1-2m)(1-4m+7m^2) \\ D &= m^6(1-m)^3 \\ P &= (1-2m)^3(3-6m+4m^2) \\ Q &= 2m(1-m)(1-2m)^2(3-7m+5m^2) \\ R &= m^2(1-m)^2(1-2m)(4-10m+7m^2) \\ S &= m^3(1-m^6) \end{aligned}\right\}, \tag{335}$$

$$s_\gamma - s_{10} = \frac{m^2a^2}{(a-m)^2}$$
$$\times\left[\frac{Aa^4-Ba^3+Ca^2-Da+E}{\{(1-2m)a-m(1-m)^2\}\{(1-2m)a-m^2(1-m)\}(Pa^2-Qa+R)}\right]^2,$$

where
$$\left.\begin{aligned} A &= (1-2m)^6 \\ B &= 2m(1-m)(1-2m)^3(2-7m+7m^2) \\ C &= 2m^2(1-m)^2(1-2m)^2(3-11m+13m^2-4m^3+2m^4) \\ D &= m^3(1-m)^3(1-2m)(4-17m+27m^2-20m^3+10m^4) \\ E &= m^4(1-m)^4(1-5m+10m^2-10m^3+5m^4) \\ P &= (1-2m)^4 \\ Q &= m(1-m)(1-2m)(1-6m+6m^2) \\ R &= -m^3(1-m)^3 \end{aligned}\right\}. \tag{336}$$

So also for $s_\gamma - s_{11}$, $s_\gamma - s_{12}$, &c.

The form of the denominator of
$$s_\gamma - s_n$$
can be inferred by putting
$$s_\gamma - s_p = s_\gamma - s_{n-p},$$
and of the numerator by putting
$$(s_\gamma - s_p)(s_\gamma - s_{n-p}) = (s_\alpha - s_\gamma)(s_\beta - s_\gamma); \tag{337}$$
and these relations serve as a check upon the preceding results.

These equations, (324) to (337), ..., provide a simple method of determining the division values of elliptic functions of the second stage in terms of a single parameter; for putting

$$s_\gamma - s_n = 0, \quad \text{or} \quad s_\gamma - s_k = s_\gamma - s_{n-k}, \tag{338}$$

gives a relation by which it is possible to express a and m in terms of a single parameter.

Thus, for instance, from the relation

$$s_\gamma - s_9 = s_\gamma - s_{10}, \tag{339}$$

we obtain the elliptic functions, sn, cn, and dn, of the nineteenth part of a period.

46. Expressed by a linear and quadratic factor,

$$S = 4s(s+x)^2 - \{(y+1)s + xy\}^2$$

$$= \left\{s - \frac{m^2a^2}{(a-m)^2}\right\}\left\{4s^2 + \frac{4(1-2m)a-1}{(a-m)^2}m^2s + \frac{m^4(1-2m)^2a^2}{(a-m)^4}\right\}, \tag{340}$$

so that, denoting the roots of the quadratic factor by s_α and s_β, we find

$$(s_\alpha - s_\gamma)(s_\beta - s_\gamma) = \frac{m^4a^2(a-m+1)}{(a-m)^3}, \tag{341}$$

$$(s_\alpha - s_\beta)^2 = m^4\frac{1-8(1-2m)a}{16(a-m)^4}, \tag{342}$$

and, according to the order of magnitude of s_α, s_γ, s_β, we may put

$$\frac{\kappa^2}{\kappa'^4}, \quad \text{or} \quad \frac{\kappa'^2}{\kappa^4}, \quad \text{or} \quad -\kappa^2\kappa'^2$$

$$= \frac{(s_\alpha - s_\gamma)(s_\beta - s_\gamma)}{(s_\alpha - s_\beta)^2}$$

$$= 16\frac{a^2(a-m)(a-m+1)}{1-8(1-2m)a}. \tag{343}$$

We may also determine Kiepert's function T (*Math. Ann.*, XXXII, p. 26) by the relation

$$T^2 = \prod_{r=1}^{r=\frac{1}{2}(n-1)} \frac{s_r - s_\gamma}{s_r - s_{2r}}. \tag{344}$$

47. According to Halphen (*F. E.*, II, p. 407) x and y are given as functions of v by the relations

$$x = \gamma_3^3(v) = \frac{\sigma^3 3v \sigma^3 v}{\sigma^8 2v}, \tag{345}$$

$$y = \gamma_4(v) = \frac{\sigma 4v \sigma^4 v}{\sigma^5 2v}. \tag{346}$$

Denoting the values when v is changed into pv by x_p and y_p, then (*F. E.*, I, p. 106)

$$x_p = \frac{\gamma_{3p}^3 \gamma_p^5}{\gamma_{2p}^8} = \frac{(s_{2p} - s_p)^3}{S_p}, \tag{347}$$

$$y_p = \frac{\gamma_{4p} \gamma_p^4}{\gamma_{3p}^5} = \sqrt{\frac{S_{2p}}{S_p}}, \tag{348}$$

and now the substitution

$$s - s_{2p} = M^2 t, \tag{349}$$

or

$$s - s_p = M^2 (t + x_p), \tag{350}$$

where

$$M^2 x_p = s_{2p} - s_p$$

$$= x^{\frac{1}{3}} \frac{\gamma_{3p}}{\gamma_{2p}^2 \gamma_p}, \tag{351}$$

or

$$M = x^{\frac{1}{3}} \frac{\gamma_{2p}^3}{\gamma_p^3 \gamma_{3p}}, \tag{352}$$

changes $S(s;\ x, y) = 4s(s+x)^2 - \{(y+1)s + xy\}^2$

into $M^6 S(t;\ x_p, y_p)$,

and makes

$$\frac{M\,ds}{\sqrt{\{S(s;\ x, y)\}}} = \frac{dt}{\sqrt{\{S(t;\ x_p, y_p)\}}}; \tag{353}$$

in this way the various permutations of the division-values of argument pv are obtained.

48. Thus, for instance, when $n=11$, and

$$\gamma_{11}=0,$$

we find
$$1-z_2=\frac{x_2}{y_2}=\frac{\gamma_6^3}{\gamma_8\gamma_4^3}$$

$$=\frac{x(y-x-y^2)^3}{y^4\{x(y-x-y^2)-(y-x)^2\}}$$

$$=\frac{z^2(1-z)}{(p-z)^2(1-z-p)}$$

$$=\frac{q^2(1+q)}{(1+c+q)^2(q-c)}$$

$$=\frac{cq(1+c)^2}{(1+c+q)^2(q-c)}$$

$$=\frac{4c(1+c)^2(\sqrt{C}-1)^2}{(1+2c+\sqrt{C})^2(-1-2c+\sqrt{C})}$$

$$=\frac{(1+c)(1+2c+2c^2-\sqrt{C})}{2c^3}.$$

But, if
$$c_2=\frac{1+2c-\sqrt{C}}{2c^2},$$

$$\sqrt{C_2}=\frac{1+3c+4c^2+c^3-(1+c)\sqrt{C}}{c^3},$$

so that
$$1-z_2=\frac{\sqrt{C_2}+1}{2},$$

while
$$1-z_1=\frac{\sqrt{C}+1}{2},$$

and z_1 is thus changed into z_2 by changing c into

$$\frac{1+2c-\sqrt{C}}{2c^2};$$

and the same substitution changes x_1 and y_1 into $M_2^{-2}x_2$ and $M_2^{-2}y_2$, where

$$M_2=x^3\frac{\gamma_4^3}{\gamma_6}=\frac{(1+2c+\sqrt{C})^2}{4(1+c)}.$$

Similarly, a change of

$$c \text{ into } \frac{-1-4c-2c^2-\sqrt{C}}{2(1+c)^2}$$

changes $x_1, y_1, z_1, \ldots$ into $M_3^{-2}x_3, M_3^{-2}y_3, z_3, \ldots$;

and so on.

Calculating the division-values

$$\wp(1, 2, 3, 4, 5)\frac{2\omega_3}{11},$$

for $$x = x_1, \quad y = y_1,$$

we find, with $$v = \frac{2\omega_3}{11},$$

$$24(1+c)^2\wp v = -2+2c+27c^2+42c^3+18c^4-2c^5-2c^6$$
$$+(6c+13c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 2v = -2-10c-33c^2-66c^3-66c^4-26c^5-2c^6$$
$$+(-6c-23c^2-28c^3-10c^4)\sqrt{C},$$

$$24(1+c)^2\wp 3v = -2-10c-33c^2-30c^3-6c^4-2c^5-2c^6$$
$$+(-6c+c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 4v = -2-22c-69c^2-102c^3-78c^4-26c^5-2c^6$$
$$+(6c+13c^2+8c^3+2c^4)\sqrt{C},$$

$$24(1+c)^2\wp 5v = -2-10c-21c^2-18c^3-6c^4-2c^5-2c^6$$
$$+(-6c-11c^2-4c^3+2c^4)\sqrt{C},$$

so that

$$24(1+c)^2 G_1 = 24\sum_{r=1}^{r=5}\wp rv$$
$$= -10-50c-129c^2-174c^3-138c^4-58c^5-10c^6$$
$$+(-6c-7c^2-8c^3-2c^4)\sqrt{C},$$

and the preceding group of substitutions merely permutes these division-values, and changes the homogeneity factor M.

49. The value of $x' = x_{4n}$ may be derived from $x = x_{2n}$ in the following manner:—

Put
$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2$$
$$= 4(s-s_\alpha)(s-s_\beta)(s-s_\gamma),$$

so that
$$s_\alpha + s_\beta + s_\gamma = \tfrac{1}{4}(y+1)^2 - 2x,$$
$$s_\beta s_\gamma + s_\gamma s_\alpha + s_\alpha s_\beta = x^2 - \tfrac{1}{2}xy(y+1),$$
$$s_\alpha s_\beta s_\gamma = -x^2y^2;$$

and put
$$s\left(\frac{\omega}{n}\right) - s\left(\frac{\omega}{2n}\right) = x_{4n} = x',$$
$$s\left(\frac{\omega}{n}\right) = x_{2n} = x,$$

so that
$$s\left(\frac{\omega}{2n}\right) = x - x'.$$

Then from the formula
$$2s\left(\frac{\omega}{2n}\right) + s\left(\frac{\omega}{n}\right) - s_\alpha - s_\beta - s_\gamma = \tfrac{1}{4}\left\{\frac{s''\dfrac{\omega}{2n}}{s'\dfrac{\omega}{2n}}\right\}^2,$$

where
$$s'^2\frac{\omega}{2n} = 4(x-x'-s_\alpha)(x-x'-s_\beta)(x-x'-s_\gamma),$$
$$\tfrac{1}{2}s''\frac{\omega}{2n} = (x-x'-s_\beta)(x-x'-s_\gamma)$$
$$+(x-x'-s_\gamma)(x-x'-s_\alpha)$$
$$+(x-x'-s_\alpha)(x-x'-s_\beta),$$

we obtain, after reduction, the equation
$$x'^4 - x(y+1)x'^2 - 2x^2x' - x^3 = 0,$$

or, putting
$$x' = \frac{x}{r},$$

$$r = \frac{x_{2n}}{x_{4n}} = \frac{s\left(\dfrac{2\omega}{n}\right) - s\left(\dfrac{\omega}{n}\right)}{s\left(\dfrac{\omega}{n}\right) - s\left(\dfrac{\omega}{2n}\right)},$$

$$r^4 + 2r^3 + (y+1)r^2 - x = 0.$$

To solve this quartic equation, write it in the form

$$(r^2+r+t)^2 = (2t-y)\,r^2+2tr+t^2+x\,;$$

when the right-hand side will be a perfect square if

$$(t^2+x)(2t-y)-t^2 = 0,$$

or

$$2t^3-t^2\,(y+1)+2tx-xy = 0,$$

or

$$4t^2\,(t^2+x)^2-\{(y+1)\,t^2+xy\}^2 = 0,$$

so that we can take, from (321), (322), (325),

$$t^2 = s_7,$$

$$t = \sqrt{s_7} = \frac{ma}{a-m},$$

$$(2t-y)\,r^2+2tr+t^2+x = \frac{a}{a-m}\,(r+m)^2,$$

and thus the quartic for r or x' may be resolved.

As a preliminary verification, take $2n = 6$; then we can put

$$s_7 = y^2, \quad t = y$$

(*Proc. Lond. Math. Soc.*, Vol. xxv, p. 216); then

$$(r^2+r+y)^2 = y\,(r+1)^2,$$

$$r^2-r\,(\sqrt{y}-1)+\sqrt{y}\,(\sqrt{y}-1) = 0,$$

$$\left(r-\frac{\sqrt{y}-1}{2}\right)^2 = \frac{(\sqrt{y}-1)(-3\sqrt{y}-1)}{4}, \text{ or } \frac{(1-\sqrt{y})(1+3\sqrt{y})}{4}.$$

This quantity $y = y_6$ is found to be connected with the parameter $a = a_{12}$ by the relation

$$y = \frac{a^2}{(1+a+a^2)^2},$$

and taking

$$\sqrt{y} = -\frac{a}{1+a+a^2},$$

$$1-\sqrt{y} = \frac{(1+a)^2}{1+a+a^2},$$

$$1+3\sqrt{y} = \frac{(1-a)^2}{1+a+a^2},$$

so that

$$r = -\frac{a+a^2}{1+a+a^2},$$

[redacted]

and this agrees with the value of

$$r = \frac{x_6}{x_{12}} = \frac{s\frac{2}{3}\omega - s\frac{1}{3}\omega}{s\frac{1}{3}\omega - s\frac{1}{6}\omega} = \frac{-x + x\dfrac{y-x}{y^2}}{x} = \frac{y-x-y^2}{y^2}$$

$$= \frac{z}{p-z} = -\frac{a+a^2}{1+a+a^2}.$$

Passing on to the case of $2n = 10$, we have (*Proc. Lond. Math. Soc.*, Vol. xxv, p. 236)

$$x = x_{10} = -\frac{a(1+a)}{(1-a)(1-a-a^2)^2},$$

$$y = y_{10} = \frac{-a(1+a)}{(1-a)(1-a-a^2)},$$

$$t = \sqrt{s_7} = \frac{-a}{(1-a)(1-a-a^2)},$$

$$\left\{r^2 + r - \frac{a}{(1-a)(1-a-a^2)}\right\}^2 = \frac{-a}{1-a-a^2}\left(r + \frac{1}{1-a}\right)^2,$$

so that, putting $\qquad A = -a + a^2 + a^3,$

$$r^2 + r - \frac{a}{(1-a)(1-a-a^2)} = \frac{(1-a)r+1}{(1-a)(1-a-a^2)}\sqrt{A},$$

$$\left(r + \frac{1}{2}\,\frac{1-a-a^2+\sqrt{A}}{1-a-a^2}\right)^2 = \frac{(1+a)(1+a^2-2\sqrt{A})}{4(1-a)(1-a-a^2)},$$

and thence $\qquad r = \dfrac{s\frac{2}{5}\omega - s\frac{1}{5}\omega}{s\frac{1}{5}\omega - s\frac{1}{10}\omega}$ and $s\left(\dfrac{\omega}{10}\right)$

can be found; so that the case of $\mu = 20$ can now be considered as solved.

With $\qquad 2n = 12,$

$$r = \frac{s\frac{1}{3}\omega - s\frac{1}{6}\omega}{s\frac{1}{6}\omega - s\frac{1}{12}\omega},$$

and now we take ("Pseudo-Elliptic Integrals," p. 248)

$$t = -\frac{a+a^2+a^3}{1-a} = \frac{A}{1-a},$$

so that $\qquad \left(r^2 + r + \dfrac{A}{1-a}\right)^2 = \left(r + \dfrac{1}{1-a}\right)^2 A,$

and thence r; this solves the case of $\mu = 24$.

With $$2n = 14,$$

$$r = \frac{\wp\frac{2}{7}\omega - \wp\frac{1}{7}\omega}{\wp\frac{1}{7}\omega - \wp\frac{1}{14}\omega},$$

and we take

$$t = \sqrt{s_7} = \frac{c\,(1+c)}{2\,(1+c-2c^2-c^3)}\{c\,(1+c)(1-2c)+(1-c)\sqrt{C}\},$$

where $$C = c\,(1+2c)(4+5c+2c^2),$$

$$y = y_{14} = c\,\frac{0+3c+6c^2-4c^3-10c^4-c^5+(1+2c-2c^2-2c^3)\sqrt{C}}{2\,(1+c)(1+c-2c^2-c^3)},$$

$$x = x_{14} = c\,(1+c)\,\frac{\begin{array}{r}0+3c+6c^2-9c^3-21c^4+0+16c^6+8c^7\\ +(1+2c-3c^2-5c^3+2c^4+4c^5)\sqrt{C}\end{array}}{(1+c-2c^2-c^3)^2},$$

so that $\wp\frac{1}{14}\omega$ can be found, which solves the case of $\mu = 28$.

In a similar way the case of $\mu = 32$. can be derived from $\mu = 16$, $\mu = 36$ from $\mu = 18$, $\mu = 44$ from $\mu = 22$, &c.

50. Considering now the transformation of order n and $2n$ together, the x, y, z, ... obtained for a transformation of order n will be the x_2, y_2, z_2, ... for the order $2n$.

Thus, starting with $$n = 5,$$

the relation $$\gamma_5 = y - x = 0, \tag{354}$$

leads to $$a = \frac{m\,(1-m)^2}{1-2m}, \tag{355}$$

$$a - m = \frac{m^3}{1-2m}, \tag{356}$$

so that $$y = x = -\left(\frac{1-m}{m}\right)^2(1-2m). \tag{357}$$

But, from the transformation of the Tenth order ("Pseudo-Elliptic Integrals," p. 235),

$$y_5 = \frac{\gamma_8}{\gamma_5^4} = -\frac{y^3z^3\,(z+p-1)}{y^5p} = \frac{a^2-a^3}{1-a}, \tag{358}$$

so that we must take $$m = \frac{1}{1+a}. \tag{359}$$

Otherwise, $$\frac{s_7-s_3}{s_7-s_1}=\frac{s_5'-s_4'}{s_5'-s_2'} \tag{360}$$

where the accented letters refer to the transformation of order $2n$; so that

$$\left(\frac{1-m}{m}\right)^2=\frac{\gamma_9\gamma_1}{\gamma_5^2\gamma_4^2}\,\frac{\gamma_5^2\gamma_3^2}{\gamma_7\gamma_8}=\frac{\gamma_9}{\gamma_7\gamma_4^2\gamma_8}=\frac{p+c}{p-z}. \tag{361}$$

Therefore, for the Tenth order,

$$\left(\frac{1-m}{m}\right)^2=\frac{p-1-a}{p-z}=a^2, \tag{362}$$

as before in (359).

With $$n=6,$$

$$x=y-y^2; \tag{363}$$

and therefore either $$m=1, \tag{364}$$

or $$a=\frac{m\,(1-m)}{2\,(1-2m)}. \tag{365}$$

With $$m=1,$$

$$\frac{a}{a-1}=\frac{s_7-s_2}{s_7-s_1}=\frac{s_6'-s_4'}{s_6'-s_2'}$$

$$=\frac{\gamma_{10}}{\gamma_8\gamma_4^3}=\frac{\lambda^4}{\gamma_4^4}, \tag{366}$$

$$\sqrt{\left(\frac{a}{a-1}\right)}=\frac{\gamma_7}{\gamma_5\gamma_4^3}=-\frac{z^2}{cpy}$$

$$=\frac{z}{(1+a)(p-z)}$$

$$=\frac{-a}{1+a+a^2} \tag{367}$$

("Pseudo-Elliptic Integrals," p. 248),

$$\alpha=\frac{-a^2}{(1+a)^2(1+a^2)}. \tag{368}$$

But, with (365),

$$a = \frac{m(1-m)}{2(1-2m)},$$

$$\frac{a}{a-m} = -\frac{1-m}{1-3m} = \frac{a^2}{(1+a+a^2)^2},$$

$$m = \frac{1+2a+4a^2+2a^3+a^4}{1+2a+6a^2+2a^3+a^4}, \tag{369}$$

$$a = \frac{a^2(1+2a+4a^2+2a^3+a^4)}{(1+2a+6a^2+2a^3+a^4)(1+2a+8a^2+2a^3+a^4)}. \tag{370}$$

51. But, without these details, we notice that the transformation is effected, in terms of a single parameter, either by putting

$$s_\gamma - s_n = \infty, \tag{371}$$

for the order n; or

$$s_\gamma - s_n = 0, \tag{372}$$

$$s_\gamma - s_{2n} = \infty, \tag{373}$$

for the order $2n$.

In this way we obtain either

$$m = 1,$$

or

$$a = \frac{m(1-m)}{2(1-2m)},$$

for the order $2n = 6$.

For the Eighth order, put

$$s_\gamma - s_4 = 0,$$

or (329)

$$a = \frac{m(1-m)}{1-2m}, \quad a-m = \frac{m^2}{1-2m}; \tag{374}$$

so that

$$x = -(1-m)(1-2m), \tag{375}$$

$$y = -\frac{(1-m)(1-2m)}{m}, \tag{376}$$

and

$$m = 1-z \tag{377}$$

(" Pseudo-Elliptic Integrals," p. 226).

For the Tenth order, put

$$s_\gamma - s_5 = 0,$$

or (330)

$$a = \frac{m^2(1-m)}{1-2m}, \tag{378}$$

$$x = -\frac{m^3(1-m)(1-2m)}{(1-3m+m^2)^2}, \tag{379}$$

$$y = \frac{m(1-m)(1-2m)}{1-3m+m^2}, \tag{380}$$

so that

$$m = \frac{1}{1-a} \tag{381}$$

(" Pseudo-Elliptic Integrals," p. 236).

For the Twelfth order, put (331)

$$s_\gamma - s_6 = 0, \quad \text{or} \quad a = \frac{m(1-m)(1-3m+3m^2)}{(1-2m)(1-2m+2m^2)}, \tag{382}$$

$$a - m = \frac{m^4}{(1-2m)(1-2m+2m^2)}, \tag{383}$$

$$x = -\frac{(1-m)(1-2m)(1-2m+2m^2)(1-3m+3m^2)}{m^4}, \tag{384}$$

$$y = -\frac{(1-m)(1-2m)(1-3m+3m^2)}{m^3}, \tag{385}$$

$$1 - z = \frac{y}{x} = \frac{1-2m+2m^2}{m} = 1 + \frac{a+a^2}{1-a}, \tag{386}$$

so that

$$m = \frac{1}{1-a} \tag{387}$$

(" Pseudo-Elliptic Integrals," p. 248).

52. For the Fourteenth order, put (332)

$$s_\gamma - s_7 = 0, \tag{388}$$

or

$$(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)\,a + m^2(1-m)^4 = 0, \tag{389}$$

so that, putting

$$(1-2m)\,a = m(1-m)\,\gamma,$$

$$(1-2m)\,\gamma^2 - (2-3m)\,\gamma + (1-m)^2 = 0. \tag{390}$$

We connect up with the results on p. 257 of "Pseudo-Elliptic Integrals," by calculating

$$\left(\frac{1-m}{m}\right)^2 = \frac{s_7-s_3}{s_7-s_1} = \frac{\gamma_{10}\gamma_4}{\gamma_8\gamma_6\gamma_8^2} = \frac{\lambda^4\gamma_4^2}{\gamma_6^2\gamma_3^2}, \tag{391}$$

$$\frac{1-m}{m} = \frac{\lambda^2\gamma_4}{\gamma_6\gamma_3} = \frac{\gamma_8\gamma_4}{\gamma_6^2\gamma_3} = -\frac{(1+c)(p-z)}{c(1-z)}$$

$$= \frac{-c+\sqrt{C}}{2(1+c)^2}, \tag{392}$$

so that

$$m = \frac{2+3c+2c^2-\sqrt{C}}{2} \tag{393}$$

and this makes

$$z_7 = -\frac{m}{c}, \tag{394}$$

$$\sqrt{s_7} = \frac{c(1+c)}{2(1+c-2c^2-c^3)}\{c(1+c)(1-2c)+(1-c)\sqrt{C}\}. \tag{395}$$

For the Sixteenth order ("Pseudo-Elliptic Integrals," p. 262), with

$$s_7-s_8 = 0, \tag{396}$$

$$\left(\frac{1-m}{m}\right)^2 = \frac{s_8-s_3}{s_8-s_1} = \left(\frac{z}{1-z}\right)^2, \tag{397}$$

so that

$$m = 1-z = \frac{1}{a^2+1}. \tag{398}$$

For the Eighteenth order (p. 265), the various relations

$$\gamma_9 = 0, \quad \text{or} \quad s_5 = s_4, \quad s_6 = s_3, \ \&c., \tag{399}$$

will be found to lead to a certain equation between a and m; and, putting

$$a = \frac{m(1-m)}{1-2m}\gamma, \tag{400}$$

$$1-2m = \frac{1}{x-1}, \tag{401}$$

$$1-2\gamma = \frac{1}{y-1}, \tag{402}$$

and

$$y = (1+q)x, \tag{403}$$

we are led to the equation

$$(q+1)^2 x^2 + (q^3+q^2-2q-1)\,x - 2q^2 = 0, \tag{404}$$

having the discriminant

$$(q^3+q^2-2q-1)^2 + 8q^2\,(q+1)^2$$
$$= q^6+2q^5+5q^4+10q^3+10q^2+4q+1 = Q, \tag{405}$$

so that x has here the same signification as on p. 266 of "Pseudo-Elliptic Integrals"; and now the rest of the identification can be effected.

53. But the Twenty-second order is of importance as affording an independent determination for this order of Kiepert's parameters

$$\xi,\ \xi_1,\ \xi_2.$$

We start by putting

$$s_4 - s_5 = s_7 - s_8, \tag{406}$$

and obtaining a quintic equation in a, from (330) and (331).

Putting
$$\frac{a}{a-m} = t^2, \quad a = \frac{mt^2}{t^2-1}, \tag{407}$$

and taking the square root, we obtain a quintic equation in t, and in m; and this, on putting

$$m = \frac{1}{1+n},$$

becomes
$$t^5 - n\,(2-n)(1+n-n^2)\,t^4 - n\,(1+n^2)\,t^3$$
$$+ n^2\,(3-n^2)\,t^2 + n^3\,(1-n+n^2)\,t - n^3 = 0, \tag{408}$$

a quintic in t, and in n.

But the relation
$$\frac{\gamma_7}{\gamma_4} = \left(\frac{\gamma_6}{\gamma_5}\right)^3,$$

which is the equivalent of $\gamma_{11} = 0$,

leads to the (a, m) equation in the form

$$\frac{(1-2m)\,a - m\,(1-m)^2}{(1-2m)(a-m)} + \frac{(1-2m)^2\,a}{m^3}$$
$$= \left\{1 + \frac{(1-2m)^2\,a}{(1-2m)\,a - m\,(1-m)^2}\right\}^3. \tag{409}$$

I am indebted to Professor E. B. Elliott for the substitution

$$1+\frac{(1-2m)^2 a}{(1-2m)\,a-m\,(1-m)^2}=\frac{1-m}{m}\,r,$$

or

$$a=\frac{m\,(1-m)}{1-2m}\,\frac{(1-m)\,r-m}{r-2m}, \tag{410}$$

$$a-m=\frac{m^2}{1-2m}\,\frac{mr+1-3m}{r-2m}, \tag{411}$$

which makes

$$s_7-s_5=\frac{1-m}{m}\,\frac{(1-m)\,r-m}{mr+1-3m}\,(r-m)^2, \tag{412}$$

$$s_7-s_6=\frac{\{(1-m)\,r-m\}^2\,(mr+1-2m)^2}{(mr+1-3m)^2\,r_5^2}, \tag{413}$$

and leads to the equation

$$\begin{aligned} 4m^4&+2\,(r^4-3r^3+r^2-3r-2)\,m^3\\ &-(r^5-r^4-8r^3+r^2-5r-1)\,m^2\\ &+r\,(r^4-4r^3-2r^2+0-1)\,m+r^4=0, \end{aligned} \tag{414}$$

a *quartic* in m; but a quintic in r.

The resolution of this quartic was effected by forming its resolving cubic

$$4s^3-g_2s-g_3=0, \tag{415}$$

and noticing that, if it has a rational root in r, this root must be of the form

$$12s=r^5+Ar^4+Br^3+Cr^2+Dr-1. \tag{416}$$

It was then found that the special numerical values of r,

$$r=1,\ -1,\ 2,\ 3,\ \text{made } 12s=-1,\ -25,\ -19,\ -97;$$

hence $\quad A=-7,\quad B=10,\quad C=-5,\quad D=1;$

and the required root of the general resolving cubic is thus given by

$$12s=r^5-7r^4+10r^3-5r^2+r-1, \tag{417}$$

and this was found to verify; as, putting $12s=e$, the resolving cubic breaks up into the linear factor

$$e-r^5+7r^4-10r^3+5r^2-r+1,$$

and the quadratic factor

$$\begin{aligned} 2e^2&+(r^5-7r^4+10r^3-5r^2+r-1)\,e\\ &-r^{10}+5r^9-6r^8-3r^7-r^6+17r^5-14r^4+3r^3-2r^2+2r-1. \end{aligned}$$

The discriminant of the quadratic factor is

$$9\,(r^{10}-6r^9+13r^8-14r^7+20r^6-28r^5+19r^4-6r^3+3r^2-2r+1)$$

$$=9\left[\{(r^5-3r^4-3r^3+4r^2+r-1)^2+5r^2(r-1)\}^2+8r^4(r-1)^2\right]$$

$$=9r^4(r-1)^2\{(H+5)^2+8\},$$

on putting $$H=\frac{r^5-3r^4-3r^3+4r^2+r-1}{r^2(r-1)}.$$

The quartic (414) can now be written

$$\{4m^2+(r^4-3r^3+r^2-3r-2)\,m-r\,(r^3-3r^2+r-1)\}^2$$
$$=r^2(r-1)^3(r^3-3r^2-r-1)(m-1)^2, \quad (418)$$

and the resolution of the quartic in m is thus effected.

54. Professor Elliott points out further that, if we put

$$y=-\frac{(2m-1)(m-r)}{r\,(r-1)(m-1)}, \quad (419)$$

then the quartic reduces to a quadratic in y,

$$y^2-(r^2-2r-1)\,y+r=0, \quad (420)$$

and further, putting $$r=\frac{x}{x+1}, \quad (421)$$

this quadratic assumes the symmetrical form

$$x^2y^2+2xy\,(x+y)+x^2+4xy+y^2+x+y=0, \quad (422)$$

or, putting $$x+y=p, \quad xy=q, \quad (423)$$

$$(p+q)^2+p+2q=0. \quad (424)$$

Thence we can deduce

$$x=\frac{2c+1+\sqrt{C}}{2c^2}, \quad y=\frac{2c+1-\sqrt{C}}{2c^2}, \quad (425)$$

where $$C=4c\,(c+1)^2+1, \quad (426)$$

so that we may put $$x=\wp\,(u-\tfrac{2}{5}\omega)-\tfrac{2}{3}, \quad (427)$$

$$y=\wp\,(u+\tfrac{2}{5}\omega)-\tfrac{2}{3}, \quad (428)$$

and y is now found to be the reciprocal of c_{22}.

55. Professor Elliott has also made a similar reduction for the equation (263), connecting

$$q = q_{23} \quad \text{and} \quad c = c_{23},$$

by writing it

$$(1+c)^2 q^2 (q-c-c^2)^2 + c^2 q (q-c-c^2)\{(1+2c)\,q-c^2-c^3\}$$
$$-c\,\{(1+2c)\,q-c^2-c^3\}^2 = 0, \qquad (429)$$

and now, if we put

$$z = \frac{q\,(q-c-c^2)}{(1+2c)\,q-c^2-c^3}, \qquad (430)$$

the quartic (429) reduces to the quadratic

$$(1+c)^2 z^2 + c^2 z - c = 0. \qquad (431)$$

Here again, by putting

$$c = \frac{1}{y}, \quad z = -x-1, \qquad (432)$$

the quadratic (431) becomes the same as (422).

The relations connecting this q and c with m and r may be written

$$r = \frac{m}{1-m}\,\frac{q+1+c}{q-c-c^2}$$

$$= -\frac{1+c}{c}\,\frac{q}{q-c^2}\,\frac{q-c-c^2}{q+1+c}; \qquad (433)$$

and the elimination of q between this and (264) leads to the relation

$$(2c+4c^2+c^3)\,r - c^2\,(1+c) + (cr+1+c)\sqrt{(4c+8c^2+4c^3+c^4)} = 0, \qquad (434)$$

or

$$c^2 r^2 - (2c+c^2)\,r - 1 - c = 0, \qquad (435)$$

a quadri-quadric relation between c and r, which becomes the same as (422) when we put, as before,

$$c = \frac{1}{y}, \quad r = \frac{x}{x+1}.$$

The elimination of m between (414) and

$$p = \frac{m(1-2m)}{r(r-3)m+r},$$

where $$p = p_{11} = 1 + c_{11},$$

is found to lead to the equation

$$p^4 - (r^4 - 4r^3 + 5r^2 + 0 + 2)\,p^3$$
$$+ (r^6 - 4r^5 + 5r^4 - 4r^3 + 6r^2 + r + 1)\,p^2$$
$$+ r(r+1)(r^3 - 2r^2 + 0 - 1)\,p + r^3 = 0. \qquad (436)$$

Expressed as the difference of two squares, preparatory to resolution, the equation may be written

$$\left[2p(p - r^2 + r - 1) - r(r^2 - 2r - 1)\{(r-2)p + 1\}\right]^2$$
$$= r^2(r-1)(r^3 - 3r^2 - r - 1)\{(r-2)p + 1\}^2. \qquad (437)$$

In Professor Elliott's procedure, this quartic equation is replaced by two quadratic relations, by putting

$$u = \frac{p(p - r^2 + r - 1)}{r(r-2)p + r}, \qquad (438)$$

and then (436) becomes

$$u^2 - (r^2 - 2r - 1)\,u + r = 0, \qquad (439)$$

the same as (420); so that, u and y, if not equal, are the two roots of this quadratic, and

$$\frac{p(p - r^2 + r - 1)}{r(r-2)p + r} + \frac{(2m-1)(m-r)}{r(r-1)(m-1)} = 0,$$

or $$\frac{p(p - r^2 + r - 1)}{r(r-2)p + r} - \frac{(2m-1)(m-r)}{r(r-1)(m-1)} = r^2 - 2r - 1.$$

Here, as before in (419), it is the relation (438) which still requires interpretation, as an elliptic function formula.

[redacted]

56. We connect up these functions m and r with the z_{11} and p_{11}, employed in the transformation of the Eleventh Order, by the relations

$$\frac{z_{11}}{p_{11}} = \frac{s_4 - s_2}{s_4 - s_1} = \frac{1-m}{m}\, r, \tag{440}$$

after reduction; while

$$z_{11} = \frac{s_3 - s_2}{s_3 - s_1} = \frac{(1-2m)\,a + m\,(1-m)^2}{(1-2m)(a-m)}$$

$$= \frac{(1-m)(1-2m)}{mr+1-3m}, \tag{441}$$

$$p_{11} = \frac{s_2 - s_1}{s_5 - s_1} = \frac{m\,(1-2m)}{(mr+1-3m)\,r}. \tag{442}$$

Expressed in terms of m and r,

$$s_\gamma - s_1 = m\,(1-m)\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{443}$$

$$s_\gamma - s_2 = (1-m)^2 \left\{ \frac{(1-m)\,r-m}{mr+1-3m} \right\}^2, \tag{444}$$

$$s_\gamma - s_3 = \frac{(1-m)^3}{m}\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{445}$$

$$s_\gamma - s_4 = m^2\,(1-m)^2 \left(\frac{r-1}{mr+1-3m} \right)^2, \tag{446}$$

$$s_\gamma - s_5 = \frac{1-m}{m}\,(r-m)^2\,\frac{(1-m)\,r-m}{mr+1-3m}, \tag{447}$$

$$s_\gamma - s_6 = \frac{\{(1-m)\,r-m\}^2\,(mr+1-2m)^2}{(mr+1-3m)^2\,r^2},$$

$$s_\gamma - s_7 = m\,(1-m)\,\frac{(1-m)\,r-m}{mr+1-3m} \left[\frac{(1-m)\,r^2-mr+m\,(1-2m)}{mr^2+(1-3m)\,r-m\,(1-2m)} \right]^2,$$

&c., &c.,

so that

$$\prod_{r=1}^{r=5} (s_\gamma - s_r) = \frac{m\,(1-m)^9\,(r-1)^2\,(r-m)^2\,\{(1-m)\,r-m\}^5}{(mr+1-3m)^7}, \tag{448}$$

while

$$s_1 - s_2 = \frac{(1-m)(1-2m)(r-2m)\{(1-m)\,r-m\}}{(mr+1-3m)^2}, \tag{449}$$

$$s_2 - s_4 = \frac{-(1-m)^2(1-2m)\,r}{(mr+1-3m)^2}, \tag{450}$$

$$s_4 - s_3 = \frac{(1-m)^2\{(1-3m+m^2)\,r^2+(1-2m)(1-3m)\,r-m\,(1-2m)^2\}}{m\,(mr+1-3m)^2}, \tag{451}$$

$$s_3 - s_5 = \frac{(1-m)(r-1)(r+1-2m)\{(1-m)\,r-m\}}{m^2\,(mr+1-3m)}, \tag{452}$$

$$s_5 - s_1 = \frac{(1-m)\,r\,(r-2m)\{(1-m)\,r-m\}}{m\,(mr+1-3m)}; \tag{453}$$

and therefore Kiepert's f is given by

$$f^{-2} = -(s_1-s_2)(s_2-s_4)(s_4-s_3)(s_3-s_5)(s_5-s_1)$$

$$= \frac{(1-m)^7(1-2m)^2 r^2 (r-1)(r-2m)^2(r+1-2m)\{(1-m)\,r-m\}^3}{m^4\,(mr+1-3m)^6}$$

$$\times \{(1-3m+m^2)\,r^2+(1-2m)(1-3m)\,r-m\,(1-2m)^2\}, \tag{454}$$

whence Kiepert's $T^2 = f^2\,\Pi\,(s_\gamma - s_r)$. (455)

57. Also, from (341),

$$(s_\alpha - s_\gamma)(s_\beta - s_\gamma) = \frac{(m-1)^3\{m\,(r+1)-1\}^2\{m^2(r-3)-m(r+2)+r\}}{\{m\,(r-3)+1\}^3}, \tag{456}$$

so that, as in Halphen's *F. E.*, III, p. 245,

$$\xi_1 = \frac{L\,(22)^4}{L\,(11)^8\,L\,(2)^4} = \frac{f\,(22)^4}{f\,(11)^8 f\,(2)^4}$$

$$= \frac{\Pi\,(s_\gamma - s_r)^2}{(s_\alpha - s_\gamma\,.\,s_\beta - s_\gamma)^5}, \tag{457}$$

$$= -\frac{m^2(m-1)^3(r-1)^4(m-r)^4\{m\,(r-3)+1\}}{\{m^2(r-3)+m\,(r+2)-r\}^5}. \tag{458}$$

58. Kiepert's parameter ξ can be calculated from the formula

$$\xi = \Delta \frac{f(22)^2 f(2)^2}{f(11)^2} = \Delta f(2)^{24} T^2 = -\frac{16T^2}{\kappa^2 \kappa'^2}, \tag{459}$$

or

$$\frac{\xi^2}{\xi_1} = \frac{(\Delta f^2)^2}{s_\alpha - s_\gamma \,.\, s_\beta - s_\gamma}; \tag{460}$$

and then we find, in terms of m and r,

$$\xi = -\frac{(r-1)(m-r)^2(2m-1)^2\{4m^2(r+1)-2m(3r+1)+r\}}{r^2(2m-r-1)(m-1)\{2m^2+m(r^2-3r-1)+r\} \times \{m^2(r-3)+m(r+2)-r\}},$$

and thence Kiepert's η, or Klein and Fricke's τ, by the relation

$$\eta + 8 = \frac{1}{\tau} = \frac{\xi^3+4\xi^2+8\xi+4}{\xi}. \tag{461}$$

59. To find the relation between Kiepert's ξ or ξ_1 and our r, we must eliminate m between these equations and the (m, r) equation (414); the work, which is very laborious, has been carried out for me by Mr. G. H. Stuart.

Contrary to anticipation, the equation for ξ_1 in terms of r is the more complicated; it is a quartic in ξ_1, but of the twenty-fifth degree in r; however, it was noticed that the coefficients of ξ_1 could all be expressed rationally in powers of

$$H = \frac{r^5-3r^4-3r^3+4r^2+r-1}{r^2(r-1)}, \tag{462}$$

so that the quartic for ξ_1 could be written

$$\begin{aligned} H^5\xi_1^4 &- (2H^5+80H^4+792H^3+2816H^2+3509H+1331)\,\xi_1^3 \\ &+ (H^5+16H^4+88H^3+184H^2-342H-2651)\,\xi_1^2 \\ &+ (8H^2+75H+143)\,\xi_1 - 1 = 0, \end{aligned} \tag{463}$$

or, resolved as the difference of two squares,

$$\begin{aligned} \{(8H^2+21H+11)\,\xi_1^2 &+ (8H^2+75H+143)\,\xi_1 - 2\}^2 \\ &= \{(H+1)\,\xi_1 - (H+9)\}\, H'^2 \xi_1^2, \end{aligned} \tag{464}$$

where, as in (167), $H'^2 = 4H^3+56H^2+220H+121$

$$= 4H^2(H-11)+(10H+11)^2. \tag{465}$$

Similarly, Mr. G. H. Stuart found that the quartic for ξ in terms of r could be written

$$H(\xi^2+4)^2+(4H-11)(\xi^2+4)\,\xi-(H^2+10H+11)\,\xi^2=0, \quad (466)$$

or

$$\xi+\frac{4}{\xi}=\frac{-4H+11+H'}{2H}. \quad (467)$$

The elimination of H between these two equations (463) and (466) will be found to lead to a reciprocal quartic in ξ_1, which breaks up into Kiepert's two quadratic equations (303) and (306) (*Math. Ann.*, XXXII, p. 92); for, from (467), in Kiepert's notation

$$H'=2\,(u^2-2)\,H-11,$$

so that (464) can be written

$$(8H^2+21H+11)\,\xi_1^2+(8H^2+75H+143)\,\xi_1-2$$
$$+\{(H+1)\,\xi_1^2-(H+9)\,\xi_1\}\{2\,(u^2-2)\,H-11\}=0, \quad (468)$$

and eliminating H between (466) and (468), two quadratics in H, will lead to the result.

60. Writing equation (467)

$$\left(\sqrt{\xi}-\frac{2}{\sqrt{\xi}}\right)^2=u^2-8=\frac{-12H+11+H'}{2H}, \quad (469)$$

then, from (278), $\quad b-11=11^2\,\dfrac{2H}{-12H+11+H'}$

$$=11^2\,\frac{12H-11+H'}{2\,(H-11)^2},$$

$$b=11\,\frac{22H^2+88H+121+11H'}{2\,(H-11)^2}. \quad (470)$$

We may distinguish this H by writing it $H(\theta)$, where θ denotes the elliptic argument; as

$$H(\theta)=\wp\,(\theta;\,g_2,g_3)-\tfrac{14}{3},$$

$$H'(\theta)=\wp'(\theta;\,g_2,g_3),$$

where (*M.F.*, II, p. 444) $\quad g_2=\dfrac{124}{3},\quad g_3=\dfrac{41\,.\,61}{27},$

$$g_2^3-27g_3^2=-11^5,\quad J=-\frac{2^6\,.\,31^3}{3^3\,.\,11^5}; \quad (471)$$

and now we find we can put

$$b = H(\theta + \tfrac{3}{5}\omega),$$

while

$$\xi + \frac{4}{\xi} = \frac{4b+77}{b-11},$$

$$u^2 = \left(\sqrt{\xi} + \frac{2}{\sqrt{\xi}}\right)^2 = \frac{8b+33}{b-11},$$

$$\bar{u}^2 = u^2 - 8 = \left(\sqrt{\xi} - \frac{2}{\sqrt{\xi}}\right)^2 = \frac{121}{b-11}.$$

61. Klein and Fricke's τ is also an elliptic function, which may be distinguished by its elliptic argument ϕ as $\tau(\phi)$; in fact,

$$H(\phi) = -11\tau(\phi), \quad H'(\phi) = 11\tau'(\phi).$$

Let ϕ' denote the argument of τ when ξ is changed into $\frac{4}{\xi}$, the effect of which is to change $2\kappa\kappa'$ into its reciprocal, or to change from a positive to a negative discriminant Δ, or from Klein's J to Kiepert's J; then

$$\tau(\phi) = \frac{\xi}{\xi^3+4\xi^2+8\xi+4},$$

$$\tau(\phi') = \frac{\xi^2}{\xi^3+8\xi^2+16\xi+16},$$

$$\tau'(\phi) = \frac{(\xi^3+2\xi^2+0-2)\,w}{(\xi^3+4\xi+8\xi+4)^2},$$

$$\tau'(\phi') = \frac{(\xi^3+0-16\xi-32)\,w}{(\xi^3+8\xi^2+16\xi+16)^2},$$

where $$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16).$$

Thence, by means of the addition formula

$$H(\phi \pm \phi') + H(\phi) + H(\phi') + 14 = \tfrac{1}{4}\left\{\frac{H'(\phi) - H'(\phi')}{H(\phi) - H(\phi')}\right\}^2,$$

or $$-11\{\tau(\phi\pm\phi') + \tau(\phi) + \tau(\phi')\} + 14 = \tfrac{1}{4}\left\{\frac{\tau'(\phi) \mp \tau'(\phi')}{\tau(\phi) - \tau(\phi')}\right\}^2, \qquad (472)$$

we find, after reduction, that

$$H(\phi+\phi')=\frac{11\xi^2+77\xi+44}{(\xi-2)^2}$$

$$=11+\frac{11^2}{\left(\sqrt{\xi}-\dfrac{2}{\sqrt{\xi}}\right)^2}$$

$$=H(\theta+\tfrac{8}{5}\omega), \tag{473}$$

so that we can take $$\phi+\phi'=\theta+\tfrac{8}{5}\omega. \tag{474}$$

Similarly, with $$u^2=\left(\sqrt{\xi}+\frac{2}{\sqrt{\xi}}\right)^2=\frac{1}{a+1}, \tag{475}$$

so that $$\frac{1}{a+1}-\frac{121}{b-11}=8, \tag{476}$$

as in (275), we find

$$H(\phi-\phi')=\frac{a^5-2a^4-5a^3+2a^2+4a+1}{a^2(a+1)^2}, \tag{477}$$

so that $$\phi-\phi'=\theta', \tag{478}$$

if θ' is the argument of the elliptic function a, which is such that

$$a(\theta')=\wp(\theta';\ g_2, g_3)-\tfrac{2}{3},$$

with $$g_2=\frac{4}{3},\quad g_3=-\frac{19}{27},\quad J=-\frac{2^7}{11};$$

an elliptic function already employed.

These relations (474) and (478) may serve as interpretations of the elliptic function properties implied in equations (419) and (438).

62. Thence $$\phi=\tfrac{1}{2}(\theta+\theta')+\tfrac{4}{5}\omega,$$

so that, starting with c in the transformation of the Eleventh Order, we may put

$$c_{11}=c=\wp\tfrac{1}{2}(\theta+\theta')-\tfrac{2}{3},\quad\text{with}\quad g_2=\tfrac{4}{3},\quad g_3=-\tfrac{19}{27};$$

$$\sqrt{C}=\sqrt{(4c^3+8c^2+4c+1)}=\wp'\tfrac{1}{2}(\theta+\theta'),$$

and then

$$-11\tau = \wp\left\{\tfrac{1}{2}(\theta+\theta')+\tfrac{4}{5}\omega\right\} - \frac{14}{3}, \quad \text{with} \quad g_2 = \frac{124}{3}, \quad g_3 = \frac{41\,.\,61}{27}.$$

$$11\tau' = \wp'\left\{\tfrac{1}{2}(\theta+\theta')+\tfrac{4}{5}\omega\right\}.$$

Putting in (462) $$r = \frac{c}{c+1}$$

gives $$H(\theta) = \frac{c^5-2c^4-5c^3+2c^2+4c+1}{c^2(c+1)^2}, \tag{479}$$

so that $c = c(\theta) = \wp(\theta) - \tfrac{2}{3}$, with $g_2 = \tfrac{4}{3}$, $g_3 = -\tfrac{19}{27}$;

and then $$c_{22} = \frac{1}{c\,(p+\tfrac{2}{5}\omega)}. \tag{480}$$

The duplication formula, for

$$H = H(\theta) \quad \text{and} \quad \tau = \tau(\theta),$$

$$H(2\theta) = -\frac{(H-11)(H^4+11H^3+11H-121)}{4H^3+56H^2+220H+121}, \tag{481}$$

or $$\tau(2\theta) = -\frac{(1+\tau)(1+\tau-11\tau^2+11\tau^3)}{1-20\tau+56\tau^2-44\tau^3}, \tag{482}$$

will often be required in the numerical applications.

63. So also the relation $$s_7 - s_{11} = 0 \tag{483}$$

may be replaced by (337), in the form

$$(s_7-s_5)(s_7-s_6) = (s_\alpha-s_7)(s_\beta-s_7);$$

and, from (447) and (456),

$$\frac{(1-m)(r-m)^2\{(1-m)r-m\}^3(mr+1-2m)^2}{m(mr+1-3m)^3r^3}$$
$$= \frac{(1-m)^3\{(1-m)r-m\}^2\{(1-m-m^2)r-2m+3m^2\}}{(mr+1-3m)^3}, \tag{484}$$

or $$(r-m)^2\{(1-m)r-m\}(mr+1-2m)^2$$
$$= m(1-m)^2\{(1-m-m^2)r-2m+3m^2\}, \tag{485}$$

a quintic in m, and in r; and Mr. James Hammond has found that this (m, r) relation becomes the same as (414) if we write

$$1-m \text{ for } m, \quad \text{and} \quad \frac{(1-m)r}{r-m} \text{ for } r.$$

64. The special numerical values in the cases of Complex Multiplication implied in the Modular Equation of the Eleventh Order provide interesting applications of the preceding theory.

Taking Kiepert's form of the modular equation (325*b*) (*Math. Ann.*, XXXII, p. 98), it will be found that the coefficient of W can be resolved into the factors

$$(\eta+6)(\eta+7)(\eta+1)(\eta+4)(\eta-8)(\eta^2+2\eta-44)(\eta^2+4\eta-16),$$

and these are found to correspond to the cases of complex multiplication where the ratio of the periods

$$\frac{K'}{K}=\sqrt{2},\ \sqrt{7},\ \sqrt{7},\ \sqrt{(19)},\ \sqrt{(43)},\ \sqrt{(10)},\ \sqrt{(35)};$$

and then $L(11)^2$ is found to be the corresponding complex multiplier, so that

$$L(11)^2=3i-\sqrt{2},\quad -2i+\sqrt{7},\quad 2i+\sqrt{7},\quad \tfrac{1}{2}\{-5i+\sqrt{(19)}\},$$
$$\tfrac{1}{2}\{-i+\sqrt{(43)}\},\quad i+\sqrt{(10)},\quad \tfrac{1}{2}\{-3i+\sqrt{(35)}\}.$$

Kiepert's notation can be connected up with that employed by Brioschi (*Annali di Matematica*, XXI, 1893, p. 309) by putting

$$L(11)=z\left\{\text{or } \frac{i\sqrt{(11)}}{z}\right\},\quad l=\frac{z^4}{11^2},\ \&c.$$

65. In a similar manner, when the ratio of the periods

$$\frac{K'}{K}=\sqrt{(22-\rho^2)}=\sqrt{(21)},\ \sqrt{(18)},\ \sqrt{(13)},\ \sqrt{6},$$

we may take $L(22)^2$ as the corresponding complex multiplier, and

$$L(22)^2=1+i\sqrt{(21)},\quad 2+i\sqrt{(18)},\quad 3+i\sqrt{(13)},\quad 4+i\sqrt{6};$$

also,

$$L(2)^{24}=\xi^6\eta'=-64\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^4,\quad 64(\sqrt{3}+\sqrt{2})^8,$$
$$-\{\sqrt{(13)}+3\}^6,\quad 64(\sqrt{2}+1)^4;$$

derived from the corresponding special values of the modulus given in the *Proc. Lond. Math. Soc.*, Vol. XIX, p. 301.

When the ratio of the periods is $\sqrt{(13)}$, we shall find that

$$\xi=2i,\quad \xi^3=-8i,\quad u^2=4,\quad v=10\sqrt{(13)},\quad w=-3\sqrt{(13)},$$
$$\sqrt{\eta'}=5\sqrt{(13)}+18,$$
$$L(2)^{24}=-\frac{16}{\kappa^3\kappa'^3}=\xi^6\eta'=-64\{5\sqrt{(13)}+18\},$$

or
$$2\kappa\kappa'=5\sqrt{(13)}-18=\left\{\frac{\sqrt{(13)}-3}{2}\right\}^3,$$

a verification of the well-known result.

Also $$\tau(\phi) = \frac{2-3i}{26}, \quad \tau'(\phi) = \frac{57+103i}{26\sqrt{(13)}},$$
$$H(\phi+\phi') = -\frac{77}{4}, \quad a(\theta') = -\frac{3}{4}, \text{ \&c.}$$

So also we find that, for
$$\text{K}'/\text{K} = \sqrt{6}, \quad \xi = -1+i\sqrt{3}, \quad \xi^3 = 8, \quad u^2 = 2, \quad v = 6,$$
$$\sqrt{\eta'} = (\sqrt{2}+1)^2;$$
$$\text{K}'/\text{K} = \sqrt{(18)}, \quad \xi = 1+i\sqrt{3}, \quad \xi^3 = -8, \quad u^2 = 6, \quad v = 98,$$
$$\sqrt{\eta'} = (\sqrt{3}+\sqrt{2})^4;$$
$$\text{K}'/\text{K} = \sqrt{(21)}, \quad \xi = i+\sqrt{3}, \quad \xi^3 = 8i,$$
$$u^2 = 4+2\sqrt{3} = (\sqrt{3}+1)^2, \quad v = 32\sqrt{7}+18\sqrt{(21)}, \quad \bar{v} = 84+48\sqrt{3},$$
$$\sqrt{\eta'} = \tfrac{1}{2}(v+\bar{v}) = (2\sqrt{7}+3\sqrt{3})(8+3\sqrt{7}) = \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3 \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2.$$

66. With $$\frac{\text{K}'}{\text{K}} = \sqrt{2},$$
$$L(2)^{24} = \xi^6\eta' = 16\left(\frac{1}{\kappa}-\kappa\right)^2 = 64,$$
$$L(2)^2 = \sqrt{2},$$
and $$\xi^2+4\xi+\frac{4}{\xi} = \eta = -6,$$
$$\xi^3+4\xi^2+6\xi+4 = 0,$$
$$(\xi+2)(\xi^2+2\xi+2) = 0,$$
$$\xi = -2, \text{ or } -1+i.$$
Then $$u = 0, \quad v = 2, \quad w = -14, \quad \eta' = 1,$$
$$\tau = \tfrac{1}{2}; \quad \tau' = \tfrac{1}{4}i\sqrt{2}.$$
Also $a = \infty$, so that $\theta' = 0$, and
$$\phi = \phi' = \tfrac{1}{2}\theta + \tfrac{1}{2}\omega;$$
$$H(\phi) = -\frac{11}{2}, \quad H(2\phi) = -\frac{33}{8};$$
$$H(\tfrac{1}{2}\theta) = -8+i\sqrt{2}, \quad H(\theta) = -5+2i\sqrt{2},$$
so that $$H(\theta)^2+10H(\theta)+33 = 0,$$
and the discriminant of the quartics (263) and (414) vanishes, so that the quartics have a pair of equal roots.

When $$H(\phi) = -\frac{33}{8}, \quad \tau(\phi) = \frac{3}{8}, \quad \tau'(\phi) = \frac{11i\sqrt{2}}{16},$$
then $$\xi = -\frac{4+\sqrt[3]{(44)}}{3}.$$

With
$$\frac{K'}{K} = \sqrt{7},$$
$$2\kappa\kappa' = \tfrac{1}{8},$$
and
$$L(2)^{24} = \xi^6\eta' = -\frac{16}{\kappa^2\kappa'^2} = -2^{12}.$$

This is satisfied by taking
$$\xi = -4, \quad u^2 = -1, \quad \eta' = -1;$$
and then
$$\tau(\phi) = \tfrac{1}{7}, \quad \tau(\phi') = 1;$$
$$\tau'(\phi) = \frac{17i\sqrt{7}}{49}, \quad \tau'(\phi') = i\sqrt{7};$$
$$H(\theta + \tfrac{5}{8}\omega_2) = -\frac{22}{9}, \quad \tau(\theta + \tfrac{5}{8}\omega_2) = \frac{2}{9};$$
$$a(\theta') = 0,$$
so that
$$\theta' = \tfrac{4}{5}\omega.$$

With
$$\frac{K'}{K} = \sqrt{(19)},$$
$$\eta = -4, \quad \tau(\phi) = \tfrac{1}{4}, \quad \tau'(\phi) = \frac{i\sqrt{(19)}}{4},$$
and
$$\xi^3 + 4\xi^2 + 4\xi + 4 = 0.$$

With
$$\frac{K'}{K} = \sqrt{(43)},$$
$$\eta = 8, \quad \tau(\phi) = \frac{1}{16}, \quad \tau'(\phi) = \frac{i\sqrt{(43)}}{32},$$
and
$$\xi^3 + 4\xi^2 - 8\xi + 4 = 0.$$

With
$$\frac{K'}{K} = \sqrt{(10)},$$
$$\xi^6\eta' = 16\left(\frac{1}{\kappa} - \kappa\right)^2 = 64\left(\frac{\sqrt{5}+1}{2}\right)^{12},$$
which we find is satisfied by
$$\xi = -3 - \sqrt{5}, \quad u^2 = -2, \quad v = 2, \quad \eta' = 1.$$

[redacted]

Then $\tau(\phi) = \dfrac{7-3\sqrt{5}}{4}, \quad \tau'(\phi) = \dfrac{11\sqrt{5}-24}{\sqrt{2}}\,i;$

$$\tau(\phi') = \frac{7+3\sqrt{5}}{4}, \quad \tau'(\phi') = \frac{11\sqrt{5}+24}{\sqrt{2}}\,i;$$

$$H(\phi+\phi') = -\frac{11}{10}, \quad \tau(\phi+\phi') = \frac{1}{10}, \quad \tau'(\phi+\phi') = \frac{11i\sqrt{(10)}}{50};$$

$$H(\phi-\phi') = -\frac{43}{18} = H(5a), \quad \text{if} \quad H(a) = -\tfrac{1}{2}.$$

With $$\frac{K'}{K} = \sqrt{(35)},$$

$$\eta = -2+2\sqrt{5}, \quad \tau(\phi) = \frac{(\sqrt{5}-1)^2}{16}, \quad \tau'(\phi) = \frac{i\sqrt{7}}{4}.$$

So also we find that

$$\frac{K'}{K} = \sqrt{(22)}$$

corresponds to

$$L(22)^2 = \sqrt{(22)}, \quad \xi = 2, \quad \tau(\phi) = \tau(\phi') = \frac{1}{22}, \quad \tau'(\phi) = \frac{7\sqrt{2}}{22};$$

$$H(\theta+\tfrac{3}{5}\omega) = \infty, \quad \theta = \tfrac{2}{5}\omega;$$

$$a(\theta') = -\tfrac{7}{8}, \quad a(\tfrac{1}{2}\theta') = -\tfrac{1}{2}.$$

But $$H(\phi) = -\tfrac{1}{2},$$

so that $$\tfrac{1}{2}\theta' = \phi,$$

and $$\phi-\phi' = 2\phi, \quad \text{or} \quad \phi = -\phi'.$$

Now $$u^3 = 8, \quad v = 198, \quad \sqrt{\eta'} = 2(\sqrt{2}+1)^6;$$

so that $$16\left(\frac{1}{\kappa}-\kappa\right)^2 = \xi^6\eta' = 64\eta',$$

$$\frac{1}{\kappa}-\kappa = 2(\sqrt{2}+1)^6;$$

agreeing with the corresponding value of the modulus.

Other numerical cases can be worked out, as further applications, corresponding to

$$\tau(\phi) = -1, \quad \phi = \tfrac{3}{8}\omega, \text{ \&c.};$$

$$H(\theta + \tfrac{3}{8}\omega) = 0, \quad \theta = 0, \quad u^2 = -3, \quad \xi = -\frac{7+\sqrt{(33)}}{2},$$

$$\tau(\phi) = \frac{2\sqrt{3} - \sqrt{(11)}}{\sqrt{(11)}}, \quad \tau'(\phi) = \frac{13\sqrt{(11)} - 24\sqrt{3}}{\sqrt{(11)}}, \text{ \&c.};$$

$$H(\theta + \tfrac{3}{8}\omega_2) = 11, \quad H(\theta) = \infty, \text{ \&c.}$$

67. The case of $\qquad \dfrac{K'}{K} = \sqrt{(11)}$

corresponds to the vanishing of Kiepert's W; and

$$W = \frac{\tau'}{\tau^3} = \sqrt{(\eta+8)}\sqrt{(\eta^3+4\eta^2-72\eta-364)}$$

$$= (\xi + 2 - 2\xi^{-2})\sqrt{(\xi^3+4\xi^2+8\xi+4)}\sqrt{(\xi^3+8\xi^2+16\xi+16)}.$$

The value $\xi = 0$, or ∞, makes $\tau = \infty$, $\eta = -8$,

and $\qquad J = -\dfrac{2^9}{3^3},$

as required in this case (*Proc. Lond. Math. Soc.*, Vol. XIX, p. 306).

When $\qquad \xi + 2 - 2\xi^{-2} = 0,$

or $\qquad \xi^3 + 2\xi^2 - 2 = 0,$

and we put $\qquad \xi^6 = 16\kappa\kappa',$

then $\qquad 2\sqrt{(\kappa\kappa')} + 2\sqrt[3]{(2\kappa\kappa')} - 1 = 0,$

the equation obtained by putting

$$\kappa = \lambda', \quad \kappa' = \lambda,$$

in Schröter's Modular Equation of the Eleventh Order.

When $\qquad \xi^3 + 4\xi^2 + 8\xi + 4 = 0,$

or $\qquad \xi^3 + 8\xi^2 + 16\xi + 16 = 0,$

then $\qquad w = 0, \quad \tau' = 0,$

and $\qquad \dfrac{K'}{K} = 2\sqrt{(11)}, \text{ or } \dfrac{\sqrt{(11)}}{2}$

(Klein-Fricke, *Modulfunctionen*, II, p. 437).

[*Note added November*, 1896.

It was considered important, as a check upon the accuracy of the formulas, to have some numerical verifications of the results in special cases of the Transformation of the Eleventh and Twenty-second Order; and this has been carried out by Mr. T. I. Dewar.

The object is to calculate the twelve values of y, namely,

$$y_\infty,\ y_0,\ y_1,\ y_2,\ \dots\ y_{10},$$

the roots of Klein's "Multiplier Equation of the Eleventh Order" (*Math. Ann.*, XV, p. 88; *M. F.*, II, p. 442), in the form

$$y^{12}+11\Delta\left(-90y^6+40.12g_2y^4-15.216g_3y^3+2.144g_2^2y^2\right)$$
$$-12g_2.216g_3\Delta y-11\Delta y-11\Delta^2=0,$$

equivalent to Kiepert's L equation (147), *Math. Ann.*, XXXI, p. 428, when we take

$$\Delta=1,\quad y=-L^2.$$

If one root y_∞ or L_∞^2 is known, the remaining eleven roots are given by the a's of § 24, from the formula

$$\frac{y_r}{y_\infty}=\frac{L_r^2}{L_\infty^2}=-\frac{1}{11}\left(1+\epsilon^{36r}a_1+\epsilon^{36.4r}a_2+\epsilon^{36.9r}a_3+\epsilon^{36.16r}a_4+\epsilon^{36.25r}a_5\right)^2,$$

$$\epsilon=e^{\frac{2}{11}\pi i},\quad r=0,\ 1,\ 2,\ \dots\ 10$$

(Klein, *Math. Ann.*, XVII, p. 567),

noticing that $a_3=a_9$ in equations (159).

First, with

$$\frac{K'}{K}=\sqrt{(11)},$$

and
$$J=-\frac{2^9}{3^3},$$

we take
$$12g_2=32,\quad 216g_3=56\sqrt{(11)},$$
$$\Delta=-1,$$
$$y_\infty=-L_\infty^2=\sqrt{(11)};$$

and then
$$\tau=\infty,\quad H=0;$$

and therefore, as in (180), we may take

$$c = -\frac{1}{1+2\cos\frac{2\pi}{11}}$$

$$= -0{\cdot}372,\ 785,\ 597,\ 771,\ 791,\ 7,$$

and this makes, in (149),

$$\sqrt{C} = \frac{2+5c+0-2c^3}{c}$$

$$= -0{\cdot}642,\ 952,\ 335,\ 136,\ 877,$$

and, in § 22,

$$z = \quad 0{\cdot}821,\ 476,\ 167,\ 568,\ 438,$$

$$y = -0{\cdot}254,\ 428,\ 804,\ 456,\ 417,$$

$$x = -0{\cdot}045,\ 421,\ 605,\ 252,\ 541,$$

$$x^{\frac{1}{4}} = -0{\cdot}356,\ 796,\ 697,\ 749,\ 900,$$

$$\lambda = -0{\cdot}467,\ 304,\ 294,\ 715,\ 509,$$

$$\lambda^{\frac{1}{11}} = -0{\cdot}933,\ 176,\ 117,\ 881,\ 270.$$

Now, in § 24, starting with

$$a_1 = -\lambda^{-\frac{2}{11}},$$

and thence determining a_3, a_9, a_5, a_4 from the relations

$$a_3 a_1^2 = -\frac{yz}{\lambda x^{\frac{1}{4}}},$$

$$a_9 a_3^2 = \frac{y^3 z^2}{\lambda^4 x^{\frac{1}{4}}},$$

$$a_5 a_9^2 = -\frac{y}{z},$$

$$a_4 a_5^2 = \frac{\lambda^5 x^{\frac{1}{4}}}{y^3 z^2},$$

Mr. Dewar finds that

$$a_1 = \quad 1{\cdot}230,\ 578,\ 018,\ 091,\ 480,$$

$$a_3 = -1{\cdot}771,\ 424,\ 180,\ 284,\ 673,$$

$$a_9 = -0{\cdot}583,\ 448,\ 985,\ 672,\ 033,$$

$$a_5 = \quad 0{\cdot}909,\ 841,\ 056,\ 781,\ 324,$$

$$a_4 = -0{\cdot}864,\ 171,\ 339,\ 453,\ 279;$$

so that

$$1+\Sigma a = -0{\cdot}078,\ 625,\ 430,\ 537,\ 181,$$

and
$$L_0^2 = \frac{(1+\Sigma a)^2}{\sqrt{(11)}}$$
$$= \quad 0{\cdot}001,\ 863,\ 927,\ 552,\ 231,$$

agreeing closely with the approximate value given by

$$L_0^2 \approx \frac{11}{12\gamma_2 . 216\gamma_3},$$
$$12\gamma_2 = 32,$$
$$216\gamma_3 = 56\sqrt{(11)}.$$

So also the imaginary roots are given by

$$L_1^2,\ L_{10}^2 = \quad 2{\cdot}323362 \pm 1{\cdot}934112i,$$
$$L_2^2,\ L_9^2 = \quad 1{\cdot}698161 \pm 0{\cdot}356919i,$$
$$L_3^2,\ L_8^2 = -2{\cdot}776133 \pm 2{\cdot}259814i,$$
$$L_4^2,\ L_7^2 = -0{\cdot}164155 \pm 3{\cdot}579350i,$$
$$L_5^2,\ L_6^2 = \quad 0{\cdot}576146 \pm 0{\cdot}247194i.$$

Next, with
$$\frac{K'}{K} = \sqrt{(22)} \quad \text{or} \quad \sqrt{\left(\frac{2}{11}\right)},$$

we take
$$\tau = \frac{1}{22}$$

and
$$a = -\frac{1}{2},$$

in (175); and therefore, with our c out of phase with this a by one twenty-fifth of a period, we can take the b and $\sqrt{B}$ the same as the c and $\sqrt{C}$ just employed with

$$\frac{K'}{K} = \sqrt{(11)};$$

and now, from (193),

$$c = \quad 27{\cdot}028,\ 919,\ 189,\ 803,\ 4,$$

or
$$= -\ 1{\cdot}063,\ 634,\ 337,\ 710,\ 340,$$

according to the sign attributed to

$$\sqrt{A} = \tfrac{1}{2}\sqrt{2}.$$

Taking $c = 27{\cdot}028,\ 919,\ 189,\ 803,\ 4,$

$$\sqrt{C} = 291{\cdot}442,\ 741,\ 535,\ 938,\ 85,$$

$$z = -\ 145{\cdot}221,\ 370,\ 767,\ 969,\ 4,$$

$$y = -\ 897{\cdot}631,\ 636,\ 212,\ 663,\ 5,$$

$$x = -\ 131252{\cdot}928,\ 291,\ 711,\ 92\,;$$

and a rapid calculation, from the formulas

$$12g_2 = \{(y+1)^2+4x\}^2-24x\,(y+1),$$

$$216g_3 = \{(y+1)^2+4x\}^3-36x\,(y+1)\,\{(y+1)^2+4x\}+216x^2,$$

showed that $\gamma_2 = \frac{g_2}{\Delta^{\frac{1}{3}}} = 11{\cdot}325,$

and therefore corresponds to

$$\gamma_2 = 775-540\sqrt{2},$$

$$\frac{K'}{K} = \sqrt{\left(\frac{2}{11}\right)}.$$

Now $\gamma_3 = -7\sqrt{(11)}(\sqrt{2}-1)^4\,(9\sqrt{2}+2),$

$$L_\infty^2 = \sqrt{(11)}(\sqrt{2}-1),$$

and $x^{\frac{1}{4}} = -50{\cdot}820,\ 195,\ 781,\ 427,\ 47,$

$$\lambda = 263{\cdot}305,\ 853,\ 647,\ 054,$$

$$\lambda^{\frac{1}{12}} = 1{\cdot}659,\ 746,\ 895,\ 262,\ 522,$$

$$a_1 = -\ 0{\cdot}218,\ 712,\ 903,\ 083,\ 518,$$

$$a_3 = 0{\cdot}773,\ 432,\ 664,\ 433,\ 080,$$

$$a_9 = -\ 2{\cdot}053,\ 977,\ 923,\ 433,\ 609,$$

$$a_5 = -\ 1{\cdot}465,\ 129,\ 712,\ 633,\ 559,$$

$$a_4 = 1{\cdot}964,\ 405,\ 935,\ 908,\ 983,$$

so that $1+\Sigma a = 0{\cdot}000,\ 018,\ 061,\ 191,\ 377.$

Taking the second value,

$$c = -1{\cdot}063,\ 634,\ 337,\ 710,\ 340,\ 0,$$
$$\sqrt{C} = 0{\cdot}991,\ 348,\ 563,\ 851,\ 883,\ 5,$$
$$z = 0{\cdot}004,\ 325,\ 718,\ 074,\ 058,\ 25,$$
$$y = 0{\cdot}004,\ 619,\ 769,\ 581,\ 744,\ 308,$$
$$x = 0{\cdot}004,\ 599,\ 785,\ 760,\ 966,\ 573;$$

and these give values to $12g_2$ and $216g_3$, which make

$$\gamma_2 = \frac{g_2}{\Delta^{\frac{1}{3}}} = 1538{\cdot}675,$$

which corresponds to

$$\gamma_2 = 775 + 540\sqrt{2}, \quad \mathrm{K}'/\mathrm{K} = \sqrt{(22)}.$$

Now $\gamma_3 = 7\sqrt{(11)}(\sqrt{2}+1)^4(9\sqrt{2}+2),$

$$L_\infty^2 = -\sqrt{(11)}(\sqrt{2}+1),$$

and $x^{\frac{1}{4}} = 0{\cdot}166,\ 307,\ 767,\ 947,\ 716,\ 8,$

$$\lambda = -0{\cdot}011,\ 305,\ 225,\ 190,\ 562,\ 9,$$
$$\lambda^{\frac{1}{11}} = -0{\cdot}665,\ 312,\ 040,\ 870,\ 516,$$
$$a_1 = 3{\cdot}395,\ 657,\ 270,\ 978,\ 186,$$
$$a_3 = -0{\cdot}081,\ 537,\ 962,\ 826,\ 480,$$
$$a_9 = 0{\cdot}614,\ 208,\ 346,\ 818,\ 869,$$
$$a_5 = -2{\cdot}830,\ 940,\ 071,\ 212,\ 028,$$
$$a_4 = -2{\cdot}077,\ 159,\ 775,\ 134,\ 698,$$

so that $1 + \Sigma a = 0{\cdot}020,\ 227,\ 808,\ 623,\ 849.$

But now, from a consideration of the approximate value of L_0^2 in Kiepert's equation (147),

$$L_0^2 \approx \frac{11}{12\gamma_2 \cdot 216\gamma_3},$$

we see that, in accordance with the general principle stated in § 5, these second values of a must be employed with the transformed modulus, corresponding to

$$\mathrm{K}'/\mathrm{K} = \sqrt{(2 \div 11)}.$$

Conversely, the first series of values of a must be employed when

$$K'/K = \sqrt{(22)};$$

and now
$$L_x^2 = -\sqrt{(11)}(\sqrt{2}+1)$$
$$= -8\cdot007,\ 040,\ 550,\ 218,\ 830,$$
$$\frac{L_0^2}{L_x^2} = \frac{(1+\Sigma a)^2}{-11}$$
$$= -0\cdot000,\ 000,\ 000,\ 029,\ 655,\ 148,\ 541,\ 5.$$
$$L_0^2 = \quad 0\cdot000,\ 000,\ 000,\ 237,\ 449,\ 976,\ 894,\ 553.$$

Also
$$L_1^2,\ L_{10}^2 = \quad 5\cdot96687 \pm 8\cdot29103i,$$
$$L_2^2,\ L_9^2 = -4\cdot75298 \pm 7\cdot52094i,$$
$$L_3^2,\ L_8^2 = \quad 9\cdot99196 \pm 3\cdot28036i,$$
$$L_4^2,\ L_7^2 = -7\cdot31464 \pm 3\cdot83824i,$$
$$L_5^2,\ L_6^2 = \quad 0\cdot11247 \pm 9\cdot63335i.$$

With
$$K'/K = \sqrt{(2 \div 11)},$$

and the second series of values of the a's,

$$L_x^2 = \sqrt{(11)}(\sqrt{2}-1),$$
$$= \quad 1\cdot373,\ 790,\ 969,\ 468,\ 031,$$
$$\frac{L_0^2}{L_x^2} = \frac{(1+\Sigma a)^2}{-11}$$
$$= \frac{0\cdot000,\ 409,\ 164,\ 241,\ 723,\ 06}{-11},$$
$$L_0^2 = -0\cdot000,\ 051,\ 100,\ 558,\ 209,\ 852,$$
$$L_1^2,\ L_{10}^2 = \quad 2\cdot5203793 \pm 1\cdot5134366i,$$
$$L_2^2,\ L_9^2 = \quad 1\cdot8423729 \pm 0\cdot0747787i,$$
$$L_3^2,\ L_8^2 = \quad 1\cdot0898557 \pm 3\cdot7294405i,$$
$$L_4^2,\ L_7^2 = -4\cdot0969314 \pm 1\cdot4719174i,$$
$$L_5^2,\ L_6^2 = -2\cdot0425644 \pm 3\cdot8031382i.]$$

[redacted]

14	53	34	13	54	33	14	53	34	13	54	33	53	53	34	13	54
46	1	26	41	6	21	46	1	26	41	6	21	46	1	26	41	6
12	55	32	15	52	35	12	55	32	15	52	35	12	55	32	15	52
44	3	24	43	4	23	44	3	24	43	4	23	44	3	24	43	4
16	51	36	11	56	31	16	51	36	11	56	31	16	51	36	11	56
42	5	22	45	2	25	42	5	22	45	2	25	42	5	22	45	2
14	53	34	13	54	33	14	53	34	13	54	33	14	53	34	13	54
46	1	26	41	6	21	46	1	26	41	6	21	46	1	26	41	6
12	55	32	15	52	35	12	55	32	15	52	35	12	55	32	15	52
44	3	24	43	4	23	44	3	24	43	4	23	44	3	24	43	4
16	51	36	11	56	31	16	51	36	11	56	31	16	51	36	11	56
42	5	22	45	2	25	42	5	22	45	2	25	42	5	22	45	2
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46	1	26	41	6	21	46	1	26	41	6	21	46	1	26	41	6
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44	3	24	43	4	23	44	3	24	43	4	23	44	3	24	43	4
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42	5	22	45	2	25	42	5	22	45	2	25	42	5	22	45	2
14	53	34	13	54	33	14	53	34	13	54	33	14	53	34	13	54
46	1	26	41	6	21	46	1	26	41	6	21	46	1	26	41	6

*The Construction of Nasik Squares of any Order.**

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The object of this paper is to give a method by which Nasik squares of the n^{th} order can be formed for all values of n; a Nasik square being defined to be "A square containing n cells in each side, in which are placed the natural numbers from 1 to n^2 in such an order that a constant sum $\frac{1}{2}n(n^2+1)$ (here called W) is obtained by adding the numbers on n of the cells, these cells lying in a variety of directions defined by certain laws."

Introduction.

Diagram A. An unlimited plane is divided into an infinite number of equal squares, here called cells, by two systems of equidistant and unlimited parallel lines at right angles to each other. Every n^{th} line is darkened, making an infinite number of squares, each containing n^2 cells.

Definitions.

A line is said to pass through a cell when it passes through its middle point, and a cell to lie on a line when its middle point lies on it.

The middle point of the corner cell which is common to the bottom row and left column of one of these squares is taken as the origin of coordinates, the axes of x and y being the lines through the cells of the bottom row and left column, which will be called the axis row and axis column, the square being called the origin square.

The coordinates of a cell are those of its middle point, the unit of length being the distance between the middle points of two contiguous cells.

The ordinates of the cells of the row next to the axis row being *one*, that row will be called the first row.

Similarly, the column next to the axis column will be called the first column. If the coordinates of a cell are 2 and 3, they will be written 2,3; and the line from the origin through that cell will be denoted thus (2,2).

Whatever numbers, letters, &c., are placed in the cells of the origin

* For Table of Contents see p. 518.

[redacted]

square, the same are supposed to be in the corresponding cells of all the outer squares, as in Diagram A. The position of an outer square is determined by the coordinates of the cell that corresponds to the origin cell, which will be marked 0,0; these will therefore be positive or negative multiples of n, the order of the squares. Hence when the coordinates of any cell are given, if such multiples of n are added to or subtracted from its abscissa and ordinate as will make each a positive number less than n, these will be its coordinates measured from the 0,0 cell of its own square. *E.g.*, when $n=5$, a cell $-13{,}28$ must have 3×5 added to its abscissa, and 5×5 taken from its ordinate, showing it to be the cell 2,3 in the outer square $-3{,}5$.

Hence, any multiple of n may be added to or subtracted from the coordinates of a cell; the effect being only to indicate a similarly situated cell in some other square; such cells will be called similar cells. When the coordinates of a cell are measured from the origin, they will be called its origin coordinates, and when from the 0,0 cell of its own square, its square coordinates. If p, q are the coordinates of a cell, the cells on (p,q) are

$$0,0 \ldots p,q \ldots 2p,2q \ldots \overline{n-1}\,.\,p,\overline{n-1}\,.\,q \ldots np,nq.$$

Hence every line from 0,0, through a cell p,q, after passing through $n-1$ cells, enters an outer square, the coordinates of whose 0,0 cell are p, q, and by symmetry, if produced, will pass through recurring groups of the same n cells; these will be called a group, and the same cells but in a different order will be regarded as the *same* group, and groups of cells will be called *different* groups, even if all but one in each are the same.

Normal Lines.

Diagram B gives the square of 5, and the 5 lines from the origin through the 5 cells of the first row and those of the axis row. These $5+1$ or 6 lines will be called in every square its normal lines. When n is a prime, these $n+1$ lines pass through $n-1$ different cells, in all through n^2-1 cells, which with the origin cell make up exactly the n^2 cells of the square.

Similar Cells.

Cells are called similar when they have the same coordinates in their squares. Hence, for two cells of two normal lines to be similar, they must be on the same row, and their abscissæ must differ by n

multiple of n. Thus, if $p,1$ and $q,1$ have similar cells on their r^{th} row, these cells will be rp,r and rq,r, and $r(p-q)=\mu n$, where p, q, and r are less than n. If n is prime, this condition cannot be satisfied, showing that no two normal lines of a square of n can in that case have similar cells; and that every cell of the square is found on some one of the $n+1$ normal lines.

Parallels.

Diagram C gives the line (2,1) with the cells through which it passes, and its 4 parallel lines that are seen to be obtained by increasing 4 times by unity the abscissæ of each cell. The cells on these 5 parallels exactly correspond to the 25 cells of the square; similarly, with any square of the n^{th} order, n being prime.

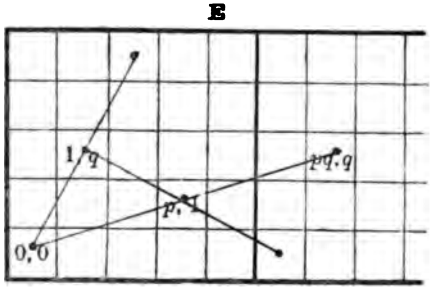
Lines through Similar Cells.

If lines are drawn through any two similar cells, not 0,0 cells, in the plane, they will have on them groups similar to the group on one of the normal lines or its parallels. We may transfer the origin to one of those similar cells. Dealing with the new square of which this origin will be its 0,0 cell, one of its cells will be similar to the other of the two similar cells. Let this be the cell 3,2, Diagram D, which has its similar cell in the adjoining square on the normal line (4,1). As the two similar cells 3,2 on the 2nd row are 5 cells apart, their cells on the 4th row will be 2×5 cells apart, and therefore similar, and so on till on the 10th row the two lines reach the 0,0 cells of outer squares. The diagram shows that it is only the 2nd, 4th, 6th, ... cells of (4,1) that are similar to the cells of 3,2 in order; but these are the same as the intermediate cells of 2,1 in a different order (Todhunter, *Algebra*, Art. 707). Similarly, we know that the group of cells on (4,1) is the same as that on the line from 0,0 through the cell 3,2 of the adjoining square 5,5, and so on; a like process shows that the lines through 0,0 and the similar cells in a column are the same. In a similar way it may be shown that, when two lines from 0,0 have two similar cells, not only is the group of cells on each the same, but the same as on the lines through any of the dissimilar cells of the two lines. *E.g.*, if the line through 4,1 and 3,2 in the origin square is drawn, as this line and the line (4,1) have similar cells on the row above 4,1, their cells on the succeeding rows will be similar, and the two lines have on them the same group of cells.

Allied Cells of the 1st Row and 1st Column.

In a square of n, Diagram D, if two lines are drawn from 0,0, one through the cell *p,1* of the 1st row, the other through the cell *1,q* of the 1st column, the cells of these two lines on the q^{th} row will be *1,q* and *pq,q*. If these two cells are similar, $pq-1=\mu n$, and the cells *p,1* and *1,q* are called allied cells. Similarly, if the lines *q,1* and *1,p* are drawn, their cells on the p^{th} row are *pq,p* and *1,p*, and the condition of their being similar cells is $pq-1=\mu n$, the same as before; this condition being satisfied, the two cells *q,1* and *1,p* are allied. It may be added that the lines through these two pairs have on them the same group of cells as on *p,1*, *1,q*, and *q,1*, *1,p*; for, in Diagram E, where

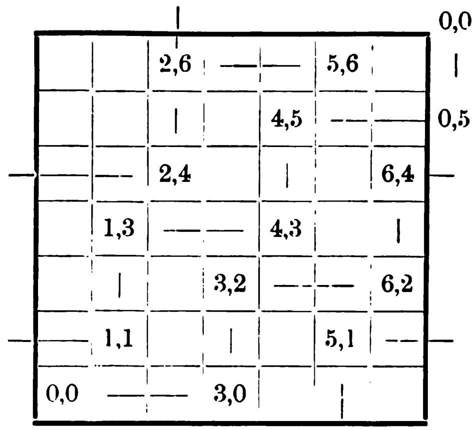
E



$p=3$, $q=2$, $pq-1=\mu 5$ is $3\times 2-1=5$, the lines through *p,1*, *1,q* and *p,1*, *pq,q*, having similar cells on the row above it, have the same group of cells on each. Now, if p is prime to n, and $p, 2p, 3p, \ldots \overline{n-1}\,.\,p$ are divided by n, all the remainders will be different (Todhunter's *Algebra*, Art. 707); if, therefore, qp be the multiple of p which, divided by n, leaves a remainder 1, there is only one such multiple, and we have $pq=\mu n+1$. When, therefore, n is prime the numbers $1, 2, 3, \ldots n-1$ can be arranged in a series of allied pairs, and no number less than n can be its own ally but 1 and $n-1$. For, if r be such a number, we shall have $r\times r=\mu n+1$, or $r^2-1=\mu n$, or $\overline{r-1}\times\overline{r+1}=\mu n$, and, as r is positive and $<n$, $r=1$ or $n-1$. Thus the numbers $2, 3, \ldots \overline{n-2}$ can without repetition be exhaustively arranged in $\frac{n-3}{2}$ different allied pairs (Chrystal's *Algebra*, p. 524, Part II.). Therefore the lines through the cells of the 1st row and 1st column have the same groups of cells, and no

line can be drawn from 0,0 that has a group of cells different from the $n+1$ normal lines. In every square $1,\overline{n-1}$ is allied to $\overline{n-1},1$; for $(n-1)^2$ is of the form $nM+1$.

F



Paths, and the way of Tracing them.

If each of the n cells of a line from 0,0, when it lies outside the origin square, has its square coordinates marked in the similar cells of the origin square, the cells thus marked will be said to be on the *path* of that *line*. The path may be traced without drawing the line into the outer squares in the following manner:—

To trace the path of the line from 0,0 through the cell 3,2 in a square of 7. Here, in Diagram F, the abscissa being 3 and ordinate 2, begin from the origin, count 3 along the axis row, and the cell 3,0 is reached; from it count 2 upward in its column, and 3,2, which is called the 1st cell in the path, is reached. Again, counting from 3,2, 3 cells along its row, we reach 6,2; and from it, counting 2 cells upward along its column, the 2nd cell 6,4 of the path is reached. As counting 3 to the right from 6,4 leads out of the square, the 3 is counted from the opposite end of its row, leading to 2,4; from this, counting 2 upward along its column, we reach 2,6, the 3rd cell of the path; 3 cells along its row lead to 5,6; and, as counting 2 cells upward takes us out of the square, the 2 is counted from the bottom of the column, bringing us to 5,1, the 4th cell of the path. Thus continuing, we reach 4,5, the 6th and last cell of the path. One step further leads us outside the square to 0,5, and from it to 0,7, the 0,0 cell of the square above it. In tracing any *path* in any square of n, or writing the cells along any path, the n^{th} cell is a 0,0 cell, a most useful test of the accuracy of our work.

To fill the Cells of a Nasik Square, n being prime.

A p is placed in the cells of a path $p,1$, and its parallels formed by placing $p_1, p_2, \dots p_{n-1}$ after each p in the $p-1$ cells of its row. Each of these $n-1$ letters will appear on the $n-1$ cells of any other path, as $(q,1)$; for when $(q,1)$ passes through a cell with p_r it cannot pass again through a cell with p_r; for, on leaving the 1st p_r, the path of p_r is that of $(p,1)$, and cannot cut the path of $(q,1)$ again. Hence we shall have $p_1, p_2, \dots p_{n-1}$ on the cells of $(q,1)$, and np on those of $(p,1)$. *Vice versa*, if in another square the parallels of $(q,1)$ are drawn, we shall have $q_1, q_2, \dots q_{n-1}$ on the cells of $(p,1)$, and nq on those of $(q,1)$. Then, if these two squares are superposed, we shall have

G

p_3q_4	p_4q_5	p_5q_6	p_6q	pq_1	p_1q_2	p_2q_3
p_6q_1	pq_2	p_1q_3	p_2q_4	p_3q_5	p_4q_6	p_5q
p_2q_5	p_3q_6	p_4q	p_5q_1	p_6q_2	pq_3	p_1q_4
p_5q_2	p_6q_3	pq_4	p_1q_5	p_2q_6	p_3q	p_4q_1
p_1q_6	p_2q	p_3q_1	p_4q_2	p_5q_3	p_6q_4	pq_5
p_4q_3	p_5q_4	p_6q_5	pq_6	p_1q	p_2q_1	p_3q_2
pq	p_1q_1	p_2q_2	p_3q_3	p_4q_4	p_5q_5	p_6q_6

H

26	54	12	41	05	33	67
45	03	37	66	24	52	11
64	22	51	15	43	07	36
13	47	06	34	62	21	55
32	61	25	53	17	46	04
57	16	44	02	31	65	23
01	35	63	27	56	14	42

a square, in the n^2 cells of which are all the combinations of $p, p_1, p_2, \dots p_{n-1}$ with $q, q_1, \dots q_{n-1}$, and where along the path $(p,1)$ we have np and Σq, and along $(q,1)$ nq and Σp, but along the other $n-1$ paths Σp and Σq.

If $\qquad p, p_1, \dots p_{n-1} = 0, 1, 2, \dots \overline{n-1}$, in any order,

and $\qquad q, q_1, \dots q_{n-1} = 1, 2, \dots n$, in any order,

and the p's are multiplied by n, the cells will be filled with the numbers 1 to n^2 in the scale of n.

In Diagram G we have the superposed square of 7, in which p and q are placed along the paths (3,1) and (4,1), and the 6 parallels of each are entered, and in Diagram H we have the same square in the septenary scale with numerical values assigned as below:—

p	p_1	p_2	p_3	p_4	p_5	p_6	and	q	q_1	q_2	q_3	q_4	q_5	q_6,
0	3	6	2	5	1	4		1	5	3	7	6	4	2.

If each p is multiplied by 7, the numbers being read in the septenary scale, we have in the 7^2 cells the numbers from 1 to 7^2, and, by these values, 34, the middle number of 1 to 7^2, is in the middle of the square, and the pairs of numbers on opposite sides of and equidistant from 34 on 6 of the lines through 34 make 2×34; we have 7×34 along each of the 6 lines from 34 through 22, 51, 43, 07, and the column and row through 34 and their parallels. The lines through 64 and 36, one having 7×4 in the place of units, and the other 7×3 in the place of 7's, have not the same summation.

In the square of 5, if the p's and q's are placed in the cells of 2,1 and 3,1, we shall have W along the other four normal lines (0,1), (1,1), (5,1) and (1,0), viz., the rows, columns, and the two diagonals, with their parallels. We may now pass from the primes, after giving what may be regarded as a Nasik square of the least prime 2.

a

p_1	p_1
p	p

b

q	q_1
q	q_1

a gives the square with p on the cells of the axis row, and p_1 on those of the 1st row; *b* gives a square with q on the cells of the axis column and q_1 on those of the 1st column; in *c* they are superposed.

c

p_1q	p_1q_1
pq	pq_1

d

11	12
01	02

e

3	4
1	2

If $p, p_1 = 0, 1$, $q, q_1 = 1, 2$, *c* becomes *d* in the binary scale, which, in the denary is *e*, with diagonal summations.

When n is not prime.

First when n is even and equals $2m$, where m is prime.

The simplest case is when $n = 2\times 3$.

Here we have, besides the 7 normal lines, 3 other lines with differ-

[redacted]

ent groups, viz., through the 1st column cells 1,2, 1,3, and 1,4, (1,5) and (5,1) having the same group. These 10 lines are given in Diagram I, where it will be seen that on none of them are any of the 4 cells 3,2, 3,4, 2,3, and 4,3; but that two of these cells, 3,2, 3,4, appear on the line (3,2), and the other two, 2,3, 4,3, on the line (2,3). The 36 cells of the square will therefore be found on these 12 lines. As when n was prime, it may be shown that the lines through any two cells (only one of which may be a 0,0 cell) have the same group of 6 cells as we have on one of these 12 lines or their parallels.

In Diagram J the 12 paths of these 12 lines are given, and on the cells of each path is placed its sign as below

0,1	1,1	2,1	3,1	4,1	5,1	1,0	1,2	1,3	1,4	2,3	3,2
κ	a	β	b	γ	c	ρ	λ	l	μ	θ	ϕ

These letters, or (as afterwards) numbers, will be called path signs, or simply signs.

Intersections of Paths.

Definition.—When a letter, as β, is placed on the cells of a path 2,1, and its parallels filled in as in Diagrams K and L, such a filled square will be indicated thus, sq.$(2,1)_\beta$, sq.(2,1), sq.(β), or simply the square of 2,1 or β; and it will be observed that each path is cut on its 2nd and 4th cell from 0,0 by 2 others and on its 3rd cell by 3 others.

K

λ_5	λ_3	λ_1	λ_5	λ_3	λ_1
λ_4	λ_2	λ	λ_4	λ_2	λ
λ_3	λ_1	λ_5	λ_3	λ_1	λ_5
λ_2	λ	λ_4	λ_2	λ	λ_4
λ_1	λ_5	λ_3	λ_1	λ_5	λ_3
λ	λ_4	λ_2	λ	λ	λ_2

L

b_3	b_4	b_5	b	b_1	b_2
b	b_1	b_2	b_3	b_4	b_5
b_3	b_4	b_5	b	b_1	b_2
b	b_1	b_2	b_3	b_4	b_5
b_3	b_4	b_5	b	b_1	b_2
b	b_1	b_2	b_3	b_4	b_5

We have now to find what two squares can be superposed that will have different combinations of signs on the cells of their paths and parallels. No two squares can be superposed if any of their paths after leaving the 0,0 cell cut each other on another cell, for then we should have the same combination in that cell as in the 0,0 cell.

The line through 0,1′ shows that sq.$(0,1)_\kappa$ cannot superpose the 5 squares of β, γ, θ or of b, ϕ; therefore it can superpose the other 6 squares a, c, ρ, λ, μ, l. As the same is true of each of the 12 letters, we have 12×6 superposed squares; but, as each appears here twice, we have 36 different pairs of squares that can be superposed.

Now we have all the 12 letters on the 3 cells 3,3, 0,3, 3,0, 4 on each, and the squares of these 12 letters may be thus grouped:

sq.$(0,1)_\kappa$ sq.$(2,3)_\theta$ sq.$(4,1)_\gamma$ sq.$(2,1)_\beta$ .... C,

sq.$(3,2)_\phi$ sq.$(1,0)_\rho$ sq.$(1,4)_\mu$ sq.$(1,2)_\lambda$ .... R,

sq.$(3,1)_b$ sq.$(1,3)_l$ sq.$(1,1)_a$ sq.$(5,1)_c$ .... D.

Denoting the upper row by C, as κ, θ, γ, β are on 0,3 of the Column,

	2nd	,,	R, ,, ϕ, ρ, μ, λ	,,	3,0	,,	Row,
and	3rd	,,	D, ,, b, l, a, c	,,	3,3	,,	Diagonal.

If the 12 paths had only cut each other on the cells 0,3, 3,0, 3,3, any one of the squares of C could have been superposed on any one of those of R or D, and any of R on those of D, and we might have had $3 \times 4 \times 4$ or 48 superposed squares; but, as they also cut each other 3 and 3 on 0,2, 2,2, 4,2, 2,0, we have 4 sets of 3 squares that cannot be superposed, namely,

sq.$(0,1)_\kappa$ sq.$(3,2)_\phi$ sq.$(3,1)_b$; sq.$(2,3)_\theta$ sq.$(1,0)_\rho$ sq.$(1,3)_l$;

sq.$(4,1)_\gamma$ sq.$(1,4)_\mu$ sq.$(1,1)_a$; sq.$(2,1)_\beta$ sq.$(1,2)_\lambda$ sq.$(5,1)_c$,

and the four groups of cells on C, R, D have been so arranged that these four groups of squares that cannot be superposed are vertically over each other.

Hence every square of C can superpose 3 squares of R and D, every square of R 3 squares of D.

We have therefore in all 3×12 or 3×6 possible pairs of squares that can be superposed, the number obtained before.

Parallels.

The parallels of a path through a cell *p,1* on the 1st row, whose sign is p, are obtained by placing $p_1, p_2, \ldots p_{n-1}$ after each p in the cells of its row, and those of a path through 1,r, a cell of the 1st column, whose sign is r, by placing r_1, r_2, ... r_{n-1} over each r in the cells of its column; and, if f is the sign of a factor path, its parallels are obtained by placing $f_1, f_2, \ldots f_{n-1}$ in the cells of the diagonal through

[redacted]

each *f*. An illustration of the first two cases is given in Diagrams K and L.

In Diagram K we have λ placed in the 6 cells of the path of 1,2; the 5 parallels of this path are made by putting over each λ in the 5 cells of its column $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$. In Diagram L b is placed in the 6 cells of the path of 3,1, and its 5 parallels drawn by putting after each b in the 5 cells of its row, b_1, b_2, b_3, b_4, b_5.

M

	C.				R.				D.			
	0,1	2,3	4,1	2,1	3,2	1,0	1,4	1,2	3,1	1,3	1,1	5,1
	κ	θ	γ	β	ϕ	ρ	μ	λ	b	l	a	c
R. sq.$(1,2)_\lambda$	$\Sigma\lambda$	$\Sigma\lambda$	$\Sigma\lambda$	II	III	III	III	6λ	$\Sigma\lambda$	$\Sigma\lambda$	$\Sigma\lambda$	II
D. sq.$(3,1)_b$	II	Σb	Σb	Σb	II	Σb	Σb	Σb	$6b$	III	III	III

Paths of Superposed Squares.

In Diagram M is a table in which sq.$(1,2)_\lambda$ and sq.$(3,1)_b$ are placed to the left of two rows of 12 cells, which show what is found on their 12 paths; the paths with their signs are indicated above. An examination of this diagram enables us to see what we have in each of the 12 paths of these two squares when superposed. II indicates three pairs of the λ's, such as will be seen to occur in the path (2,1) of K and (0,1) of L. III indicates two pairs of triplets seen in (1,0), the axis row of K, and the diagonals of L.

If λ and $\lambda_3 = 1$ and 6, λ_1 and $\lambda_4 = 2$ and 5, λ_2 and $\lambda_5 = 3$ and 4, each pair $= 7$, and 3 pairs $= 21 = \Sigma\lambda$.

If b and $b_3 = 0$ and 5, b_1 and $b_4 = 1$ and 4, b_2 and $b_5 = 2$ and 3, each pair $= 5$, and 3 pairs $= 15 = \Sigma b$.

There cannot be values of λ's and b's given to make two triplets equal, as $\Sigma\lambda$, Σb are odd.

The superposed squares with the above values of the letters, and the b's multiplied by 6, give a Nasik square whose cells are filled with 1 to 36 in the senary scale, and which has *W* along 0,1, 2,3, 4,1 2,1, the four paths of C.

In Diagram N is given the complete table from which Diagram M was taken. By it we are able to tell what is found on the 12 paths of each of the 36 possible squares obtained by superposing pairs of squares.

The diagram shows that all the 12 squares have III on one of the paths 0,1, 1,0, 1,1, 5,1; therefore no two superposed squares can give a Nasik square with W along its rows, columns, and diagonals; such a square will be called a CDR square. It will be shown at the end that we can have such a square if not limited to consecutive numbers.

All that has been so far said of the square of 2×3 will now be shown to be true of a square of $2\times m$, m being prime.

Diagram O represents a square of n, where $n = 2m$, m being prime. The cells on the 1st row have $+$, $2, 4 \ldots \overline{n-3}$, $\overline{n-1}$, and 1, $3, 5 \ldots, \overline{n-1}$, $\overline{n-2}$, the former being called the even, the latter the odd, m cells; the cells of the 1st column have in the even cells $\times$, $\alpha\beta$, ... π, $\rho\sigma$, and in the odd middle cell $1,n$ is placed l, the rest being left empty, for, as will now be shown, each of them is allied to an odd cell of the 1st row.

Here, m being prime, we have seen that $2, 3, 4, \ldots \overline{m-3}$, $\overline{m-2}$ can be exhaustively arranged in $\frac{m-3}{2}$ pairs of allied numbers. If pq is such a pair

$$pq = tm+1,$$

p and q will be said to be allied with respect to m; and, if

$$pq = 2tm+1,$$

p and q will be said to be allied with respect to $2m$.

i. When p and q are both even, $\overline{p+m}\,.\,\overline{q+m}$ is odd, and of the form $mt+1$; therefore mt is even, and $\overline{p+m}$ and $\overline{q+m}$ are allied with respect to $2m$.

ii. When p is odd and q even, then $p\times\overline{q+m}$ is odd and of the form $mt+1$; therefore mt is even, and p and $\overline{q+m}$ are allied with respect to $2m$.

iii. When p and q are each odd, then $mt+1$ is odd and mt even; therefore p and q are allied with respect to $2m$.

Now, from i. and ii., we see that from any allied pair with respect to m, one or both of whose components are even, we can obtain an allied pair with respect to $2m$, by adding m to the even component or components.

If we take the series

$$(a)\;\; 2, 3, 4, 5, 6, \ldots \overline{m-4}, \overline{m-3}, \overline{m-2},$$

and add m to each of the even numbers, we have, by rearranging them, the series of odd numbers

$$(b)\ 3,\ 5,\ 7,\ \ldots\ \overline{m-4},\ \overline{m-2}\ ;\ \overline{m+2},\ \overline{m+4},\ \overline{m+6},\ \ldots\ \overline{2m-3}.$$

The allied pairs with respect to m in series (a) which come under Case iii. remain allied pairs with respect to $2m$ in series (b). The allied pairs with respect to m in series (a) which come under Case i. or ii. give rise to allied pairs with respect to $2m$ in series (b).

There are evidently no omissions or repetitions; therefore the series (b) can be exhaustively arranged in pairs allied with respect to $2m$ or n.

Now, if p represents the number in any cell of the 1st row, the m^{th} cell of $(p,1)$ on the m^{th} row will be mp,m. If p is even, this is the cell $0,m$; if odd, the cell m,m. The m^{th} cell of $(1,l)$ through the middle cell l of the 1st column is m,ml; and, as l is odd, this is the cell m,m. Again, if q represents any of the even cells of the 1st column, the m^{th} cell of $(1,q)$ will be m,mq; and, as q is even, this is the cell $m,0$. So far we have $m+1$ lines through the m odd cells of the 1st row, and the cell $1,l$ of the 1st column passing through the cell m,m; the m lines through the m even cells of the 1st row passing through the cell $0,m$, and the m lines through the m even cells of the 1st column passing through the cell $0,m$.

The formula $pq = \mu n+1$ for allied cells; as μn is even, pq is odd; therefore neither p nor q can be even, which shows that no even cell of the 1st column can be allied to an even cell of the 1st row. With respect to the two factor lines $(m,2)$ and $(2,m)$ whose path signs are ϕ and θ, their r^{th} cells are $rm,2r$, and $2r,rm$.

If r is even, these are even cells of the axis column and axis row respectively; and, if r is odd, they are even cells of the m^{th} column and m^{th} row respectively.

Hence we have ϕ on the cell $m,0$, and θ on the cell $0,m$, and $m+1$ different paths passing through the three cells $0,m$, m,m, and $m,0$.

On the cell $0,2$, we have the three path signs $+$, m, ϕ, being one from each of the three groups of $m+1$ path signs on $0,m$, m,m, and $m,0$. No two of the squares sq.$(+)$, sq.(m), and sq.(ϕ) can be superposed, so that sq.$(+)$ can superpose all the path squares of m,m but sq.(m) and all those of $m,0$ but sq.(ϕ). We have also three path signs on the cell $2,0$, and on the 2nd cell of all the other paths three path signs, two of which are easily obtained as in Diagram O, while the third is obtained by following the paths of $\beta \ldots \pi\rho\sigma$, each of which passes through one of the cells between α and ϕ, and they are repeated in the same order in the cells from ϕ to σ.

Squares of Higher Orders.

To explain the way of forming squares of any order, it will suffice to take the six representative squares of 2^t, 9, 15, 18, 24, and 30. As the diagrams of their paths and intersections (as in Diagram J) are too large to be given entire, it will suffice to give the path signs in the chief cells of some one path, as they are all symmetrical and can be produced one from another. For convenience the path of the axis column will be taken, and the odd cells of the first column written 3, 5, 7, ... &c. The square formed by superposing two squares sq.$(p,1)_\alpha$ and sq.$(q,1)_\beta$ will be denoted thus, $\frac{\text{sq.}(p,1)_\alpha}{\text{sq.}(r,1)_\beta}$ or $\frac{p,1}{r,1}$, or simply $\frac{\alpha}{\beta}$; and a square that has W on its rows, columns, and diagonals and their parallels will be called a CDR square.

When $n = 2m$, *m being* 2^t.

Here each of the 2^t odd cells is allied to one of the odd cells of the 1st row: we have therefore in the cells $0,2^t$, $2^t,2^t$, $2^t,0$, the 2^t path signs 2, 4, 6, ..., 1, 3, 5, ..., and α, β, γ, ..., 3×2^t paths, and 3×2^{2t} superposed squares. In every one of these squares there is a path with II's, and others with IV's, VIII's, ...; but, as the same pairs of r's that are in the II's occur in the IV's, if we take the r's so as to make each of the different pairs equal to $(2^{t+1}+1)$, we shall have Σr, or its equivalent on all the paths but that which has nr.

This condition is satisfied when

$t = 1$ or $n = 4$, if $r r_1 r_2 r_3 \;\; = 1, 2, 4, 3,$

$t = 2$ or $n = 8$, if $r_1 r_2 \dots r_7 = 1, 7, 6, 4, 8, 2, 3, 5,$

$t = 3$ or $n = 16$, if $r_1 r_2 \dots r_{15} = 9, 16, 2, 14, 4, 5, 11, 7, 8, 1, 15, 3,$
13, 12, 6, 10.

Various squares of $n = 2\times 2^t$ are CDR squares. As this is the only square of the form $2m$ that has only m path signs in C, R, and D, these m signs will be denoted by C′, R′, and D′, and the extra signs in each will be denoted by the same letter μ, so in every case we shall have C, R, D represented by C′μ, R′μ, D′μ.

[redacted]

When $n = 2m$, m being any Prime.

This case has already been considered, but it may be noticed in connexion with what follows that n has only two factors, m and 2, which gives rise to the two paths $2,m$ and $m,2$, the first passing through $0,m$, the other through $m,0$; and all the odd cells of the first column are allied to cells of the first row, except $1,m$, the path of which passes through the cell m,m. Thus the one pair of factors corresponds to the one unallied cell; a curious relation, which holds, as will be seen, in the squares of a higher order, except when m is the product of prime factors (exclusive of 2).

When $n = 2m$, m being 6.

Here n is of the form $4m'$, and the two cells $\overline{2m'\pm 1,1}$ are allied to the two cells $\overline{1,2m'\pm 1}$, for $(2m'\pm 1)^2$ is of the form $4m'M+1$, or $nM+1$. In the square of 2×3, μ was 1; μ is here 2, for we have two factor products 2×3 and 3×4; these give rise to the four factor paths

$$\overbrace{2,3\quad 3,2}\quad \text{and}\quad \overbrace{3,4\quad 4,3},$$

the signs of which will be □ Δ $<$ $>$.

The 6th cells on these four paths are

$$\overbrace{12,18\quad 18,12}\quad \text{and}\quad \overbrace{18,24\quad 24,18}$$

which are, omitting the 12's,

$$\overbrace{0,6\quad 6,0}\quad \text{and}\quad \overbrace{6,0\quad 0,6}$$

Hence the eight path signs of C and R are C′, □, $>$, and R′, Δ, $<$.

Now 5, 7, 11 are allied to 5, 7, 11; so there are two unallied cells, 3, 9, the paths of which, passing through the cell 6,6, make D′, 3, 9 for the path signs of D. The number of paths will therefore be 3×8.

The diagram of the paths and their intersections has on its 1st column,

in 0,6, the 6th cell, the 8 path signs + 2 4 6 8 10 □ $>$
,, 0,4, ,, 4th ,, ,, 6 ,, + 3 6 9 Δ $<$
,, 0,3, ,, 3rd ,, ,, 4 ,, + 4 8 $>$
,, 0,2, ,, 2nd ,, ,, 2 ,, + 6;

this shows that in sq.$(0,1)_+$ we have

6 II's on the path of 6,

4 III's „ 3 paths of 4 8 $>$,

3 IV's „ 4 „ 3 6 9 Δ $<$,

2 VI's „ 7 „ 2 4 6 8 10 $\square$ $>$.

Hence, if the r's be so chosen that

$$6 \text{ II's} = 4 \text{ III's} = 3 \text{ IV's} = 2 \text{ VI's} = \Sigma(r), \text{ or } \frac{12 \times 13}{2},$$

there will be $\Sigma(r)$, or its equivalent, on all the twenty-four paths but that of r. This can be done with all but the 4 III's; for 6×13 is not divisible by 4. We see from the above that sq.$(0,1)_+$ cannot be superposed by sq.(3), sq.(9) of D, or by sq.(Δ), sq.($<$) of R; therefore the number of superposed squares will be $3 \times 8 \times 6$. As each of two superposed squares has III's on three of its paths, the superposed square will have III's on six of its paths, and there will be W on $6 \times 13 - 8$ paths.

If $r_1, r_2, \ldots r_{11}$ are made equal, in order, to

1 7 3 9 8 11 12 6 10 4 5 2,

the above conditions will be satisfied.

If it be required to find how many of these superposed squares are CDR squares, we must observe how many have III's on any of the four paths 0,1, 1,0, 1,1, 11,1.

The diagram of paths shows that

the path signs on the 3rd cell of the path 0,1 are $+$ 4 8 $>$

„ „ „ 1,0 „ $\times$ β δ $<$

„ „ „ 1,1 „ 1 5 9 $\bar{9}$

„ „ „ 11,1 „ 11 3 7 $\bar{3}$

the squares therefore of 4, 8, and $>$ will have III on the path 0,1; and similarly for 1,0, 1,1, and 11,1. Therefore a superposed square in which any of the twelve paths

3 4 5 7 8 9 $\bar{3}$ $\bar{9}$ β δ $>$ $<$

have been used cannot be a CDR square. The other eight path signs that can be used to make one are

$$2\ 6\ 10\ \alpha\ \gamma\ \epsilon\ \square\ \Delta.$$

The diagram of paths shows which squares of these eight paths cannot be superposed; the rest form the following twelve CDR squares:—

2	2	2	6	6	6	10	10	10	□	□	□
Δ	ε	γ	α	ε	γ	α	Δ	γ	α	Δ	ε

$$n = 2m, \textit{ where } m = 9.$$

This square of 18 is the next to be dealt with. Here, in forming the square of intersections, when the nine path signs of C′, R′, and D′ have been placed on the cells of their several paths we find many cells empty. To fill these we turn to the factor paths. The factors 2×3 and 2×9 give four paths that fill all the empty cells but 12. The two empty cells nearest the origin are 3,4 and 4,3, and we find that their path signs placed on the cells of 3,4 and 4,3 exactly fill the 12 empty cells. (This shows that, as 3×4 are factors, not of 18 but 2×18, the factors of $2n$ are to be taken, as well as those of n.)

Now the 9th cells on the 6 paths

$$\overbrace{2{,}3\ \ 3{,}2}\ \ \overbrace{3{,}4\ \ 4{,}3}\ \text{ and }\ \overbrace{2{,}9\ \ 9{,}2}$$

are

$$\overbrace{18{,}27\ \ 27{,}18}\ \ \overbrace{27{,}36\ \ 36{,}27}\ \ \overbrace{18{,}81\ \ 81{,}18},$$

or, omitting every 18, the 6 cells

$$\overbrace{0{,}9\ \ 9{,}0}\ \ \overbrace{9{,}0\ \ 0{,}9}\ \ \overbrace{0{,}9\ \ 9{,}0}.$$

Three of these paths pass through the cell 0,9, and three through the cell 9,0; thus adding three extra path signs to C′ and R′.

Calling the path signs of the above six factor paths, in order,

$$\square,\ \Delta,\ <,\ >,\ \text{and}\ \sqsubset,\ \sqsupset,$$

we have for C $\quad$ C′, □, >, ⊏,

and for R $\quad$ R′, Δ, <, ⊐,

12 signs in both.

Now the allied cells of the 1st column are

5 7 11 13 and 17 to 11 13 5 7 and 17,

and the 3 unallied cells are 3 9 15; therefore for D we have

D′ 3 9 15.

Here therefore $\mu = 3$, and we have 3×12 paths.

Now on the	9th	cell of	0,1,	we have	+ 2 4 6 8 10 12 14 16 □ > ⊏
„	6th	„	„	„	+ 6 12 9 ⊐ 3 15 Δ <
„	3rd	„	„	„	+ 6 12
„	2nd	„	„	„	+ 9 ⊐

We find from this that in sq.$(0,1)_+$ we have

II's	on the	paths of	9 ⊐,
III's	„	„	6 12,
VI's	„	„	3 15 Δ <,
IX's	„	„	2 4 8 10 14 16 □ > ⊏.

The r's therefore must be so chosen that 9 II's, 6 III's, 3 VI's, and 2 IX's $= \frac{18 \times 19}{2}$; this can be done for all but the 6 III's, as $\frac{1}{6} \times \frac{18 \times 19}{2}$ is not an integer.

The path signs on the 6th cell show that sq.$(0,1)_+$ cannot superpose

sq.(3) sq.(9) sq.(15) of D, or sq.(Δ) sq.(<) sq.(⊐) of R.

The number of superposed squares will therefore be $3 \times 12 \times 9$. The above conditions are satisfied if the 18 r's in succession are

1 16 5 12 9 17 4 13 8 18 3 14 7 10 2 15 6 11.

On the 3rd cells of 1, 17, +, ×, we have 7, 13, 5, 11, 6, 14, γ, ζ; hence every superposed square where none of these eight path signs have been used will be a CDR square.

When $n = 2m$, *m being* 12.

We have 5, 7, 11, 13, 17, 19, 23 each allied to the corresponding cells on the 1st row, and 3, 9, 15, 21 not allied.

Here then we have for D

D′ 3 9 15 21.

The eight factor paths are

$$\overbrace{2,3\ \ 3,2}\quad \overbrace{3,4\ \ 4,3}\quad \overbrace{3,8\ \ 8,3}\quad \text{and}\quad \overbrace{2,9\ \ 9,2},$$

to which we give the path signs

□ Δ < > ⊕ ⊖ ♂ ☌

The 12th cells on these eight paths will be

$$\overbrace{24,36\ \ 36,24}\quad \overbrace{36,48\ \ 48,36}\quad \overbrace{36,96\ \ 96,36}\quad \text{and}\quad \overbrace{24,108\ \ 108,24},$$

which are

$$\overbrace{0,12\ \ 12,0}\quad \overbrace{12,0\ \ 0,12}\quad \overbrace{12,0\ \ 0,12}\quad \text{and}\quad \overbrace{0,12\ \ 12,0}.$$

Therefore (2,3), (4,3), (8,3), (2,9) pass through the cell 0,12,
and (3,2), (3,4), (3,8), (9,2) „ „ 12,0,
and we have for C C′, □, >, ⊖, ♂,
and for R′ R′, Δ, <, ⊕, ☌.
Here then $\mu = 4$, and there are 3×16 paths.

Now, on the cell *0,m* we have

+ 2 4 6 8 10 12 14 16 18 20 22 □ > ⊖ ♂

on the cell 0,8 we have + 3 6 9 15 18 21 12 Δ < ⊕ ☌
„ 0,6 „ + 4 8 16 20 12 > ⊖
„ 0,4 „ + 6 18 12
„ 0,3 „ + 8 16 ⊖
„ 0,2 „ + 12

Here the r's must be chosen so as to make 12 II's, 8 III's, 6 IV's, and 2 XII's $= \frac{24 \times 25}{2}$; this may be done, except for 8 III, for III would be $\frac{1}{8} \times \frac{24 \times 25}{2}$, a fraction.

We see also that sq.$(0,\dot{1})_+$ cannot superpose any of the eight squares of the path signs

$$3\ \ 9\ \ 15\ \ 21\ \ \Delta\ \ <\ \ \oplus\ \ ♂$$

The number of superposed squares will therefore be $3 \times 16 \times 3$.

The path signs of 1, 23, $+$, $\times$ on the 3rd cell of each are

on the path of	1 ...	9	$\dot{9}$	17
„ „	23 ...	15	$\dot{15}$	7
„ „	$+$ ...	8	16	$\ominus$
„ „	$\times$ ...	8	$\dot{16}$	$\oplus$

All the superposed squares that are not formed from any of these 16 path signs will be CDR squares.

When $n = 2m$, *where* $m = 3 \times 5$.

This square introduces a peculiarity not met with in even squares of a lower order, but is a type of many similar squares of a higher order. For hitherto the two paths arising from two factors of n passed, one through the cell $0,m$, the other through $m,0$; the number of such extra sign paths in C or R being the same as those of the unallied cells of the 1st column, whose paths went through the cell m,m, thus making the number of extra path signs of C, R, and D the same; but now we have here the factors 3×5 giving rise to the two paths 3,5 and 5,3 and the 15th cells on these two paths are 45,75 and 75,45, each of which is the cell 15,15; thus we have two extra path signs for D. We will take ⳩, ⚒ as the signs of

$$\overbrace{3,5\ \ 5,3}.$$

The other factor paths are

$$\overbrace{2,3\ 3,2}\ \overbrace{3,4\ 4,3}\ \overbrace{2,5\ 5,2}\ \overbrace{4,5\ 5,4}\ \overbrace{5,6\ 6,5}\ \overbrace{2,15\ 15,2}\ \overbrace{3,10\ 10,3}\ \overbrace{2,27\ 27,2}\ \overbrace{3,26\ 26,3}$$

the path signs being

□ Δ < > ⊤ ⊥ Ɔ ɔ ⊢ ⊣ ( ) ⚸ ⚷ ⚳ ⚴ ✱ ✤

(The paths $\overbrace{2,27\ 27,2}$ and $\overbrace{3,26\ 26,3}$ are identical with the paths

$$\overbrace{2,3\ 3,2}\ \ \overbrace{3,4\ 4,3}$$

traced inward from 0,24, the 0,0 cell of the upper square.)

As odd multiples of 15, when the 30's are cast out, become 15,
and even „ „ „ „ 0,
when the 30's are cast out from the 15th cell of all these paths, we get

0,15 for the 15th cell of the paths □ > ⊤ Ɔ ⊣ ☾ ℘ ʔ ✤

which are the 9 extra path signs of C,

and 15,0 for the 15th cell of the paths Δ < ⊥ ɔ ⊢ ☽ ↥ ♀ ✱

which are the 9 extra path signs of R.

Hence the number of paths is 3×24.

Here the cells 7 11 13 17 19 23 29

are, in order, allied to 13 11 7 23 19 17 29;

and, the unallied cells being

3 5 9 15 21 25 27,

these, including the path signs of 3,5 and 5,3, give for D in the cell 15,15,

D′ 3 5 9 15 21 25 27 ✢ ✗

Now we have on the cell 0,15 C′ □ > ⊤ Ɔ ⊣ ☾ ℘ ʔ ✤

„ „ 0,10 + 3 6 9 12 15 18 21 24 27

„ „ 0,6 + 5 10 15 20 25 ℘ ɔ ⊥ ✗ ☽ ⊢

„ „ 0,5 + 6 12 18 24 ⊣

„ „ 0,3 + 10 20 ℘

„ „ 0,2 + 15 ☽

This shows that sq.$(0,1)_+$ cannot superpose any squares of the twelve path signs

3 5 9 15 21 25 27 ɔ ⊥ ⊢ ☽ ✗

The number of superposed squares will therefore be $3 \times 24 \times 12$.

The path signs on the 3rd cells of 1,1, 29,1, +, and × are

for the cell 1,1 11 21 21

„ „ 29,1 9 19 9

„ „ 0,1 10 20 ℘

„ „ 1,0 ϵ λ ↥

All the superposed squares not formed from any of these sixteen path signs will be CDR squares.

When n is of the form $2m+1$.

The single primes having been considered, n will now be the powers or product of simple primes or their powers.

When $n=3$, the square may be filled by p, placing p on the cells of 2,1, and r on the cells of 1,1; this gives the diagram

p_2r_2	$p\,r$	p_1r_1
p_1r	p_2r_1	pr_2
$p\,r_1$	p_1r_2	p_2r

Making $r, r_1, r_2 = 2, 1, 3$ and $p, p_1, p_2 = 0, 2, 1$, we have below, in the ternary scale, the smallest odd Nasik square, which is the ordinary magic square with W on the rows, columns, and two principal diagonals.

13	1	22
21	12	3
2	23	11

When $n=9$.

We have	on the cells	0,3	+	3,1	6,1
	„ „	3,3	1	4,1	7,1
	„ „	9,3	2	5,1	8,1
	„ „	3,0	×	1,3	1,6.

Here we have 4×3 paths, and, as the square of any of the three path signs of 0,3 can superpose the squares of all the other nine path signs, the number of superposed squares will be 6×9.

The square of every path sign will have 3 III's on two of its paths; therefore every superposed square will have 3 III's on four of its paths, but, as 3 III's can be made to equal $\frac{15\times 16}{2}$, we shall have

W on $4\times 3-2$ paths, if $r\ r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ r_7\ r_8$

are made to be $5\ 4\ 3\ 6\ 8\ 7\ 1\ 0\ 2$.

Superposed squares will be CDR squares, if $+$, $\times$, 1,1, 8,1 have not been used in forming them. Here, and when n is the square of a prime, we have four groups of path signs instead of the three when n is given.

When $n = 3 \times 5$.

In forming a diagram of the paths and their intersections, we find them 24 in number, and that they draw together into 4 groups of 6, and 6 groups of 4 path signs: these are

	on the cell	0,5	the group	$+$	3	6	9	12	$\bar{+}$
	" "	5,5	"	1	4	7	10	13	$\bar{15}$
	" "	10,5	"	2	5	8	11	14	$\bar{5}$
	" "	5,0	"	$\times$	$\bar{3}$	$\bar{6}$	$\bar{9}$	$\bar{12}$	$\bar{\times}$
and	" "	0,3	"	$+$	5	10	$\bar{\times}$		
	" "	3,3	"	1	6	11	$\bar{6}$		
	" "	6,3	"	2	7	12	$\bar{3}$		
	" "	9,3	"	3	8	13	$\bar{12}$		
	" "	12,3	"	4	9	14	$\bar{9}$		
	" "	3,0	"	$\bar{5}$	$\bar{10}$	$\bar{+}$			

These 4 groups of 6 path signs, and the 6 groups of 4, each contain the 15 path signs of the 1st row; and the two groups on 5,0 and 3,0 of the axis row contain $\bar{3}$ $\bar{6}$ $\bar{9}$ $\bar{12}$ $\bar{5}$ $\bar{10}$ of the 1st column.

The omitted cells of the 1st column are

$\bar{2}$ $\bar{4}$ $\bar{7}$ $\bar{8}$ $\bar{11}$ $\bar{13}$,

which, being allied to 8 4 13 2 11 7,

leave the 24 path signs of the square. The path signs on 0,3, 0,5 show that sq.$(0,1)_+$ cannot superpose the squares of the 8 path signs

3 6 9 12 5 10 $\bar{+}$ $\bar{\times}$;

there are therefore 15 squares it can superpose. Similarly for each of the other 23 path signs. The number of superposed squares will therefore be $\frac{1}{2} \times 24 \times 15$, the $\frac{1}{2}$, for among the 24×25 combinations of paths every pair as 3, 5 appears again as 5, 3.

In the square of every path sign there are on 5 of its paths 3 V's, and on 3 paths 5 III's, but, as the 3 V's and 5 III's can be made equal to $\frac{15 \times 16}{2}$, we can have W on 22 paths of this square if we make $r, r_1, r_2, \ldots r_{14}$ equal, in order, to

1 5 4 3 2 12 9 7 6 14 11 10 13 15 8.

Every square in which $+$, $\times$, 1, 14 are not used in its construction will be a CDR square.

Sliding of Rows or Columns over each other.

The effect of sliding the rows, either way, one cell over the other, is that, while the ordinates of all the cells are unchanged, the cells of the 1st, 2nd, and 3rd, ... rows are moved respectively over 1, 2, 3, ... cells, their abscissæ being increased or diminished by 1, 2, 3, ...

Paths of 6^2	0,1	2,3	4,1	2,1	... C (page 495)
	3,2	1,0	1,4	1,2	... R
	3,1	1,3	1,1	5,1	... D

The squares in Diagram A are formed by superposing sq.(2,3) of C and sq.(3,2) of R; this gives a square with the four paths of D.

Sliding its rows to the right over one cell, as the cell 2,3 is on the 3rd row, its abscissa 2 will be increased by 3; thus the cell becomes 5,3. The cells on the path of 5,3 are

0,0 5,3 4,0 3,3 2,0 1,3 0,0;

so the path of 5,3 is the path of 1,3 of D. Proceeding in the same way, we find the path of 3,2 of R becomes that of 5,2 which is the path 1,4 of the same group R. The superposing sq.(1,3) of D and sq.(1,4) of R give a square with W on its four paths of C. This square is seen at C in Diagram P.

Again, when the columns of D are moved up or down one cell over each other, the abscissæ are unchanged, but, the 1st, 2nd, 3rd, ... columns being respectively moved over 1, 2, 3, ... cells, the ordinates of the cells on these columns are increased by 1, 2, 3, ... If then the columns are moved upwards one cell over each other, the abscissæ of 2,3 and 3,2 being unchanged, the cell 2,3 of C changes to 2,5, the path of which is that of 4,1 of the same group C, and the cell 3,2 of R changes to 3,5, the path of which is that of 3,1 of D. The sq.(4,1) of C and sq.(3,1) of D when superposed give R (Diagram P) a square that has W on the four paths of R.

[illegible]

[illegible]

[illegible]

[illegible] a group of [illegible]

$$\begin{matrix}2.3\\3.2\end{matrix} \text{ becomes } \begin{matrix}4.1\\[illegible]\end{matrix} \quad \begin{matrix}2.1\\3.2\end{matrix} \quad \begin{matrix}2.3\\3.1\end{matrix} \quad \begin{matrix}4.1\\[illegible]\end{matrix} \quad \begin{matrix}2.1\\3.1\end{matrix} \quad \begin{matrix}2.3\\3.2\end{matrix}$$

a group of six recurring different squares.

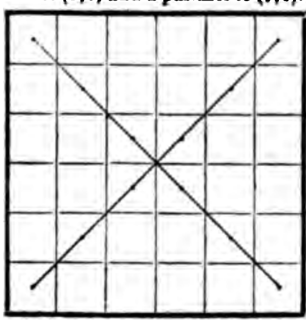
[illegible] curious to find that, while D becomes R by sliding its columns [illegible] cell, by sliding them another cell it becomes D again, but with [illegible] arrangement of the figures; and the same oscillation takes [illegible] C and D by moving its rows; but, if we slide the rows of [illegible] columns of C, we find R and C again but differently arranged. [illegible] is a bridge for passing from C to R by first making C into D [illegible] rows, then making D into C by moving its columns.

[illegible]

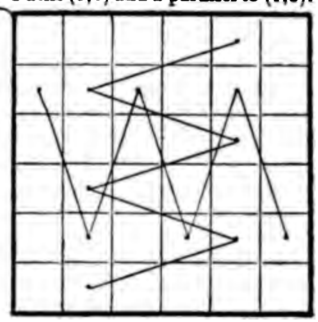
The diagrams in Q show a simpler way of passing from C to R, for they show the paths of D, C, and R; and that by turning the two squares of C through a right angle they become the two squares of R. The symmetry of the paths of D leaves it unchanged after turning in this way. A comparison of the coordinates of C and R shows the same.

Q

Paths (1,1) and a parallel to (5,1).

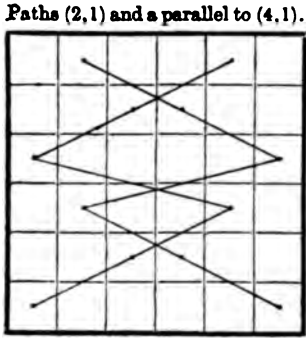


Paths (3,1) and a parallel to (1,3).

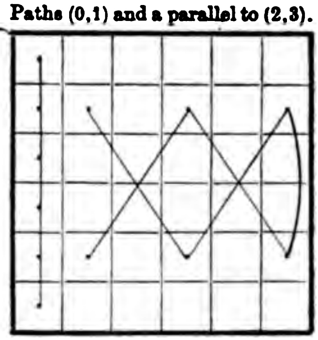


D

Paths (2,1) and a parallel to (4,1).

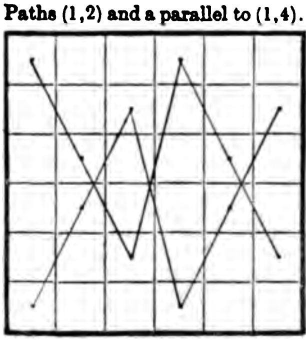


Paths (0,1) and a parallel to (2,3).

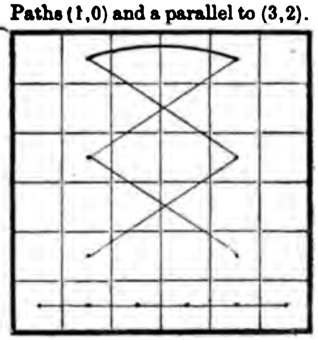


C

Paths (1,2) and a parallel to (1,4).



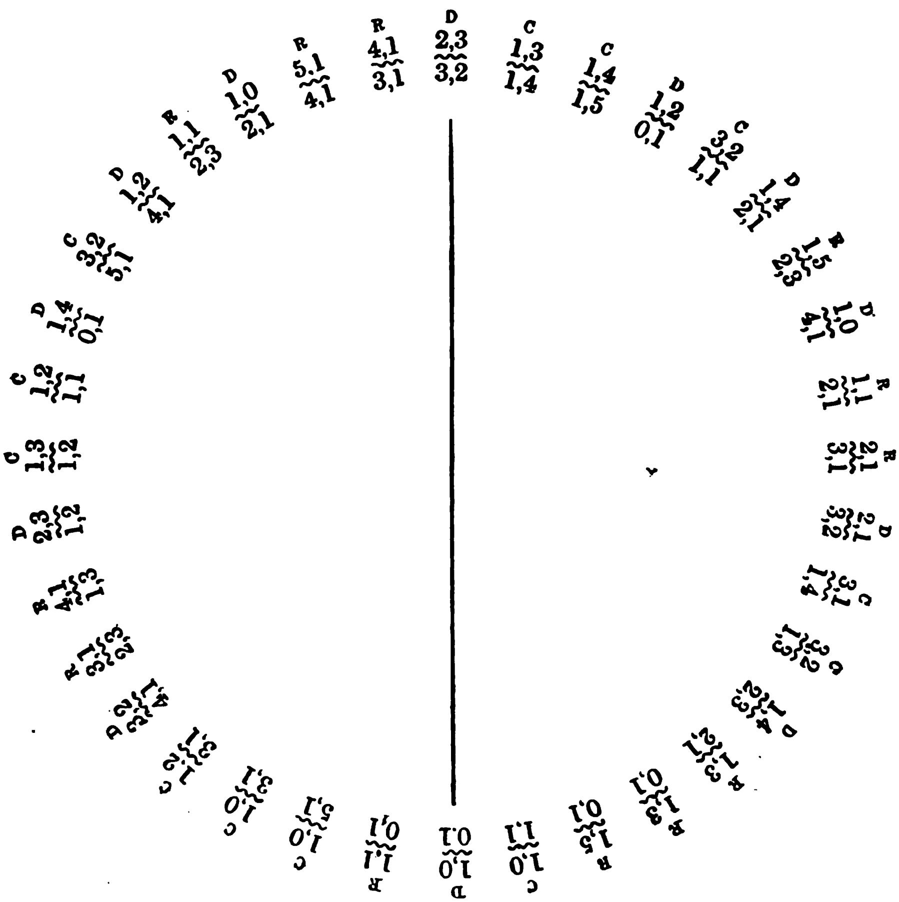
Paths (1,0) and a parallel to (3,2).



R

If any one of the 36 superposed squares is formed, all the 35 other squares can be got from it by Diagram R, which is obtained by commencing from the square $\underset{3,2}{\overset{2,3}{\sim}}$ and moving alternately first the rows over one cell, then the columns one cell of the new square; after 35 such alternations of a row and column, the series would enter the square $\underset{3,2}{\overset{2,3}{\sim}}$ from which we began. Any square as $\underset{1,1}{\overset{1,2}{\sim}}$ might have

R



been taken, and, commencing from it in the same way, after 35 changes $\frac{1,2}{1,1}$ would be reached again. All the pairs of cells that are equidistant from $\frac{2,3}{3,2}$ or $\frac{1,0}{0,1}$ at the ends of the diameter through them are reciprocal, that is, one is obtained from the other by turning it through a right angle; this is seen by comparing their coordinates. In all the changes through the 36 squares, the *actual* numbers that give *W* along the four paths of the initial square remain the same, but they lie along constantly changing paths, *e.g.*, $\overset{\text{D}}{\frac{2,3}{3,2}}$ on the 3rd change of rows and columns becomes again $\overset{\text{D}}{\frac{1,2}{0,1}}$, that is, a square with the 4 paths of D;

but the numbers that were lying along			(1,1)	now lie on	(1,3)
”	”	”	(5,1)	”	(1,1)
”	”	”	(1,3)	”	(5,1)
”	”	”	(3,1)	unchanged	(3,1)

Connexion between factors of n and allied cells.

On page 506 we have seen that when a factor product as 2×3 is even one of the two paths it leads to passes through $0,\frac{n}{2}$, the other through $\frac{n}{2},0$, but when the product is odd, as 3×5, its two paths pass through $\frac{n}{2},\frac{n}{2}$. This leads to the following general equation. If EF represents the number of even factor products, OF the number of odd ones, and UC the number of unallied cells,

$$\text{OF} = \text{UC} + \text{EF}.$$

When $n = 6$, 12, 18, 24, EF is zero, and $\text{OF} = \text{UC}$, a singular connexion between factors of n and the allied cells.

Summary.

Order of Square.	Extra Path Signs.	Paths.	Superposed Squares.	Allied Cells.	Unallied Cells.	Factor Pairs.	CDR Squares.
$n=2\times 2^t$	$\mu=0$	3×2^t	3×2^{2t}	2^t-1	0	0	CDR
$n=2\times 3$	$\mu=1$	3×4	$3\times 4\times 3$	0	1	1	None
$n=2\times 6$	$\mu=2$	3×8	$3\times 8\times 6$	3	2	2	CDR
$n=2\times 9$	$\mu=3$	3×12	$3\times 12\times 9$	5	3	3	CDR
$n=2\times 12$	$\mu=4$	3×16	$3\times 16\times 8$	7	4	4	CDR
$n=2\times 15$	$\mu=9$	3×24	$3\times 24\times 12$	9	9	9	CDR
$n=9$		4×3	$6\times 3\times 3$				CDR
$n=15$		24	12×15				CDR

All that has been said of, or done to, 6^2 applies to all even squares.

Squares with Nasikal Summations but not Consecutive Numbers.

It has been seen that a Nasik square of 6 could not be formed with W on its rows, columns, and two diagonals and their parallels, on account of paths having 2 III's, which, being even, could not equal $\frac{6\times 7}{2}$. If, however, we do not limit the square to consecutive numbers, it is easy to make it a CDR square, and not only when $n=6$, but for all even values of n. If we make

$$r, r_1, r_2, r_3, r_4, r_5 = 1, 7, 5, 3, 6, 2,$$

and
$$p, p_1, p_2, p_3, p_4, p_5 = 0, 6, 4, 2, 5, 1,$$

and place the p's in the cells of the path (2,3) and its parallels,

and „ r's „ „ (3,2) „ „

we have in the Diagram S, a CDR square, in the 36 cells of which are placed, in the septenary scale, the numbers

67 66 65 ... 63 62 61 57 56 55 ... 53 52 51 47 46 45 ... 43 42 41

01 02 03 ... 05 06 07 11 12 13 ... 15 16 17 21 22 23 ... 25 26 27

the 18 pairs of which, one over the other, make 68 or 50 in the denary. We have here W along the 10 paths

0,1 1,1 2,1 3,1 4,1 5,1 1,0 1,2 1,3 1,4

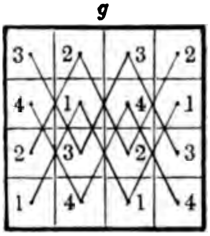
and their parallels. A curious property of the square thus filled is that, if we take the six numbers on the cells of any one of the four paths of D, viz., 1,1, 5,1, 3,1, 1,3 and their parallels, we find that the two unit figures of the 1st and 4th, the 2nd and 5th, and the 3rd and 6th numbers make 8, and the two others in the place of 7's make 6.

On the Possibility of other Methods of forming Nasik Squares for ALL *Values of n.*

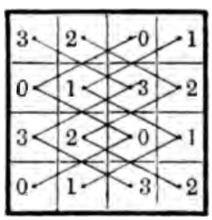
Even if the n^2 elements of a Nasik square of n, with all its paths, be obtained, in algebraic form, yet, in the case especially of high values of n, there remains the difficulty, perhaps insuperable, of deducing the consecutive numbers from 1 to n^2, an essential requisite in a Nasik square. The method employed in this paper avoids this difficulty. That a Nasik square when $n = 4$ may be formed by another kind of superposed squares is shown in Diagram T.

T

g



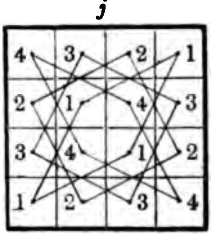
h



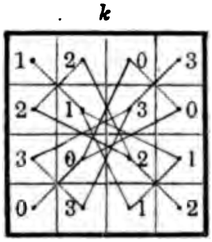
i

33	22	3	12
4	11	34	21
32	23	2	13
1	14	31	24

j



k



l

14	23	2	31
22	11	34	3
33	4	21	12
1	32	13	24

g is a square with 1 on the cells of (1,2) and 2, 4, 3 on its 3 parallels,
h „ „ 0 „ „ (2,1) „ 1, 3, 3 „ „

i is the superposed square, filled with 1 ... 16 in the quaternary. j, k are two squares, the way of filling which explains itself, and l is the square with these two squares superposed. This square could not be got by the method of this paper. If the lower two 2's and 3's are placed outside the square and above it, we have the four 1's, 2's, 3's, 4's at the angles of four similar diamonds, and, if the 0, 1, 2, 3 in the corner of k are put outside the square at the opposite corner, we have 1, 2, 3, 4 on the angles of four similar diamonds.

The way in which squares like l were obtained is as follows:— The numbers from 1 to 36 are changed to

$$\pm 1 \quad \pm 3 \quad \pm 5 \quad \pm 7 \quad \pm 9 \quad \pm 11 \quad \pm 13 \quad \pm 15.$$

These are arranged as in Diagram U,

U

1	−3	11	−9
−5	7	−15	13
−11	9	−1	3
15	−13	5	−7

where we have zero along the rows, columns, diagonals, and their parallels, 17 is added to each, and the sum halved, which gives Diagram V; this changed to the quaternary is Diagram W, which

V

9	7	14	4
6	12	1	15
3	13	8	10
16	2	11	5

W

21	13	32	04
12	24	01	33
03	31	14	22
34	02	23	11

suggested j, k. The same method gives CDR squares of $n = 2^t$, but they have no other paths along which W can be obtained, though they have several curious properties in compensation.

As every square filled with 1, 2, ... n^2 in the scale of n must have in it a number of the form $np+r$, W will be of the form $n\Sigma p+\Sigma r$, in every path where W is required. The method adopted here is to make $n\Sigma p$ *and* Σr constant, viz., $n\dfrac{n\,.\,\overline{n-1}}{2}$ *and* $\dfrac{n\,.\,\overline{n+1}}{2}$, while making their *sum* $\dfrac{n\,(n^2+1)}{2}$ constant gives scope for the possible formation of other Nasik squares, not obtainable by the method of this paper.

Explanation of Diagram A.

This diagram is part of an infinite number of squares of 6, each of which is filled in the same way with the consecutive numbers from 1 to 36 in the scale of 6. The six red lines show the directions in which lines can be drawn from any cell, as the one with 1 on it, through six numbers, along each of which there are one of four different groups of figures with the same sum. These lines are 1,34..., 1,54..., 1,14..., 1,36.... The other two lines 1,52... and 1,16... have on them the same numbers as those on 1,54... and 1,36... in a reverse order; so the former pair will not be now considered. The four lines through 34, 54, 14, and 36 correspond to the directions of the four path lines of D, 1,1, 3,1, 5,1, 1,3. The thirty-six numbers are now arranged in what may be called conjugate pairs, as each pair, as 34, 23, has the same sum 57; between each pair a letter is placed, to show that particular pair.

56	55	54	53	52	51	46	45	44	43	42	41	36	35	34	33	32	31
α	a	β	b	γ	c	d	δ	e	ϵ	f	ζ	η	g	θ	h	ι	k
1	2	3	4	5	6	11	12	13	14	15	16	21	22	23	24	25	26

The diagram shows that each of the four red lines of D contains the pair α, and two other pairs, the pairs on 1,34..., 1,54..., 1,14..., and 1,36... being α, θ, δ, α, β, γ, α, ϵ, ι, and α, η, ζ; or nine different conjugate pairs. If each of these four lines is reckoned as the first of its group of six parallels, we find, by drawing the parallels through the numbers 26, 41, 6, 21, 46 of the row through 1, that

	the odd parallels of			1,34... are	α, θ, δ	and the even ones are		a, e, k
					β, ζ, ι			c, f, h
					γ, ϵ, η			b, d, g
	,,	,,	,,	1,54... ,,	α, β, γ	,,	,, ,,	g, h, k
and				1,52...	δ, ϵ, ζ			a, b, c
					η, θ, ι			d, e, f
	,,	,,	,,	1,14... ,,	α, δ, θ	,,	,, ,,	a, e, k
					β, ζ, ι			c, f, h
					γ, ϵ, η			b, d, g
	,,	,,	,,	1,36... ,,	α, ζ, η	,,	,, ,,	b, e, h
and				1,16...	γ, δ, ι			c, d, k
					β, ϵ, θ			a, f, g

From this it appears that the three conjugate pairs on each of the red lines and its odd parallels comprise all the conjugate pairs of the red lines together. The same numbers as on the red lines and their parallels will be found on the lines from 1 through $\overbrace{34\ 12}$ $\overbrace{54\ 52}$ $\overbrace{14\ 32}$ and $\overbrace{36\ 16}$ and their parallels in every surrounding square; from which it follows that, if two lines are drawn from 1 in any direction at however small an inclination to each other, as they must at last enclose one of the squares, there will be eight lines that may be drawn from 1 through the above eight numbers in that square with W on each and their parallels, and still more lines as the number of enclosed squares increases. All that has been here said of the square of 6 is true of the squares of 10, 12, 14, ...; but the limit of this paper does not admit of the squares of higher order to be dealt with. A Nasik square may now be thus described:—"A square whose cells are filled with consecutive numbers from 1 to n^2, and which is surrounded by an unlimited number of squares similarly filled, and in such a manner that an infinite number of parallel lines can be drawn from any cell, in an infinite number of different directions, each of which has on it a recurring group of n numbers whose sum is the same."

TABLE OF CONTENTS.

On some general Formulæ for the Potentials of Ellipsoids, Shells, and Discs. *By* E. W. Hobson. Read April 23rd, 1896.
Received October 16th, 1896.

The problem of the determination of the potential of solid ellipsoids and of ellipsoidal shells at an external or an internal point, is one which has engaged the attention of many celebrated mathematicians. Among the various methods by which the potential of a uniform solid ellipsoid of gravitating matter at an external or internal point has been found, perhaps the most ingenious and interesting is that of Dirichlet. His method is that of multiplying the integral to be found by a discontinuous factor, which factor is equal to unity throughout the ellipsoid and to zero at all external points; the volume integral taken throughout the ellipsoid, which expresses the potential, is thus replaced by an integral to be taken throughout all space, and the simplification thus made in the limits of integration facilitates the reduction of the volume integral to a single integral.

Denoting by dv an element of volume of the ellipsoid, by r the distance of the element from the point at which the potential is to be found, the expression for the potential, the density being taken as unity, is

$$\iiint \frac{dv}{r},$$

in which the integral is taken throughout the volume inside the ellipsoid. This integral Dirichlet multiplies by

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\theta\cos f\theta}{\theta}\,d\theta,$$

where $f=1$ denotes the equation of the ellipsoid; this discontinuous factor has the value unity if $f<1$, and zero if $f>1$; thus the potential is

$$\frac{2}{\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{dv}{r}\,\frac{\sin\theta\cos f\theta}{\theta}\,d\theta,$$

which is then reduced to a single integral.

It occurred to me some three years ago that, by the use of a discontinuous factor of a more general form than that of Dirichlet, formulæ might be obtained for the potentials of ellipsoids of variable density and with any law of force; and, in fact, the principal results given in the present communication were obtained by the use of the expression

$$\frac{\iota e^{-\frac{1}{2}(\iota r\pi)}}{2\sin r\pi}\int_\infty^\infty (-\psi)^{r-1}\,e^{\iota f\psi}\sin\psi\,d\psi,$$

in which the integral is taken along a path from ∞ on the positive side of the ψ axis round the point $\psi = 0$, to $\psi = \infty$ on the negative side of the ψ axis; the real part of the above expression is equal to $\frac{\pi}{2\Gamma(1-r)}\,\frac{1}{(1-f)^r}$, if $f<1$, and to zero, if $f>1$.

After reading the modified form of Dirichlet's method given by Kronecker (*Vorlesungen*, Vol. I.), in which the factor

$$\int_{-\infty}^{\infty}\frac{e^{c(q+\iota z)}}{q+\iota z}\,dr,$$

q denoting a positive quantity, is used, I found that my results could be obtained more simply by the use of the expression

$$\int_{-\infty}^{\infty}\frac{e^{c(q+\iota z)}}{(q+\iota z)^{\lambda}}\,dz,$$

which is equal to

$$\frac{2\pi}{\Pi(\lambda-1)}\,c^{\lambda-1},$$

or zero, according as c is positive or negative, λ, q being positive quantities; this factor is accordingly used in the present paper.

I have found it convenient to consider the case of an n-dimensional elliptic "disc" in a space of $n+1$ dimensions; the determination of the potential of such a "disc" includes that of an n-dimensional solid ellipsoid as the special case in which the $n+1^{\text{th}}$ coordinate of the point at which the potential is found is put equal to zero; when the general formula has been found we can put $n=3$ for an ordinary solid ellipsoid, $n=2$ for an ordinary elliptic disc, and $n=1$ for a bar. In the same way a "ring" in $n+1$-dimensional space may be regarded as an ellipsoidal "shell" in n dimensions.

Many of the results in the present paper agree with results obtained by very different methods by Dr. Routh in his comprehensive memoir "On the Attraction of Ellipsoids for certain Laws of Force other than the Inverse Square."*

A particular case of the general formulæ has been given by Mr. Dyson,† who obtained his formulæ by verifying that an expression suggested by the consideration of simple cases satisfied the characteristic conditions.

The formulæ given by Cayley in his memoir on "Prepotentials,"‡ and in his other papers on the subject, are special cases of the formulæ below.

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* *Phil. Trans.*, Vol. CLXXXVI. (1895), A. † *Quarterly Journal*, Vol. XXV.
‡ *Phil. Trans.*, Vol. CLXV. (1875).

The Potential of an n-Dimensional Elliptic Disc which is in an n+1-Dimensional Space, the Law of Force being that of the inverse m+1th Power of the Distance.

1. Let $\xi_1, \xi_2, \ldots \xi_n$ denote the coordinates of any point of the n-dimensional elliptic "disc" whose equation is

$$\frac{\xi_1^2}{a_1^2}+\frac{\xi_2^2}{a_2^2}+\ldots+\frac{\xi_n^2}{a_n^2}=1,$$

and which is supposed to be in a space of $n+1$ dimensions, the coordinates of any point of which may be denoted by $x_1, x_2, \ldots x_\mu, h$, so that the disc is in the plane $h=0$; we shall first reduce the integral

$$W=\iiint\ldots\frac{\left[1-\frac{\xi_1^2}{a_1^2}-\frac{\xi_2^2}{a_2^2}-\ldots-\frac{\xi_n^2}{a_n^2}\right]^{\lambda-1}}{\left\{(x_1-\xi_1)^2+\ldots+(x_n-\xi_n)^2+h^2\right\}^{\frac{1}{2}m}}\,e^{a_1\xi_1+a_2\xi_2+\ldots+a_n\xi_n}\,d\xi_1\,d\xi_2\ldots d\xi_n,$$

where the integral is taken throughout the "disc," to a single integral; the quantities m, λ, n will be restricted to satisfy the conditions $m>0,\ \frac{n-m}{2}+\lambda-1 \geqq 0,\ \lambda-1\geqq 0.$

Using the transformation

$$\frac{1}{\left\{\Sigma\,(x-\xi)^2+h^2\right\}^{\frac{1}{2}m}}=\frac{1}{\Pi\left(\frac{m}{2}-1\right)}\int_0^\infty t^{\frac{1}{2}m-1}e^{-t\,[\Sigma(x-\xi)^2+h^2]}\,dt,$$

the n-fold integral W takes the form

$$\frac{1}{\Pi\left(\frac{m}{2}-1\right)}\iint\ldots\iint_0^\infty t^{\frac{1}{2}m-1}e^{-h^2t}\prod_1^n e^{-t(x_r-\xi_r)^2+a_r\xi_r}\left(1-\Sigma\frac{\xi_r^2}{a_r^2}\right)^{\lambda-1} dt\,d\xi_1\,d\xi_2\ldots d\xi_n.$$

Now it is known* that, provided λ is positive, the value of Cauchy's integral

$$\int_{-\infty}^{\infty}\frac{e^{c\,(q+\iota z)}}{(q+\iota z)^\lambda}\,dz,$$

where q is a positive quantity, is $\frac{2\pi}{\Pi\,(\lambda-1)}\,c^{\lambda-1}$, or zero, according as c is positive or negative; its value is zero when $c=0$, provided $\lambda>1$. We shall employ this as a discontinuous factor, to transform W into

* See Meyer's *Vorlesungen über die Theorie der bestimmten Integrale*, p. 196.

an integral in which the values of $\xi_1, \xi_2, \ldots \xi_n$ are taken each from $+\infty$ to $-\infty$. Let

$$c = 1 - \Sigma \frac{\xi_r^2}{a_r^2};$$

then W is equivalent to

$$\frac{1}{2\pi} \frac{\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}-1\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \int_0^{\infty} dt \int_{-\infty}^{\infty} \frac{t^{\frac{1}{2}m-1} e^{q+\iota z}}{(q+\iota z)^{\lambda}} e^{-t(h^2+\Sigma x^2)}$$

$$\prod_1^n e^{-t\xi^2+2tx\xi+a\xi-(q+\iota z)\xi^2/a^2}\, dz\, d\xi_1\, d\xi_2 \ldots d\xi_n;$$

all those elements of this integral vanish which correspond to points outside the "disc"; thus the integration may be taken throughout all the n-dimensional space instead of through the "disc" only.

On changing the order of the integrations and evaluating the integrals with respect to the ξ, by means of the formula

$$\int_{-\infty}^{\infty} e^{-t\xi^2+2tx\xi+a\xi-(q+\iota z)\xi^2/a^2}\, d\xi = \sqrt{\frac{\pi}{t+\frac{q+\iota z}{a^2}}}\, e^{(tx+\frac{1}{2}a)^2/[t+(q+\iota z)/a^2]},$$

we have

$$W = \frac{1}{2\pi} \frac{\Pi(\lambda-1)\,\pi^{\frac{1}{2}n}}{\Pi\left(\frac{m}{2}-1\right)} \int_0^{\infty} dt \int_{-\infty}^{\infty} \frac{t^{\frac{1}{2}m-1}}{(q+\iota z)^{\lambda}} e^{(q+\iota z)\left[1-\Sigma\left\{x^2/\left(a^2+\frac{q+\iota z}{t}\right)\right\}-h^2/\frac{q+\iota z}{t}\right]}$$

$$\Pi\left\{\frac{1}{\sqrt{\;+\frac{q+\iota z}{a^2}}} e^{\Sigma\left[a^2/\left(t+\frac{q+\iota z}{a^2}\right)+\left\{atx/\left(t+\frac{q+\iota z}{a^2}\right)\right\}\right]}\right\} dz.$$

Let

$$\frac{q+\iota z}{t} = \theta;$$

then

$$W = \frac{1}{2\pi} \frac{\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}-1\right)} a_1 a_2 \ldots a_n \pi^{\frac{1}{2}m} \int_0^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{\left\{(a^2+\theta)(a_2^2+\theta)\ldots(a_n^2+\theta)\right\}^{\frac{1}{2}}}$$

$$e^{\Sigma[(aa^2x)/(a^2+\theta)]}\, Q\, d\theta,$$

where

$$Q = \int_{-\infty}^{\infty} \frac{e^{(q+\iota z)\left(1-\Sigma\frac{x^2}{a^2+\theta}-\frac{h^2}{\theta}\right)}}{(q+\iota z)^{\frac{1}{2}(n-m)+\lambda}} e^{1/[4(q+\iota z)]\,\Sigma[(a^2a^2\theta)/(a^2+\theta)]}\, dz$$

$$= \int_{-\infty}^{\infty} \frac{e^{(q+\iota z)\left(1-\Sigma\frac{x^2}{a^2+\theta}-\frac{h^2}{\theta}\right)}}{(q+\iota z)^{\frac{1}{2}(n-m)+\lambda}} \left\{1+\sum_1^{\infty} \frac{1}{s!} \frac{A^s}{(q+\iota z)^s}\right\} dz,$$

where

$$A = \tfrac{1}{4}\Sigma \frac{a^2a^2\theta}{a^2+\theta}.$$

The value of the integral

$$\int_{-\infty}^{\infty} \frac{e^{(q+\iota z)\left(1-\Sigma\frac{x^2}{a^2+\theta}-\frac{h^2}{\theta}\right)}}{(q+\iota z)^{\frac{1}{2}(n-m)+\lambda+s}}\,dz,$$

is
$$\frac{2\pi}{\Pi\left(\frac{n-m}{2}+\lambda+s-1\right)}\left\{1-\Sigma\frac{x^2}{a^2+\theta}-\frac{h^2}{\theta}\right\}^{\frac{1}{2}(n-m)+\lambda+s-1},$$

or zero, according as $1-\Sigma\dfrac{x^2}{a^2+\theta}-\dfrac{h^2}{\theta}$ is positive or negative; hence

$$Q=\frac{2\pi}{\Pi\left(\frac{n-m}{2}+\lambda-1\right)}U^{\frac{1}{2}(n-m)+\lambda-1}$$
$$\left\{1+\frac{U\theta V}{2\,.\,n-m+2\lambda}+\frac{U^2\theta^2V^2}{2\,.\,4\,.\,n-m+2\lambda\,.\,n-m+2\lambda+2}+\dots\right\},$$

where
$$U=1-\Sigma\frac{x^2}{a^2+\theta}-\frac{h^2}{\theta},\quad V=\Sigma\frac{\alpha^2a^2}{a^2+\theta},$$

hence

$$W=\frac{\Pi(\lambda-1)\,\pi\quad a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}-1\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}U^{\frac{1}{2}(n-m)+\lambda-1}$$
$$\left\{1+\frac{U\theta V}{2\,.\,n-m+2\lambda}+\frac{U^2\theta^2V^2}{2.4.n-m+2\lambda.n-m+2\lambda+2}+\dots\right\}d\theta,$$

where θ_0 is the positive root of the equation $U=0$, and P denotes

$$\{(a_1^2+\theta)(a_2^2+\theta)\dots(a_n^2+\theta)\}^{\frac{1}{2}}.$$

Let $F\left(\dfrac{x_1}{a_1},\ \dfrac{x_2}{a_2},\ \dots\ \dfrac{x_n}{a_n}\right)$ be a function which, for all points within the elliptic "disc," is capable of representation in a convergent series of positive integral powers of $x_1, x_2, \dots x_n$, which series is given by Maclaurin's theorem; thus if $x_1', x_2', \dots x_n'$ denote the zero values of $x_1, x_2, \dots x_n$ at the centre,

$$F\left(\frac{x_1}{a_1},\frac{x_2}{a_2},\dots\frac{x_n}{a_n}\right)=e^{x_1(\partial/\partial x_1')+x_2(\partial/\partial x_2')+\dots+x_n(\partial/\partial x_n')}F\left(\frac{x_1'}{a_1},\dots\frac{x_n'}{a_n}\right),$$

where $x_1' \dots x_n'$ are each put equal to zero after the operation is performed. In W, let

$$\iota=\frac{\partial}{\partial x'},$$

and let W operate upon

$$F\left(\frac{x_1'}{a_1}, \frac{x_2'}{a_2}, \ldots \frac{x_n'}{a_n}\right);$$

we see then that the value V of

$$\frac{1}{m}\iiint\ldots\frac{\left\{1-\Sigma\frac{\xi^2}{a^2}\right\}^{\lambda-1}}{\left\{\Sigma(x-\xi)^2+h^2\right\}^{\frac{1}{2}m}}F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \ldots \frac{\xi_n}{a_n}\right)d\xi_1 d\xi_2 \ldots d\xi_n,$$

where the integral is taken throughout the "disc," is

$$\frac{\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)}\tfrac{1}{2}\pi^{\frac{1}{2}m}a_1a_2\ldots a_n\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}U^{\frac{1}{2}(n-m)+\lambda-1}$$

$$\left\{1+\frac{U\theta D}{2.n-m+2\lambda}+\frac{U^2\theta^2D^2}{2.4.n-m+2\lambda.n-m+2\lambda+2}\right.$$

$$\left.+\frac{U^3\theta^3D^3}{2.4.6.n-m+2\lambda.n-m+2\lambda+2.n-m+2\lambda+4}+\ldots\right\}$$

$$F\left(\frac{a_1x_1}{a_1^2+\theta}, \ldots, \frac{a_nx_n}{a_n^2+\theta}\right)d\theta \quad \ldots\ldots\ldots\ldots(1),$$

where $\quad P=\left\{(a_1^2+\theta)(a_2^2+\theta)\ldots(a_n^2+\theta)\right\}^{\frac{1}{2}},$

$$U=1-\frac{x_1^2}{a_1^2+\theta}-\frac{x_2^2}{a_2^2+\theta}-\ldots-\frac{x_n^2}{a_n^2+\theta}-\frac{h^2}{\theta},$$

$$D=\frac{a_1^2+\theta}{a_1^2}\frac{\partial^2}{\partial x_1^2}+\frac{a_2^2+\theta}{a_2^2}\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{a_n^2+\theta}{a_n^2}\frac{\partial^2}{\partial x_n},$$

and θ_0 is given by the equation

$$\frac{x_1^2}{a_1^2+\theta}+\frac{x_2}{a^2+\theta}+\ldots+\frac{x_n^2}{a_n^2+\theta}+\frac{h^2}{\theta}=1,$$

which obviously has one positive root, since the expression on the left-hand side of the equation is zero when θ is infinite, and is infinite when $\theta=0$, and continually increases as θ diminishes from infinity to zero.

The expression (1) represents the potential of the n-dimensional disc when the law of force is that of the inverse $(m+1)^{\text{th}}$ power of the distance, and the law of density is given by

$$\rho=\left\{1-\frac{\xi_1^2}{a_1^2}-\frac{\xi_2^2}{a_2^2}-\ldots-\frac{\xi_n^2}{a_n^2}\right\}^{\lambda-1}F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \ldots \frac{\xi_n}{a_n}\right)\ldots(2);$$

m is not necessarily integral, but is subject to the restrictions that it is positive and such that $\frac{n-m}{2}+\lambda-1$ is positive.

In case the function F contains an infinite number of terms, it must be such that the series contained in the integral in (1) is absolutely convergent.

It has been pointed out by Kronecker that Dirichlet's investigation of the potential of a homogeneous ellipsoid is not free from objections; the same objections hold in regard to the preceding investigation. Kronecker* gives an elaborate discussion of the legitimacy of the various operations, including that of the inversion of order of integration; his discussion is applicable to the more general theorem investigated above; it should, however, be observed that a considerable part of Kronecker's investigation has been rendered unnecessary by the introduction of the finite quantity h, which in his work is taken to be zero.

2. In the particular case $n = 2$, we have from (1) the potential of a two-dimensional elliptic disc; in this case, writing x, y for x_1, x_2, and a, b for a_1, a_2, we see that the potential of the elliptic disc

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

of which the density is given by

$$\rho=\mu\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\lambda-1}F\left(\frac{x}{a},\ \frac{y}{b}\right)$$

at an external point x, y, z, is

$$\frac{\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\lambda-\frac{m}{2}\right)}\frac{\mu\pi ab}{2}\int_{\theta_0}^{\infty}\frac{\theta^{-\frac{1}{2}m}\,U^{\lambda-\frac{1}{2}m}}{\sqrt{(a^2+\theta)(b^2+\theta)}}$$

$$\cdot\left\{1+\frac{U\theta D}{2.2-m+2\lambda}+\frac{U^2\theta^2D^2}{2.4.2-m+2\lambda.2-m+2\lambda+2}+\dots\right\}$$

$$F\left(\frac{ax}{a^2+\theta},\ \frac{by}{b^2+\theta}\right)d\theta \quad \dots\dots\dots\dots\dots\dots(3),$$

* See *Vorlesungen über die Theorie der einfachen und der vielfachen Integrale*, pp. 317–341.

where
$$D = \frac{a^2+\theta}{a^2}\frac{\partial^2}{\partial x^2} + \frac{b^2+\theta}{b^2}\frac{\partial^2}{\partial y^2},$$

$$U = 1 - \frac{x^2}{a^2+\theta} - \frac{y^2}{b^2+\theta} - \frac{z^2}{\theta},$$

and θ_0 is the positive root of the equation $U = 0$; $\lambda - \frac{m}{2}$ must be positive.

For an internal point we have $z = 0$, and $\theta_0 = 0$, since

$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

is negative for all points inside the disc; thus the lower limit of the integral is zero.

We see that the potential of a uniform gravitating disc is

$$2\mu ab \int_{\theta_0}^{\infty} \frac{U^{\frac{1}{2}}\, d\theta}{\sqrt{\theta}\sqrt{(a^2+\theta)(b^2+\theta)}},$$

where $\theta_0 = 0$ for an internal point; this agrees with the expression given by Cayley.*

From (3) we obtain the general theorem that the external potential of a disc of density

$$\mu\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\lambda-1},$$

when the law of force is that of the inverse $(2\lambda+1)^{\text{th}}$ power of the distance, is

$$\frac{1}{\lambda}\,\frac{\mu\pi ab}{2}\int_{\theta_0}^{\infty} \frac{d\theta}{\theta^{\lambda}\sqrt{(a^2+\theta)(b^2+\theta)}},$$

and thus that the external level surfaces are ellipsoids confocal with the disc.

The Potential of an n-Dimensional Solid Ellipsoid.

8. If in the theorem (1) we put $h = 0$, we have an expression for the potential of the solid ellipsoid

$$\frac{\xi_1^2}{a_1^2} + \frac{\xi_2^2}{a_2^2} + \ldots + \frac{\xi_n^2}{a_n^2} = 1$$

* See *Proc. Lond. Math. Soc.*, Vol. VI. (1875).

of which the density is given by

$$\rho = \left(1 - \frac{\xi_1^2}{a_1^2} - \frac{\xi_2^2}{a_2^2} - \dots - \frac{\xi_n^2}{a_n^2}\right)^{\lambda-1} F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \dots, \frac{\xi_n}{a_n}\right),$$

the law of force being that of the inverse $(m+1)^{\text{th}}$ power of the distance, m being subject to the same restrictions as before.

For an external point, the limit θ_0 of the integral is given by

$$\Sigma \frac{x^2}{a^2+\theta_0} = 1\,;$$

for an internal point we have $\theta_0 = 0$, since, as h approaches the value zero, θ_0 must do the same, their ratios being given by

$$\Sigma \frac{x^2}{a^2} = 1 - \text{Lt.}\, \frac{h^2}{\theta_0},$$

the expression $1 - \Sigma \dfrac{x^2}{a^2}$ having a positive value for all internal points.

The potential of the solid ellipsoid is therefore

$$\frac{\Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 a_3 \dots a_n \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} U^{\frac{1}{2}(n-m)+\lambda-1}$$

$$\left\{1 + \frac{U\theta D}{2.\overline{n-m+2\lambda}} + \frac{U^2\theta^2 D^2}{2.4.\overline{n-m+\ \lambda}.\overline{n-m+2\lambda+2}} + \dots\right\}$$

$$F\left(\frac{a_1 x_1}{a_1^2+\theta}, \dots \frac{a_n x_n}{a_n^2+\theta}\right) d\theta \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4),$$

where $U = 1 - \Sigma \dfrac{x^2}{a^2+\theta}$, $\lambda - 1 \geqq 0$, $m > 0$,

$D = \Sigma \dfrac{a^2+\theta}{a^2} \dfrac{\partial^2}{\partial x^2}$, $\dfrac{n-m}{2} + \lambda - 1 \geqq 0$,

and θ_0 is the positive root of the equation

$$\Sigma \frac{x^2}{a^2+\theta} = 1,$$

and is zero for an internal point.

In the particular case $n = 3$, $m = 1$, the formula agrees with that obtained by Mr. Dyson*, who obtained the expression by the

* See *Quarterly Journal of Math.*, Vol. xxv.

consideration of particular cases, and then verified that the expression satisfied the characteristic equation

$$\nabla^2 V = -4\pi\rho.$$

When the density is given by

$$\rho = \mu \left(1 - \Sigma \frac{\xi^2}{a^2}\right)^{\lambda-1},$$

the formula (4) reduces to

$$\frac{\mu \Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right) \Pi\left(\frac{n-m}{2}+\lambda-1\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} U^{\frac{1}{2}(n-m)+\lambda-1}\, d\theta.$$

For example, let $n = 3$; we have then for the ordinary three-dimensional ellipsoid

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1,$$

the law of density being given by

$$\rho = \mu \left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2}\right)^{\lambda-1},$$

$$V = \frac{\mu \Pi(\lambda-1)}{\Pi\left(\frac{m}{2}\right) \Pi\left(\frac{1}{2}+\lambda-\frac{m}{2}\right)} \tfrac{1}{2}\pi^{\frac{3}{2}} abc \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(1-m)}}{\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}} \left\{1 - \frac{x^2}{a^2+\theta} - \frac{y^2}{b^2+\theta} - \frac{z^2}{c^2+\theta}\right\}^{\frac{1}{2}(1-m)+\lambda} d\theta.$$

When $\lambda = \frac{1}{2}(m-1)$, we see that the external level surfaces are confocal ellipsoids.

The Potentials of infinitely thin Homœoidal Rings and Ellipsoidal Shells.

4. In the expression (1), change $a_1, a_2, \dots a_n$ into $qa_1, qa_2, \dots qa_n$, and also θ, θ_0 into $q^2\theta, q^2\theta_0$; the expression then becomes, in the case $\lambda = 1$,

$$\frac{1}{\Pi\left(\frac{m}{2}\right) \Pi\left(\frac{n-m}{2}\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-}}{P} U_q^{\frac{1}{2}(n-m)} \left\{1 + \frac{U_q \theta D}{2.n-m+2} + \dots\right\} F d\theta,$$

where

$$U_q = q^2 - \Sigma \frac{x^2}{a^2+\theta} - \frac{h^2}{\theta},$$

and θ_0 is the positive root of the equation $U_q = 0$; P, D have the same meanings as before. If we take the differential of this expresion with respect to q, we obtain the potential of an indefinitely thin homœoidal ring, bounded by the discs corresponding to the values q, $q+dq$ of q; this is, since the differential with respect to θ_0 is zero,

$$\frac{1}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)}\pi^{\frac{1}{2}n}a_1a_2\dots a_n\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}\,q\,dq\,U_q^{\frac{1}{2}(n-m)-1}$$
$$\left\{1+\frac{U_q\theta D}{2.n-m}+\frac{U_q^2\theta^2D^2}{2.4.n-m.n-m+2}+\dots\right\}Fd\theta \;\dots(a).$$

Now let $q = 1$, and suppose p is given by

$$\frac{1}{p^2} = \Sigma\frac{\xi^2}{a^4};$$

then pdS is an element of "area" of the ring, when dS is an element of "arc" of the bounding ellipse; we have

$$\frac{dp}{p} = \frac{dq}{q} = dq;$$

hence, if matter of density

$$\mu p F\left(\frac{\xi_1}{a_1}, \dots \frac{\xi_n}{a_n}\right)$$

is distributed along the "arc" of the ellipse

$$\Sigma\frac{\xi^2}{a^2} = 1,$$

the potential at any point whose coordinates are $x_1, x_2, \dots x_n, h$ is

$$\frac{\mu\pi^{\frac{1}{2}n}a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}U^{\frac{1}{2}(n-m)-1}$$
$$\left\{1+\frac{U\theta D}{2.n-m}+\frac{U^2\theta^2D^2}{2.4.n-m.n-m+2}+\dots\right\}F\left(\frac{a_1x_1}{a_1^2+\theta}, \dots\right)d\theta\dots(5),$$

where U, D, θ_0 have the same meanings as in (1).

If in (5) we put $h = 0$, the formula gives the potential of an ellipsoidal shell of density

$$\mu p F\left(\frac{\xi_1}{a_1}, \dots \frac{\xi_n}{a_n}\right),$$

the lower limit θ_0 being, in this case, zero for an internal point. The particular case $n=3$, $m=1$ of this last theorem was given by Mr. Dyson (*loc. cit.*).

Extension of the Formulæ for the Potentials of an Elliptic Disc and a Solid Ellipsoid.

5. If we multiply the expression (a) by $\mu\phi(q^2)$, and integrate with respect to q, between the limits 0 and 1, we have an expression for the potential of an elliptic "disc" of density

$$\rho = \mu\phi\left(\frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}\right) F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \dots \frac{\xi_n}{a_n}\right).$$

The expression for the potential is

$$\frac{\mu\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)} \int_0^1 \phi(q^2)\, q\, dq \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} U^{\frac{1}{2}(n-m)-1} \left\{1+\frac{U_q \theta D}{2.n-m}+\dots\right\} F d\theta,$$

where
$$q^2 - \Sigma \frac{x^2}{a^2+\theta_0} - \frac{h^2}{\theta_0} = 0;$$

on changing the order of integration, the integral with respect to q^2 must be taken from

$$\Sigma \frac{x^2}{a^2+\theta} + \frac{h^2}{\theta},$$

which we shall denote by σ, to unity, and θ must be taken from the value of θ_0 given by

$$1 - \Sigma \frac{x^2}{a^2+\theta_0} - \frac{h^2}{\theta_0} = 0,$$

up to ∞; thus the expression for the potential of the "disc" is

$$\frac{\mu\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{2\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)} \int_{\theta_0}^{\infty} \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} \int_{\sigma}^{1} \phi(q^2)\, d(q^2) (q^2-\sigma)^{\frac{1}{2}(n-m)-1} \left\{1+\frac{(q^2-\sigma)\,\theta D}{2.n-m}+\dots\right\} F d\theta.$$

Let
$$q^2 = \sigma + (1-\sigma)\, v;$$

then the limits for v are 0 and 1; hence the potential is

$$\frac{\mu\pi^{\frac{1}{2}n}a_1 a_2 \dots a_n}{2\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}(1-\sigma)^{\frac{1}{2}(n-m)}$$

$$\int_0^1 \phi\,(\sigma+\overline{1-\sigma}\,v)\,v^{\frac{1}{2}(n-m)-1}\,dv$$

$$\left\{1+\frac{(1-\sigma)\,v\theta D}{2.n-m}+\frac{(1-\sigma)^2 v^2\theta^2 D^2}{2.4.n-m.n-m+2}+\dots\right\}F\left(\frac{a_1 x_1}{a_1^2+\theta},\dots\right)d\theta$$

$$\dots\dots\dots\dots(6).$$

This is a generalization of a theorem due to Boole,* and Cayley, which is obtained by putting $F=1$, when the expression becomes

$$\frac{\mu\pi^{\frac{1}{2}n}a_1 a_2 \dots a_n}{2\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}(1-\sigma)^{\frac{1}{2}(n-m)}$$
$$\int_0^1 \phi\,(\sigma+\overline{1-\sigma}\,v)\,v^{\frac{1}{2}(n-m)-1}\,d\theta,$$

the potential of a "disc" of density

$$\mu\phi\left(\frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}\right).$$

The potential of an ellipsoid of density

$$\mu\phi\left(\frac{\xi_1^2}{a_1^2}+\dots+\frac{\xi_n^2}{a_n^2}\right)F\left(\frac{\xi_1}{a_1},\ \dots\ \frac{\xi_n}{a_n}\right)$$

is obtained from (6) by putting $h=0$, in which case, as before, θ_0 has the value zero for an internal point.

It may be observed that, if $n-m>3$, the component attractions parallel to the x_1 axis may be obtained from (6) by changing m into $m+1$ and putting

$$F=a_1\left(\frac{\xi_1}{a_1}-1\right),$$

so that
$$F\left(\frac{a_1 x_1}{a_1^2+\theta},\ \dots\right)=a_1\left(\frac{x_1}{a_1^2+\theta}-1\right).$$

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* See, for references, Cayley's memoir on "Prepotentials," *Phil. Trans.*, 1875, p. 774.

The Potentials of Rings and Shells in case $m > n-2$.

6. The formulæ hitherto obtained for the potentials of rings and shells are subject to the restriction $m < n-2$; others may, however, be deduced for the case $m > n-2$.

Since
$$\frac{d^r}{d(h^2)^r}\frac{1}{\{\Sigma(x-\xi)^2+h^2\}^{\frac{1}{2}m}}$$
$$=(-1)^r\frac{m}{2}\left(\frac{m}{2}+1\right)\dots\left(\frac{m}{2}+r-1\right)\frac{1}{\{\Sigma(x-\xi)^2+h^2\}^{\frac{1}{2}(m+2r)}},$$

we have, from (5), for the potential of an elliptic ring, when the law of force is the inverse $(m+2r+1)^{\text{th}}$ power of the distance,

$$V=\frac{\mu\pi^{\frac{1}{2}n}a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}+r\right)\Pi\left(\frac{n-m}{2}-1\right)}(-1)^r\frac{d^r}{d(h^2)^r}$$
$$\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}U^{\frac{1}{2}(n-m)-1}\left\{1+\frac{U\theta D}{2.n-m}+\dots\right\}F\left(\frac{a_1x_1}{a_1^2+\theta},\dots\right)d\theta.$$

We thus have to calculate the values of expressions of the form

$$(-1)^r\frac{d^r}{d(h^2)^r}\int_{\theta_0}^{\infty}\theta^{\frac{1}{2}(n-m)+s-1}U^{\frac{1}{2}(n-m)+s-1}D^sF\left(\frac{a_1x_1}{a_1^2+\theta},\dots\right)d\theta;$$

first suppose s such that

$$\frac{n-m}{2}+s-1\geqq r;$$

since U vanishes when $\theta=\theta_0$, we can differentiate under the integral sign, and we thus find, as the result of the differentiation,

$$\left(\frac{n-m}{2}+s-1\right)\left(\frac{n-m}{2}+s-2\right)\dots\left(\frac{n-m}{2}+s-r\right)$$
$$\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)+s-r-1}}{P}U^{\frac{1}{2}(n-m)+s-r-1}D^sF\left(\frac{a_1x_1}{a_1^2+\theta},\dots\right)d\theta.$$

We thus obtain, as part of the value of V, the expression

$$V_1=\frac{\pi^{\frac{1}{2}n}a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}+r\right)}\int_{\theta_0}^{\infty}\sum_{s_0}\frac{\theta^{\frac{1}{2}(n-m)+s-r-1}}{P}\frac{U^{\frac{1}{2}(n-m)+s-r-1}}{2^{2s}\Pi(s)\Pi\left(\frac{n-m}{2}+s-r-1\right)}$$
$$D^sF\left(\frac{a_1x_1}{a_1+\theta},\dots\right)d\theta\ \dots\dots\dots(7),$$

where the summation is taken for all values of s such that

$$s+\frac{n}{2}-\frac{m}{2}-r-1 \geqq 0.$$

The complete value of V is V_1+V_2, where

$$V_2=\frac{\pi^{\frac{1}{2}n}a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}+r\right)\Pi\left(\frac{n-m}{2}-1\right)}(-1)^r\frac{d^r}{d(h^2)^r}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}U^{\frac{1}{2}(n-m)-1}$$

$$\left\{1+\frac{U\theta D}{2.n-m}+\dots+\frac{U^{s_0-1}\theta^{s_0-1}D^{s_0-1}}{2.4\dots 2s_0-2.n-m\dots n-m+2s_0-2}\right\}$$

$$F\left(\frac{a_1x_1}{a_1^2+\theta},\ \dots\right)d\theta\ \dots\dots\dots\dots(8);$$

this value of V_2 can be simplified in special cases.

7. Suppose $n-m$ is an even integer; then in that case

$$s_0=\frac{m-n}{2}+r+1,$$

and thus

$$V_1=\frac{\pi^{\frac{1}{2}n}a_1a_2\dots a_n}{\Pi\left(\frac{m}{2}+r\right)}\frac{1}{2^{2s_0}}\int_{\theta_0}^{\infty}\frac{1}{P}$$

$$\left\{\frac{D^{s_0}}{\Pi(s_0)\,\Pi(0)}+\frac{\theta UD^{s_0+1}}{2^2\Pi(s_0+1)\,\Pi(1)}+\frac{\theta^2U^2D^{s_0+2}}{2^4\Pi(s_0+2)\,\Pi(2)}+\dots\right\}$$

$$F\left(\frac{a_1x_1}{a_1^2+\theta},\ \dots\right)d\theta.$$

It will be observed that this expression vanishes unless F contains terms of degrees equal to or higher than $2s_0$.

To evaluate V_2, consider the expression

$$\frac{d^r}{d(h^2)^r}\int_{\theta_0}^{\infty}\frac{1}{P}\theta^{\frac{1}{2}(n-m)-1+t}U^{\frac{1}{2}(n-m)-1+t}D^tF\left(\frac{a_1x_1}{a_1^2+\theta},\ \dots\right)d\theta,$$

where $t<s_0$; this term is equal to

$$\frac{d^{s_0-t}}{d(h^2)^{s_0-t}}\int_{\theta_0}^{\infty}\frac{\theta^{r+t-s_0}}{P}\frac{d^{r-(s_0-t)}}{d(h^2)^{r-(s_0-t)}}U^{r-(s_0-t)}D^tF\left(\frac{a_1x_1}{a_1^2+\theta},\ \dots\right)d\theta$$

$$=\frac{d^{(s_0-t)}}{d(h^2)^{(s_0-t)}}(-1)^{r-(s_0-t)}\Pi(r-s_0+t)\int_{\theta_0}^{\infty}\frac{1}{P}D^tF\left(\frac{a_1x_1}{a_1^2+\theta},\ \dots\right)$$

now, since
$$\Sigma \frac{x^2}{a^2+\theta_0} + \frac{h^2}{\theta_0} = 1,$$

we have
$$\left\{ \Sigma \frac{x^2}{(a^2+\theta_0)^2} + \frac{h^2}{\theta_0^2} \right\} d\theta_0 - \frac{1}{\theta_0}\, d(h^2) = 0;$$

hence
$$\frac{d}{d(h^2)} = \frac{1}{\theta_0 \Sigma \dfrac{x^2}{(a^2+\theta_0)^2} + \dfrac{h^2}{\theta_0}} \frac{d}{d\theta_0} = \frac{1}{1-\Sigma \dfrac{a^2x^2}{(a^2+\theta_0)^2}} \frac{d}{d\theta_0};$$

hence the term is equal to

$$(-1)^{r-s_0+t+1}\,\Pi\,(r-s_0+t) \left\{ \frac{1}{1-\Sigma \dfrac{a^2x^2}{(a^2+\theta_0)^2}} \frac{d}{d\theta_0} \right\}^{s_0-t-1}$$
$$\frac{1}{P_0\left(1-\Sigma \dfrac{a^2x^2}{(a^2+\theta_0)^2}\right)} D_0^t F\left(\frac{a_1x_1}{a_1^2+\theta_0}, \dots\right),$$

where
$$P_0 = \{(a_1^2+\theta_0)(a_2^2+\theta_0) \dots (a_n^2+\theta_0)\}^{\frac{1}{2}},$$
$$D_0 = \frac{a_1^2+\theta_0}{a_1^2} \frac{\partial^2}{\partial x_1^2} + \dots + \frac{a_n^2+\theta_0}{a_n^2} \frac{\partial^2}{\partial x_n^2};$$

therefore the value of V_2 is given by

$$V_2 = \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\dfrac{m}{2}+r\right)} (-1)^{\frac{1}{2}(n-m-2r)}$$
$$\left\{ \left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m+2r-n)} \frac{1}{P_0Q_0} - \left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m+2r-n)-1} \frac{D_0}{2^2.P_0Q_0} \right.$$
$$+ \left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m+2r-n)-2} \frac{D_0^2}{2^4.2!\,P_0Q_0} - \dots + \dots$$
$$\left. + (-1)^{\frac{1}{2}(m+2r-n)} \frac{D_0^{\frac{1}{2}(m+2r-n)}}{2^{m+2r-n}\,\Pi\left(\dfrac{m+2r-n}{2}\right)} \right\} F\left(\frac{a_1x_1}{a_1^2+\theta_0}, \dots\right),$$

where
$$Q_0 = 1 - \Sigma \frac{a^2x^2}{(a^2+\theta_0)^2}.$$

Now write $m+2r$ for m; we have then as the expression for the potential of a distribution of matter of density

$$\rho = \mu p F\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \dots \frac{\xi_n}{a_n}\right),$$

on the "arc" of an elliptic ring of n dimensions, when the law of force is such that $m-n$ is even and >-2, the expression

$$V = \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{m}{2}\right)} \frac{1}{2^{m-n+2}} \int_{\theta_0}^{\infty} \frac{1}{P}$$

$$\left\{\frac{D^{s_0}}{\Pi(s_0)\,\Pi(0)} + \frac{\theta U D^{s_0+1}}{2^2\Pi(s_0+1)\,\Pi(1)} + \frac{\theta^2 U^2 D^{s_0+2}}{2^4\Pi(s_0+2)\,\Pi(2)} + \dots\right\}$$

$$F\left(\frac{a_1 x_1}{a_1^2+\theta}, \dots\right) d\theta + \frac{\pi^{\frac{1}{2}n} a_1 \dots a_n}{\Pi\left(\frac{m}{2}\right)} (-1)^{\frac{1}{2}(m-n)}$$

$$\left\{\left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m-n)} \frac{1}{P_0 Q_0} - \left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m-n)-1} \frac{D_0}{2^2.1!\,P_0 Q_0} + \dots\right.$$

$$\left. + (-1)^{\frac{1}{2}(m-n)} \frac{D_0^{\frac{1}{2}(m-n)}}{2^{m-n}\,\Pi\left(\frac{m-n}{2}\right) P_0 Q_0}\right\} F\left(\frac{a_1 x_1}{a_1^2+\theta_0}, \dots\right) \dots (9),$$

where $$s_0 = \frac{m-n}{2}+1.$$

The potential of an ellipsoidal thin shell is obtained from (7) by putting $h=0$; for internal points θ_0 must be put equal to zero after the differentiations are performed.

8. In the case in which the density is constant, $m-n$ being still an even integer, the potential is

$$\frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{m}{2}\right)} (-1)^{\frac{1}{2}(m-n)} \left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m-n)} \frac{1}{P_0 Q_0};$$

in the special case $m=n$, this becomes

$$\frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{n}{2}\right)} \frac{1}{\sqrt{(a_1^2+\theta_0)\dots(a_n^2+\theta_0)}} \frac{1}{1 - \frac{a_1^2 x_1^2}{(a_1^2+\theta)^2} - \dots - \frac{a_n^2 x_n^2}{(a_n^2+\theta)^2}} \dots (10);$$

and in the case of an ellipsoidal shell, this becomes for an internal point

$$\frac{\pi^{\frac{1}{2}n}}{\Pi\left(\frac{n}{2}\right)} \frac{1}{1-\frac{x_1^2}{a_1^2}-\dots-\frac{x_n^2}{a_n^2}} \quad \dots\dots\dots\dots (11).$$

In particular, we have, for $n = 3$, Townsend's expressions

$$\tfrac{1}{3}\pi \frac{abc}{a'b'c'} \frac{1}{\theta_0\left(\frac{x^2}{a'^4}+\frac{y^2}{b'^4}+\frac{z^2}{c'^4}\right)}$$

and

$$\tfrac{1}{3}\pi \frac{1}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$$

for the external and internal potentials of an ordinary ellipsoidal shell when the law of force is that of the inverse fourth power of the distance.

Similar formulæ hold in the case $n = 2$, $m = 2$, for a homœoidal ring.

In the general case $m = n$, the potential is

$$V = \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{n}{2}\right)} \frac{1}{P_0 Q_0} F\left(\frac{a_1 x_1}{a_1^2+\theta_0}, \dots\right)$$

$$+ \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{4\Pi\left(\frac{n}{2}\right)} \int_{\theta_0}^{\infty} \frac{1}{P} \left(\frac{D}{\Pi(1)\,\Pi(0)} + \frac{\theta U D^2}{2^3\Pi(2)\,\Pi(1)} + \dots\right)$$

$$F\left(\frac{a_1 x_1}{a_1^2+\theta}, \dots\right) d\theta.$$

When F is a linear function, say

$$F\left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \dots\right) = A_1 \frac{x_1}{a_1} + A_2 \frac{x_2}{a_2} + \dots,$$

we have

$$V = \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{n}{2}\right)} \frac{A_1 \frac{a_1 x_1}{a_1^2+\theta_0} + A_2 \frac{a_2 x_2}{a_2^2+\theta_0} + \dots}{\sqrt{(a_1^2+\theta_0)(a_2^2+\theta_0)\dots}\left(1-\Sigma\frac{a^2 x^2}{(a^2+\theta_0)^2}\right)},$$

and at an internal point

$$V = \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{n}{2}\right)} \frac{\Sigma A \frac{x}{a}}{1 - \Sigma \frac{x^2}{a^2}}.$$

The Internal Potentials of Shells of Uniform Density.

9. A convenient expression for the internal potential of a shell of constant density, for the case in which $m-n$ is a positive integer, can be deduced from (11).

Taking μp as the density of distribution over the "surface," we have, since

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}\right) \frac{1}{\{\Sigma (n-\xi)^2\}^{\frac{1}{2}m}} = m(m-n+2) \frac{1}{\{\Sigma (x-\xi)^2\}^{\frac{1}{2}m+1}},$$

$$\nabla^{2r} \frac{1}{\{\Sigma (x-\xi)^2\}^{\frac{1}{2}m}} = m(m+2) \dots (m+2r-2)(m-n+2) \dots (m-n+2r) \frac{1}{\{\Sigma (x-\xi)^2\}^{\frac{1}{2}m+r}},$$

the potential of the shell when the law of force is that of the inverse $(n+2r+1)^{\text{th}}$ power of the distance is

$$\mu \frac{n}{n+2r} \frac{1}{n(n+2)\dots(n+2r-2)\, 2.4 \dots 2r} \frac{\pi^{\frac{1}{2}n}}{\Pi\left(\frac{n}{2}\right)} \nabla^{2r} \frac{1}{1-\Sigma \frac{x^2}{a^2}},$$

or, writing $m = n+2r,$

$$\mu \frac{1}{m} \frac{1}{(n+2)\dots(m-2)\, 2.4 \dots 2(m-n)} \frac{\pi^{\frac{1}{2}n}}{\Pi\left(\frac{n}{2}\right)} \nabla^{m-n} \frac{1}{1-\Sigma \frac{x^2}{a^2}},$$

when $m-n$ is even and positive.

Now, taking the differentiation theorem*

$$f_p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\right) \phi\, (x_1^2 + \dots + x_n^2)$$

$$= \Big\{ 2^p \phi^{(p)} (x_1^2 + \dots + x_n^2) + \frac{2^{p-1}}{2} \phi^{(p-1)} (x_1^2 + \dots + x_n^2) \nabla^2 + \frac{2^{p-2}}{2.4} \phi^{(p-2)} (x_1^2 + \dots + x_n^2) \nabla^4 + \dots \Big\} f_p(x_1, x_2, \dots x_n),$$

---

* See *Proc. Lond. Math Soc.*, Vol. xxiv., p. 67.

we have, on writing $\frac{x}{a}$ for x, and

$$f_p(x_1, x_2, \dots x_n) = \left\{\Sigma \frac{x^2}{a^4}\right\}^{\frac{1}{2}p}, \quad p = m-n,$$

$$\nabla^{m-n}\phi\left(\Sigma_n \frac{x^2}{a^2}\right) = \left\{2^{m-n}\phi^{(m-n)}\left(\Sigma \frac{x^2}{a^2}\right) + \frac{2^{m-n-1}}{2}\phi^{(m-n-1)}\left(\Sigma \frac{x^2}{a^2}\right)\delta^2\right.$$

$$\left. + \frac{2^{m-n-2}}{2.4}\phi^{(m-n-2)}\left(\Sigma \frac{x^2}{a^2}\right)\delta^4 + \dots\right\}\left(\Sigma \frac{x^2}{a^4}\right)^{\frac{1}{2}(m-n)},$$

where
$$\delta^2 = \sum_1^n a^2 \frac{\partial^2}{\partial x^2}.$$

Writing
$$\phi\left(\Sigma \frac{x^2}{a^2}\right) = \frac{1}{1-\Sigma \frac{x^2}{a^2}},$$

we have
$$\nabla^{m-n}\phi\left(\Sigma \frac{x^2}{a^2}\right)$$

$$= 2^{m-n}\Pi(m-n)\left\{\frac{1}{\left(1-\Sigma \frac{x^2}{a^2}\right)^{m-n+1}} + \frac{1}{2.2m-2n}\,\frac{1}{\left(1-\Sigma \frac{x^2}{a^2}\right)^{m-n}}\delta^2\right.$$

$$\left. + \frac{1}{2.4.2m-2n.2m-2n-2}\,\frac{1}{\left(1-\Sigma \frac{x^2}{a^2}\right)^{m-n-1}}\delta^4 + \dots\right\}\left(\Sigma \frac{x^2}{a^4}\right)^{\frac{1}{2}(m-n)};$$

hence the required potential is

$$\mu \frac{\Pi(m-n)\,\pi^{\frac{1}{2}n}}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{m-n}{2}\right)}\,\frac{1}{H^{m-n+1}}\left\{1 + \frac{H\delta}{2.2m-2n} + \frac{H^2\delta^2}{2.4.2m-2n.2m-2n-2} + \dots\right\}$$

$$\left(\frac{x_1^2}{a_1^4} + \frac{x_2^2}{a_2^4} + \dots\right)^{\frac{1}{2}(m-n)} \quad \dots\dots(12$$

where
$$H = 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \dots - \frac{x_n^2}{a_n^2},$$

$$\delta = a_1^2\frac{\partial^2}{\partial x_1^2} + a_2^2\frac{\partial^2}{\partial x_2^2} + \dots + a_n^2\frac{\partial^2}{\partial x_n^2}.$$

In the case $n = 3$, this becomes

$$\mu \frac{2^{m-1}\pi}{m(m-2)}\,\frac{1}{H^{m-2}}$$

$$\left\{1 + \frac{H\delta}{2.2m-6} + \frac{H^2\delta^2}{2.4.2m-6.2m-8} + \dots\right\}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}(m-3)},$$

where
$$H = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2},$$
$$\delta = a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2} + c^2 \frac{\partial^2}{\partial z^2}.$$

This agrees with Dr. Routh's formula (*loc. cit.*).

In the case $n = 2$, we have

$$\mu \frac{\Pi(m-2)\pi}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{m}{2}-1\right)} \frac{1}{H^{m-1}}$$
$$\left\{1 + \frac{H\delta}{2.2m-4} + \frac{H^2\delta^2}{2.4.2m-4.2m-6} + \dots\right\} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{\frac{1}{2}m-1}$$

for the potential inside a circular ring, the law of force being any odd inverse power of the distance, greater than the first, the line density on the arc being μp.

10. When $\frac{n-m}{2} - 1$ is positive, the internal potential of a homogeneous shell is

$$\frac{\mu \pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)} \int_0^\infty \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} \left(1 - \Sigma \frac{x^2}{a^2+\theta}\right)^{\frac{1}{2}(n-m)-1} d\theta.$$

Excluding the case considered in Art. 9, in which $n-m$ is an even integer, we can deduce the potential for all values of m greater than $n-2$; the potential for the law of force, the inverse $(m+2r+1)^{\text{th}}$ power of the distance, is

$$\mu \frac{m}{m+2r} \frac{1}{m(m+2)\dots(m+2r-2)(m-n+2)\dots(m-n+2r)}$$
$$\nabla^{2r} \frac{\pi^{\frac{1}{2}n} a_1 a_2 \dots a_n}{\Pi\left(\frac{m}{2}\right)\Pi\left(\frac{n-m}{2}-1\right)} \int_0^\infty \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} \left(1 - \Sigma \frac{x^2}{a^2+\theta}\right)^{\frac{1}{2}(n-m)-1} d\theta$$

$$= \frac{\mu \pi^{\frac{1}{2}n} a_1 a_2 \dots a_n . \Pi\left(\frac{n-m}{2} - r - 1\right)}{2^{2r}\Pi\left(\frac{m}{2}+r\right)\Pi\left(\frac{n-m}{2}-1\right)\Pi\left(\frac{n-m}{2}-1\right)} (-1)^r$$
$$\int_0^\infty \frac{\theta^{\frac{1}{2}(n-m)-1}}{P} \nabla^{2r} \left(1 - \Sigma \frac{x^2}{a^2+\theta}\right)^{\frac{1}{2}(n-m)-} d\theta.$$

Carrying out the differentiation as in Art. 9, the expression becomes

$$\frac{\mu\pi^{\frac{1}{2}n}a_1\dots a_n}{\Pi\left(\frac{m}{2}+r\right)\Pi\left(\frac{n-m}{2}-1\right)}\,\frac{\Pi\left(\frac{n-m}{2}-r-1\right)}{\Pi\left(\frac{n-m}{2}-1-2r\right)}\,(-1)^r\int_c^\infty\frac{\theta^{\frac{1}{2}(n-m)-1}}{P}\,U^{\frac{1}{2}(n-m)-1-2r}$$

$$\left\{1-\frac{U\delta^2}{2.n-m-4r}+\frac{U^2\delta^4}{2.4.n-m-4r.n-m-4r+2}-\dots\right\}\left(\Sigma\frac{x^2}{(a^2+\theta)^2}\right)^r d\theta,$$

where
$$\delta^2=\Sigma\,(a^2+\theta)\frac{\partial^2}{\partial x^2},$$

$$U=1-\Sigma\frac{x^2}{a^2+\theta}.$$

Extension of Formula for the Potential of a Solid Ellipsoid.

11. We shall now find the potential of a uniform solid ellipsoid in the case $n=m$; the formula (4) holds when $\lambda=1$, and $n-m$ is positive, say, $=2\kappa$; the potential is then, if the density is μ,

$$\frac{\mu}{\Pi\left(\frac{n}{2}-\kappa\right)\Pi(\kappa)}\,\tfrac{1}{2}\pi^{\frac{1}{2}n}a_1a_2\dots a_n\int_{\theta_0}^\infty\frac{\theta^{\kappa-1}}{P}\,U^\kappa d\theta.$$

For an external point the integral remains convergent when κ becomes zero; the potential at an external point is therefore

$$\frac{\mu}{2\Pi\left(\frac{n}{2}\right)}\,\pi^{\frac{1}{2}n}a_1a_2\dots a_n\int_{\theta_0}^\infty\frac{1}{\theta P}\,d\theta.$$

The external level surfaces are therefore confocal ellipsoids.

When $n=3$, we have for the external potential of a solid ellipsoid, the law of force being that of the inverse fourth power,

$$\tfrac{2}{3}\mu\pi abc\int_{\theta_0}^\infty\frac{d\theta}{\theta\sqrt{(a^2+\theta)(b^2+\theta)(c^2+\theta)}}.$$

When $n=2$, we have for the external potential of a solid disc, the law of force being that of the inverse cube,

$$\tfrac{1}{2}\mu\pi ab\int_{\theta_0}^\infty\frac{d\theta}{\theta\sqrt{(a^2+\theta)(b^2+\theta)}}.$$

For an internal point, for which $\theta_0 = 0$, the integral is not convergent when $\kappa = 0$, the potential becoming infinite; we shall, however, calculate the finite excess of the potential at any point over that at the centre; this may then be used instead of the potential in the ordinary sense; we have

$$\int_0^\infty \frac{\theta^{\kappa-1}}{P} U^\kappa d\theta = \left[\frac{\theta^\kappa U^\kappa}{\kappa P}\right]_0^\infty - \int_0^\infty \frac{\theta^\kappa}{\kappa} \frac{d}{d\theta} \frac{U^\kappa}{P} d\theta;$$

hence

$$V - V_0 = -\frac{\mu}{\Pi\left(\frac{n}{2}-\kappa\right)\Pi(\kappa)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \ldots a_n \int_0^\infty \theta^\kappa \frac{d}{d\theta} \frac{U^\kappa - 1}{\kappa P} d\theta,$$

which gives, when $\kappa = 0$,

$$V - V_0 = -\frac{\mu}{\Pi\left(\frac{n}{2}\right)} \tfrac{1}{2}\pi^{\frac{1}{2}n} a_1 a_2 \ldots a_n \int_0^\infty \frac{d}{d\theta}\left(\frac{1}{P}\log_e U\right) d\theta,$$

or $$V - V_0 = \frac{\mu\pi^{\frac{1}{2}n}}{2\Pi\left(\frac{n}{2}\right)} \log_e\left(1 - \Sigma\frac{x^2}{a^2}\right) \qquad (13)$$

Thus the internal level surfaces are similar ellipsoids.

In the case of a solid three-dimensional ellipsoid, the force being that of the inverse fourth power of the distance, we have

$$V - V_0 = \tfrac{2}{3}\mu\pi \log_e\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right).$$

For a two-dimensional disc, the law of force being that of the inverse cube, the potential is given by

$$V - V_0 = \tfrac{1}{2}\mu\pi \log_e\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

12. When the law of force is that of the inverse $(n+3)^{\text{th}}$ power of the distance, the external potential of a homogeneous ellipsoid is

[redacted]

obtained by putting $h=0$, after the operation

$$-\frac{2}{n+2}\,\frac{d}{d\,(h^2)}\,\frac{\mu}{2\Pi\left(\frac{n}{2}\right)}\,\pi^{\frac{1}{2}n}\,a_1 a_2 \dots a_n \int_{\theta_0}^{\infty} \frac{1}{\theta P}\,d\theta$$

is performed; since

$$\left(\Sigma\,\frac{x^2}{(a^2+\theta_0)^2}+\frac{h^2}{\theta_0^2}\right)d\theta_0=\frac{1}{\theta_0}\,d\,(h^2),$$

this expression becomes

$$\frac{\mu}{2\Pi\left(\frac{n}{2}+1\right)}\,\pi^{\frac{1}{2}n}\,a_1 a_2 \dots a_n\,\frac{1}{\theta_0^2\,\Sigma\,\frac{x^2}{(a^2+\theta_0)^2}\,P}$$

If we perform on this expression the operation

$$\frac{n+2}{n+2r+2}\,\frac{1}{(n+2)(n+4)\dots(n+2r)\,4.6\dots 2r+2}\,\nabla^{2r},$$

we obtain the potential when the law of force is that of the inverse $(n+2r+3)^{\text{th}}$ power of the distance; the potential is then

$$\frac{\mu}{2^{2r+1}\,\Pi\left(\frac{n}{2}+r+1\right)\Pi\,(r+1)}\,\pi^{\frac{1}{2}n}\,a_1 a_2 \dots a_n\,\nabla^{2r}\,\frac{1}{P\theta_0^2\,\Sigma\,\frac{x^2}{(a^2+\theta_0)^2}}\quad\dots(14),$$

which agrees with an expression obtained otherwise by Cayley.*

13. The internal potential when the law of force is that of the inverse $(n+2r+3)^{\text{th}}$ power of the distance is obtained in a similar manner by operating on the expression in (13); the potential is equal to

$$\frac{n}{n+2r+2}\,\frac{1}{n(n+2)\dots(n+2r)\,2.4\dots 2r+2}\,\nabla^{2r+2}\,\frac{\mu\pi^{\frac{1}{2}n}}{2\Pi\left(\frac{n}{2}\right)}\,\log_e\left(1-\Sigma\,\frac{x^2}{a^2}\right).$$

Carrying out the operation by the differentiation theorem as in

* See *Collected Works*, Vol. I.

Art. 9, we find for the potential the expression

$$-\frac{\mu\pi^{\frac{1}{2}n}\,\Pi(2r+2)}{2\Pi\left(\frac{n}{2}+r\right)\Pi(r+1)}\left\{\frac{1}{\left(1-\Sigma\frac{x^2}{a^2}\right)^{2r+2}}-\frac{1}{\left(1-\Sigma\frac{x^2}{a^2}\right)^{2r+1}}\frac{\delta^2}{2.4r+4}+\frac{1}{\left(1-\Sigma\frac{x^2}{a^2}\right)^{2r}}\frac{\delta^4}{2.4.4r+4.4r+6}-\dots\right\}\left(\Sigma\frac{x^2}{a^4}\right)^{r+1}\quad\dots\dots(15),$$

where $$\delta^2=\Sigma a^2\frac{\partial^2}{\partial x^2}.$$

14. The general formula (1) may be extended, as in Arts. 6 and 7, to cases in which the condition

$$\frac{n-m}{2}+\lambda-1\geqq 0$$

is not satisfied, by operating on the expression by means of

$$\left(\frac{\partial}{\partial(h^2)}\right)^{2r},$$

whence a formula similar to (8) can be obtained.

The result is as follows:—If the law of force is that of the $(m+2r+1)^{\text{th}}$ power of the distance, the potential is V_1+V_2, where

$$V_1=\frac{\mu\Pi(\lambda-1)}{2\Pi\left(\frac{m}{2}+r\right)}\pi^{\frac{1}{2}n}a_1a_2\dots a_n$$

$$\int_{\theta_0}^{\infty}\frac{1}{P}\sum_{s=s_0}^{s=\infty}\frac{\theta^{\frac{1}{2}(n-m)+s-r-1}\,U^{\frac{1}{2}(n-m)+\lambda+s-r-1}\,D^s}{2^{2s}\Pi(s)\,\Pi\left(\frac{n-m}{2}+\lambda+s-r-1\right)}F\left(\frac{a_1x_1}{a_1^2+\theta_1},\ \dots\ \frac{a_nx_n}{a_n^2+\theta}\right)d\theta,$$

s_0 being the least integer, such that

$$s_0+\frac{n}{2}-\frac{m}{2}+\lambda-r-1\geqq 0;$$

[redacted]

$$V_2 = \frac{\mu\Pi(\lambda-1)}{2\Pi\left(\frac{m}{2}+r\right)\Pi\left(\frac{n-m}{2}+\lambda-1\right)}\pi^{\frac{1}{2}n}a_1a_2\ldots a_n$$

$$(-1)^r\frac{d^r}{d(h^2)^r}\int_{\theta_0}^{\infty}\frac{\theta^{\frac{1}{2}(n-m)}}{P}U^{\frac{1}{2}(n-m)+\lambda-1}\left\{1+\frac{U\theta D}{2.n-m+2\lambda}+\ldots\right.$$

$$\left.+\frac{U^{s_0-1}\theta^{s_0-1}D^{s_0-1}}{2.4\ldots 2s_0-2.n-m+2\lambda\ldots n-m+2\lambda+2s_0-2}\right\}F\left(\frac{a_1x_1}{a_1+\theta},\ \ldots\right)d\theta.$$

15. As in Art. 7, we can find the potential of a solid ellipsoid or elliptic disc, when $m-n$ is an even positive integer, λ being also integral; the result is as follows:—

$$V=\frac{\mu\Pi(\lambda-1)}{2\Pi\left(\frac{m}{2}\right)}\frac{\pi^{\frac{1}{2}n}a_1a_2\ldots a_n}{2^{m-n-2\lambda+2}}$$

$$\int_{\theta_0}^{\infty}\frac{1}{\theta^{\lambda}P}\left\{\frac{D^{s_0}}{\Pi(s_0)\Pi(0)}+\frac{\theta UD^{s_0+1}}{2^2\Pi(s_0+1)\Pi(1)}+\frac{\theta^2U^2D^{s_0+2}}{2^4\Pi(s_0+2)\Pi(2)}+\ldots\right\}$$

$$F\left(\frac{a_1x_1}{a_1^2+\theta},\ \ldots\right)d\theta+\frac{\mu\Pi(\lambda-1)}{2\Pi\left(\frac{m}{2}\right)}\pi^{\frac{1}{2}n}a_1a_2\ldots a_n(-1)^{\frac{1}{2}(n-m)+\lambda}$$

$$\left\{\left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m-n)-\lambda}\frac{1}{\theta_0^{\lambda}P_0Q_0}-\left(\frac{1}{Q_0}\frac{d}{d\theta_0}\right)^{\frac{1}{2}(m-n)-\lambda-1}\frac{D_0}{2^2.1!\theta^{\lambda}P_0Q_0}+\ldots\right\}$$

$$F\left(\frac{a_1x_1}{a_1^2+\theta_0},\ \ldots\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

where
$$s_0=\frac{m-n}{2}-\lambda+1.$$

It is to be observed that, when the degree of F is less than s_0, the definite integral in the above expression vanishes. This potential applies to the disc and to the ellipsoid; in the latter case $h=0$.

The Associated Dynamics of a Top and of a Body under no Forces.

By A. G. Greenhill.

Read March 12th, 1896. Received August 19th, 1896.

In discussing the important theorem enunciated by Jacobi (*Gesammelte Werke*, II, p. 480) that the motion of a Top, that is, of a solid body of revolution moving under gravity about a fixed point in its axis, can be resolved into two constituents, which represent the motion about the fixed point of two other bodies moving under no forces, two methods have been adopted.

We may either begin with the two associated states of motion of the bodies under no forces, and build up with them the motion of a top—this is the method employed by

Halphen, *Comptes Rendus*, C, 1885;

Darboux, *Cours de Mécanique* (Despeyrous, II, Note XX);

Routh, *Quarterly Journal of Mathematics*, XXIII, p. 34;

Marcolongo, *Annali di Matematica*, XXII, 1894.

Or we may follow Jacobi's own method, and, starting with the motion of the top, we may show how this may be resolved into the associated states of motion of two bodies moving under no forces; this was the procedure followed in my own paper on "The Dynamics of a Top," in the *Proceedings of the London Mathematical Society*, Vol. XXVI, p. 215.

M. Darboux has shown, in his Notes to Despeyrous' *Cours de Mécanique*, that the motion of the axis of the top may be imitated by one of the generating lines OC of a deformable hyperboloid of one sheet, if the other generating line OG through the fixed point O is held in a vertical position, while the opposite parallel generator HY_1 is guided so that the point H on it, which is at the other end of the diameter of the hyperboloid through the fixed point, describes a certain herpolhode, which lies in a horizontal plane and is normal to the hyperboloid (Fig. 1).

According to the plan of the present paper, we begin with a discussion of the deformation of the hyperboloid, and then show how this deformation is associated with the motion of a top and with two states of motion *à la Poinsot*, under no forces; in this way we avoid giving precedence to one dynamical problem over the other.

The method adopted by Poinsot in his celebrated *Théorie nouvelle de la rotation de corps* (Paris, 1852), of giving the geometrical interpretation of the various analytical formulas, has been followed, in its extension to these new developments of dynamics; and thereby many curious theorems are introduced in connexion with the ellipse.

1. The deformable hyperboloid is constructed, in Henrici's manner, by drawing the focal ellipse given by the equation

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{0}=1, \tag{1}$$

and then by placing a pair of thin rods along each tangent line of this ellipse, and knotting together the rods belonging to opposite systems at their points of crossing.

Take the tangents at any two points P_1 and P_2, intersecting in H, and cutting the axes of the ellipse in T_1, V_1 and T_2, V_2 (Fig. 1); draw the perpendiculars OY_1 and OY_2 upon them from the centre O; draw the planes through H perpendicular to HP_1 and HP_2, and drop the perpendiculars OG and OC from O upon these planes.

We shall find that, in the associated dynamical problems, the elliptic functions which determine the motion have a modulus

$$\kappa=\frac{\beta}{\alpha}, \tag{2}$$

the ratio of the axes of the focal ellipse; so that the complementary modulus κ' is the excentricity of the ellipse.

Denoting by K and K' the corresponding quarter periods of the elliptic functions, we put

$$AOY_1=\operatorname{am}\{(p+r)\,\mathrm{K}',\,\kappa'\}, \tag{3}$$

$$AOY_2=\operatorname{am}\{(p-r)\,\mathrm{K}',\,\kappa'\}; \tag{4}$$

and r is negative, if AOY_2 is drawn greater than AOY_1.

Then the excentric angles of P_1 and P_2, measured from the minor axis OB, are

$$\left.\begin{array}{l}\operatorname{am}\{(1-p-r)\,\mathrm{K}',\,\kappa'\},\\ \operatorname{am}\{(1-p+r)\,\mathrm{K}',\,\kappa'\}.\end{array}\right\} \tag{5}$$

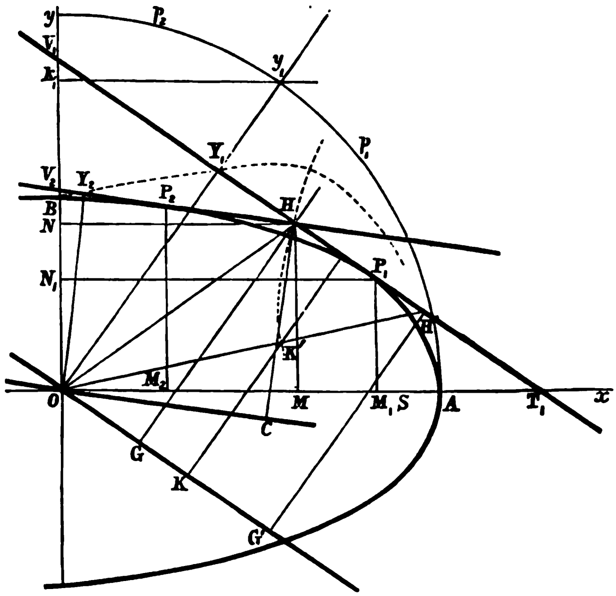


FIG. 1.

Dropping the letter K′, the tangents at P_1 and P_2 are given by

$$x \,\text{cn}\,(p \pm r) + y \,\text{sn}\,(p \pm r) = a \,\text{dn}\,(p \pm r); \tag{6}$$

and therefore, at their point of intersection H,

$$\left.\begin{aligned} \frac{x}{a} &= \frac{\text{sn}\,(p+r)\,\text{dn}\,(p-r) - \text{sn}\,(p-r)\,\text{dn}\,(p+r)}{\sin\{\text{am}\,(p+r) - \text{am}\,(p-r)\}} = \frac{\text{cn}\,p\,\text{dn}\,r}{\text{dn}\,p\,\text{cn}\,r}, \\ \frac{y}{a} &= \frac{\text{cn}(p-r)\,\text{dn}\,(p+r) - \text{cn}\,(p+r)\,\text{dn}\,(p-r)}{\sin\{\text{am}\,(p+r) - \text{am}\,(p-r)\}} = \frac{\kappa^2\,\text{sn}\,p}{\text{dn}\,p\,\text{cn}\,r}. \end{aligned}\right\} \tag{7}$$

If the confocal ellipsoid and hyperboloid of two sheets through H are given by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{\beta^2+\lambda} + \frac{z^2}{\lambda} = 1, \tag{8}$$

$$\frac{x^2}{a^2+\nu} + \frac{y^2}{\beta^2+\nu} + \frac{z^2}{\nu} = 1, \tag{9}$$

we must put

$$a^2+\lambda = a^2\frac{\text{dn}^2 r}{\text{cn}^2 r}, \qquad \beta^2+\lambda = a^2\frac{\kappa^2}{\text{cn}^2 r}, \qquad \lambda = a^2\frac{\kappa^2\,\text{sn}^2 r}{\text{cn}^2 r}, \tag{10}$$

$$a^2+\nu = a^2\frac{\kappa'^2\,\text{cn}^2 p}{\text{dn}^2 p}, \quad \beta^2+\nu = -a^2\frac{\kappa^2\kappa'^2\,\text{sn}^2 p}{\text{dn}^2 p}, \quad \nu = -a^2\frac{\kappa^2}{\text{dn}^2 p}. \tag{11}$$

Denoting the angle Y_1OY_2, or GOC, or the angle between the tangents at P_1 and P_2, by θ_3, then

$$\theta_3 = \text{am}\left\{(p+r)\,\text{K}', \kappa'\right\} - \text{am}\left\{(p-r)\,\text{K}', \kappa'\right\}, \tag{12}$$

$$\cos\theta_3 = \frac{1-2\,\text{sn}^2 r + \kappa'^2\,\text{sn}^2 p\,\text{sn}^2 r}{1-\kappa'^2\,\text{sn}^2 p\,\text{sn}^2 r}. \tag{13}$$

We find also that

$$OY_1 = a\,\text{dn}\,(p+r), \tag{14}$$

$$OY_2 = a\,\text{dn}\,(p-r); \tag{15}$$

$$OH^2 = a^2 + \lambda + \beta^2 + \nu$$

$$= a^2\left(\frac{\text{dn}^2 r}{\text{cn}^2 r} - \kappa^2\kappa'^2\frac{\text{sn}^2 p}{\text{dn}^2 p}\right), \tag{16}$$

$$OG^2 = HY^2 = OH^2 - OY_1^2$$

$$= a^2\left\{\frac{\text{dn}^2 r}{\text{cn}^2 r} - \kappa^2\kappa'^2\frac{\text{sn}^2 p}{\text{dn}^2 p} - \text{dn}^2\,(p+r)\right\}, \tag{17}$$

$$OC^2 = HY_1^2 = OH^2 - OY_2^2$$

$$= a^2\left\{\frac{\text{dn}^2 r}{\text{cn}^2 r} - \kappa^2\kappa'^2\frac{\text{sn}^2 p}{\text{dn}^2 p} - \text{dn}^2\,(p-r)\right\}. \tag{18}$$

Various other lengths, required in the sequel, may be added here, expressed by means of elliptic functions; we write $p+r$, &c., for the arguments $(p+r)\,\text{K}'$, &c., for brevity; thus

$$\left.\begin{aligned}
Y_1P_1 &= a\kappa'^2\,\frac{\text{sn}\,(p+r)\,\text{cn}\,(p+r)}{\text{dn}\,(p+r)}, \\
Y_1T_1 &= a\,\frac{\text{sn}\,(p+r)\,\text{dn}\,(p+r)}{\text{cn}\,(p+r)}, \\
Y_1V_1 &= a\,\frac{\text{cn}\,(p+r)\,\text{dn}\,(p+r)}{\text{sn}\,(p+r)}, \\
T_1V_1 &= a\,\frac{\text{dn}\,(p+r)}{\text{sn}\,(p+r)\,\text{cn}\,(p+r)}, \\
P_1V_1 &= a\,\frac{\text{cn}\,(p+r)}{\text{sn}\,(p+r)\,\text{dn}\,(p+r)}, \\
P_1T_1 &= a\kappa^2\,\frac{\text{sn}\,(p+r)}{\text{cn}\,(p+r)\,\text{dn}\,(p+r)};
\end{aligned}\right\} \tag{19}$$

and the argument must be changed to $p-r$ when the suffix is changed from 1 to 2; also

$$HP_1,\ HP_2 = \frac{a}{\mathrm{dn}\,(p\pm r)}\,\frac{\kappa^2\,\mathrm{sn}\,r}{\mathrm{dn}\,p\ \mathrm{cn}\,r},$$

because
$$OY_1.HP_1 = OY_2.HP_2 = \text{area } OP_1HP_2$$
$$= a^2\,\frac{\kappa^2\,\mathrm{sn}\,r}{\mathrm{dn}\,p\ \mathrm{cn}\,r}. \tag{21}$$

So also we find

$$HV_1,\ HV_2 = \frac{HN}{\sin\,(OV_1Y_1 \text{ or } OV_2Y_2)}$$
$$= \frac{a}{\mathrm{sn}\,(p\pm r)}\,\frac{\mathrm{cn}\,p\ \mathrm{dn}\,r}{\mathrm{dn}\,p\ \mathrm{cn}\,r},$$
$$HT_1,\ HT_2 = \frac{a}{\mathrm{cn}\,(p\pm r)}\,\frac{\kappa^2\,\mathrm{sn}\,p}{\mathrm{dn}\,p\ \mathrm{cn}\,r},$$
$$HY_1,\ HY_2 = \frac{\cos\{\mathrm{am}(p+r)-\mathrm{am}(p-r)\}\,\mathrm{dn}(p\pm r)-\mathrm{dn}(p\mp r)}{\sin\{\mathrm{am}(p+r)-\mathrm{am}(p-r)\}}. \tag{22}$$

Thus, for instance,

$$\left.\begin{aligned} P_1V_1.Y_1T_1 &= P_2V_2.Y_2T_2 = a^2 = OA^2,\\ P_1T_1.Y_1V_1 &= P_2T_2.Y_2V_2 = \beta^2 = OB^2,\\ P_1Y_1.T_1V_1 &= P_2Y_2.T_2V_2 = a^2\kappa^2 = OS^2. \end{aligned}\right\} \tag{23}$$

2. Now when the model of the hyperboloid is deformed so as to leave the plane of its focal ellipse, keeping the axes fixed in their original direction, its equation may be written

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{\beta^2+\mu}+\frac{z^2}{\mu}=1; \tag{24}$$

and, to the modulus κ, we can put

$$\left.\begin{aligned} a^2+\mu &= a^2\,\mathrm{dn}^2\,q\mathrm{K},\\ \beta^2+\mu &= a^2\kappa^2\,\mathrm{cn}^2\,q\mathrm{K},\\ \mu &= -a^2\kappa^2\,\mathrm{sn}^2\,q\mathrm{K}. \end{aligned}\right\} \tag{25}$$

During the deformation a point H will move normally to the hyperboloid, and will describe a line of curvature on the confocal ellipsoid and hyperboloid of two sheets drawn through the original position of H in the plane xy of the focal ellipse; and thus λ and ν, or p and r will not vary for a point H during the deformation.

[redacted]

The coordinates of H will be given by

$$\left.\begin{aligned} x^2 &= \frac{\alpha^2+\lambda\,.\,\alpha^2+\mu\,.\,\alpha^2+\nu}{\alpha^2(\alpha^2-\beta^2)}, \\ y^2 &= \frac{\beta^2+\lambda\,.\,\beta^2+\mu\,.\,\beta^2+\nu}{\beta^2(\beta^2-\alpha^2)}, \\ z^2 &= \frac{\lambda\mu\nu}{\alpha^2\beta^2}; \end{aligned}\right\} \tag{26}$$

or

$$\left.\begin{aligned} x &= a\,\frac{\operatorname{cn} p\mathrm{K}' \operatorname{dn} q\mathrm{K} \operatorname{dn} r\mathrm{K}'}{\operatorname{dn} p\mathrm{K}' \operatorname{cn} r\mathrm{K}'}, \\ y &= a\,\frac{\kappa^2 \operatorname{sn} p\mathrm{K}' \operatorname{cn} q\mathrm{K}}{\operatorname{dn} p\mathrm{K}' \operatorname{cn} r\mathrm{K}}, \\ z &= a\,\frac{\kappa^2 \operatorname{sn} q\mathrm{K} \operatorname{sn} r\mathrm{K}}{\operatorname{dn} p\mathrm{K}' \operatorname{cn} r\mathrm{K}'}, \end{aligned}\right\} \tag{27}$$

the modulus of the elliptic function of the argument of $q\mathrm{K}$ being κ, and of the arguments $p\mathrm{K}'$ and $r\mathrm{K}'$ being the complementary modulus κ'.

Also

$$\begin{aligned} OH^2 &= x^2+y^2+z^2 \\ &= \alpha^2+\lambda+\beta^2+\mu+\nu \\ &= a^2\left(\frac{\operatorname{dn}^2 r\mathrm{K}'}{\operatorname{cn}^2 r\mathrm{K}'}+\kappa^2 \operatorname{cn}^2 q\mathrm{K}-\frac{\kappa^2}{\operatorname{dn}^2 p\mathrm{K}'}\right). \end{aligned} \tag{28}$$

We find also that the perpendiculars OY_1 and OY_2, drawn from O to the generating lines through H in its new position, are given by

$$\left.\begin{aligned} OY_1^2 &= a^2\left\{\operatorname{dn}^2(p+r)\mathrm{K}'-\kappa^2 \operatorname{sn}^2 q\mathrm{K}\right\}, \\ OY_2^2 &= a^2\left\{\operatorname{dn}^2(p-r)\mathrm{K}'-\kappa^2 \operatorname{sn}^2 q\mathrm{K}\right\}; \end{aligned}\right\} \tag{29}$$

so that

$$\left.\begin{aligned} OG^2 = HY_1^2 &= a^2\left\{\frac{\operatorname{dn}^2 r\mathrm{K}'}{\operatorname{cn}^2 r\mathrm{K}'}-\frac{\kappa^2\kappa'^2 \operatorname{sn}^2 p\mathrm{K}'}{\operatorname{dn}^2 p\mathrm{K}'}-\operatorname{dn}^2(p+r)\mathrm{K}'\right\}, \\ OC^2 = HY_2^2 &= a^2\left\{\frac{\operatorname{dn}^2 r\mathrm{K}'}{\operatorname{cn}^2 r\mathrm{K}'}-\frac{\kappa^2\kappa'^2 \operatorname{sn}^2 p\mathrm{K}'}{\operatorname{dn}^2 p\mathrm{K}'}-\operatorname{dn}^2(p-r)\mathrm{K}'\right\}, \end{aligned}\right\} \tag{30}$$

and therefore HY_1 and HY_2 remain constant during the deformation, and thus Y_1 and Y_2 are fixed points on the generating lines through H; hence Darboux's theorem, that the planes through H, perpendicular to the generating lines HY_1 and HY_2, are tangent to concentric spheres, with radii OG and OC.

The points P_1, T_1, V_1 will also be fixed points on the generating line $T_1P_1V_1$; and P_2, T_2, V_2, will be fixed points on the generating line $T_2P_2V_2$; because these are the points in which these generating lines meet the three principal planes xOy, xOz, yOz.

3. Denoting the constant length OG or HY_1 by δ_1, suppose a concentric coaxial quadric surface is drawn, touching the plane through H perpendicular to OG at the point H, or having the generator HP_1 as the normal line at P_1.

Then, since the normal line $T_1HP_1Y_1V_1$ to this new quadric at H meets the principal planes in V_1, T_1, P_1, and since δ_1 is the length of the perpendicular from the centre upon the tangent plane at H of this quadric surface, it follows, by a well-known theorem of Solid Geometry, that the squares of the semi-axes of this quadric surface are numerically equal to the rectangles

$$\delta_1 . HV_1, \quad \delta_1 . HT_1, \quad \delta_1 . HP_1; \tag{31}$$

and, these being constant, the quadric is a fixed surface.

Writing its equation

$$A_1x^2+B_1y^2+C_1z^2 = D_1\delta_1^2, \tag{32}$$

and noticing that the squares of the semi-axes must be taken positive or negative according as Y_1 and the point of intersection with the principal plane are on the same or opposite sides of H, our Fig. 1 gives a hyperboloid of two sheets, in which

$$\frac{D_1}{A_1}\delta_1^2 = \delta_1 . HV_1,$$

or

$$\frac{D_1}{A_1} = \frac{HV_1}{\delta_1}, \tag{33}$$

and

$$\frac{D_1}{B_1} = -\frac{HT_1}{\delta_1}, \tag{34}$$

$$\frac{D_1}{C_1} = -\frac{HP_1}{\delta_1}. \tag{35}$$

To obtain an ellipsoid, P_2 must be taken beyond B, and H beyond V_1, or *vice versa*.

Similarly, denoting the distance OC or HY_2 by δ_2, a fixed coaxial quadric surface can be drawn touching at H the plane perpendicular to the generating line HP_2; and, writing its equation

$$A_2x^2+B_2y^2+C_2z^2 = D_2\delta_2^2, \tag{36}$$

our Fig. 1 gives a hyperboloid of one sheet, in which

$$\frac{D_2}{A_2} = \frac{HV_2}{\delta_2}, \tag{37}$$

$$\frac{D_2}{B_2} = -\frac{HT_2}{\delta_2}, \tag{38}$$

$$\frac{D_2}{C_2} = \frac{HP_2}{\delta_2}. \tag{39}$$

These two quadric surfaces (32) and (36) intersect in the curve described by H, which is a line of curvature on the fixed confocal ellipsoid or hyperboloid of two sheets through H; and the perpendiculars from O upon the tangent planes of these quadric surfaces along their line of intersection are of constant length δ_1 and δ_2, so that the line of intersection is a *polhode* curve on each quadric surface.

Thus any line of curvature on a hyperboloid of one sheet, common to an ellipsoid and a confocal hyperboloid of two sheets, is a polhode curve on two coaxial quadric surfaces, the normals to these surfaces being the generating lines of the confocal hyperboloid of one sheet which pass through the point; so that the tangent line of the polhode is normal to the hyperboloid of one sheet.

The lines of curvature on the ellipsoid, or on the hyperboloid of two sheets, are polhode curves on imaginary surfaces.

These two quadrics (32) and (36) are the momental quadrics of Jacobi's two associated bodies, moving about O under no forces; but, as these quadrics are unrestricted in shape, the bodies must be composed of matter which is capable of having negative density, the fiction employed in the Two-Fluid Theory of Electricity.

It is the advantage of this geometrical procedure that we can see at a glance the nature of these two momental quadrics; and also that we exclude the imaginary cases which arise when the subject is treated symmetrically by an analytical procedure.

4. Calling the planes through H, perpendicular to the generators through H, or to OG and OC, the invariable planes of G and C, then H describes a herpolhode in each of these planes.

Considering, for instance, the herpolhode of H in the invariable plane of G, and denoting the polar coordinates of H with respect to G by ρ, ϖ, with the suffix 1 when we wish to distinguish between G and C, then, as shown in the *Proc. Lond. Math. Soc.*, Vol. XXVI, p. 244, from purely geometrical considerations of a quadric surface, the differential relation of the herpolhode of H is

$$\frac{d\varpi}{d\rho^2} = \frac{\delta}{\sqrt{R}} + \frac{A-D.B-D.C-D}{ABC}\,\frac{\delta^3}{\rho^2\sqrt{R}}, \tag{40}$$

where
$$R = -4(\rho^2-\rho_a^2)(\rho^2-\rho_b^2)(\rho^2-\rho_c^2), \tag{41}$$

$$\left.\begin{aligned} \rho_a^2 &= \frac{B-D.D-C}{BC}\,\delta^2, \\ \rho_b^2 &= \frac{C-D.D-A}{CA}\,\delta^2, \\ \rho_c^2 &= \frac{A-D.D-B}{AB}\,\delta^2. \end{aligned}\right\} \tag{42}$$

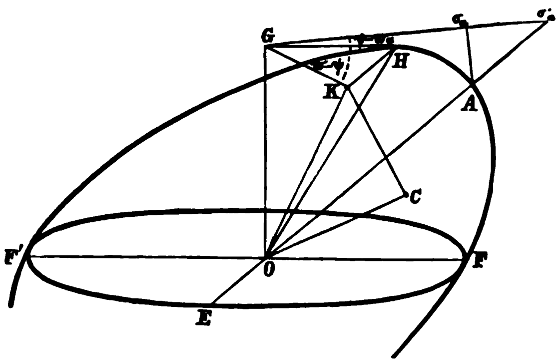


FIG. 2.

For, if the quadric surface OHF (Fig. 2) rolls about its centre O on the fixed plane GHK, the resultant angular velocity is about OH, the radius to H, the point of contact; and this angular velocity can be resolved into two components, one about OG, the perpendicular to the plane GHK, and the other about OF, the radius parallel to GH.

This second component angular velocity makes the plane OEF, fixed in the quadric and parallel to the plane GHK, or conjugate to OH, move so that OF is the line of intersection with its consecutive position; and therefore, if H moves to a consecutive position H' on the quadric surface, the plane OHH' is conjugate to OF.

If HK is the tangent of the herpolhode curve formed by H in the plane GHK, then HK is the ultimate direction of HH'; and thus HK is parallel to OE, the conjugate line to the plane OHF.

Draw the plane OGK perpendicular to HK or OE; then, since OE, OF, OH form a conjugate system of the quadric, the squares of whose semi-axes are

$$\frac{D}{A}\delta^2,\quad \frac{D}{B}\delta^2,\quad \frac{D}{C}\delta^2,$$

therefore, by fundamental theorems of quadric surfaces,

$$OE^2+OF^2+OH^2=\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2, \tag{43}$$

$$OE^2.\,OF^2.\sin^2 EOF+OK^2.\,OE^2+OF^2.\,OG^2$$
$$=\left(\frac{1}{BC}+\frac{1}{CA}+\frac{1}{AB}\right)D^2\delta^4, \tag{44}$$

$$OG^2.\,OE^2.\,OF^2.\sin^2 EOF=\frac{1}{ABC}D^3\delta^6. \tag{45}$$

Thence, putting as before

$$OG=\delta,\quad GH=\rho,$$

and putting $GK=p$, for a moment, so that

$$\sin EOF=\sin GHK=p/\rho, \tag{46}$$

we find, by solution of these equations,

$$p^2.OE^2=\left(\rho^2+\frac{A-D.B-D.C-D}{ABC}\delta^2\right)\delta^2, \tag{47}$$

$$p^2.OF^2=\left\{\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2\right\}p^2$$
$$-\frac{A-D.B-D.C-D}{ABC}\delta^4-\delta^2\rho^2. \tag{48}$$

Writing equation (45) in the form

$$p^4.\,OE^2.\,OF^2=\frac{D^3\delta^4}{ABC}p^2\rho^2, \tag{49}$$

and eliminating OE and OF,

$$\left(\rho^2\delta^2 + \frac{A-D.B-D.C-D}{ABC}\delta^4\right)$$

$$\times\left[\left\{\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2\right\}p^2-\frac{A-D.B-D.C-D}{ABC}\delta^4-\delta^2\rho^2\right]$$

$$= \frac{D^3}{ABC}\delta^4 p^2\rho^2,$$

$$p^2 = \frac{\left(\rho^2+\frac{A-D.B-D.C-D}{ABC}\delta^2\right)^2\delta^2}{\left\{\left(\frac{D}{A}+\frac{D}{B}+\frac{D}{C}-1\right)\delta^2-\rho^2\right\}\left(\rho^2+\frac{A-D.B-D.C-D}{ABC}\delta^2\right)-\frac{D^3}{ABC}\delta^2\rho^2}, \quad (50)$$

the relation connecting p and ρ in the herpolhode.

Thence

$$\left(\rho^2\frac{d\varpi}{d\rho^2}\right)^2 = \tfrac{1}{4}\tan^2 GHK = \tfrac{1}{4}\frac{p^2}{\rho^2-p^2}$$

$$= \frac{\left(\rho^2+\frac{A-D.B-D.C-D}{ABC}\delta^2\right)^2\delta^2}{R},$$

or

$$\frac{d\varpi}{d\rho^2} = \frac{\delta}{\sqrt{R}} + \frac{A-D.C-D.C-D}{ABC}\,\frac{\delta^3}{\rho^2\sqrt{R}}. \quad (40)$$

5. We reduce this relation (40) to our standard form of the Elliptic Integral of the Third Kind

$$I(v) = \tfrac{1}{2}\int\frac{P(v)(\sigma-s)-\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} + \{P(v)-\zeta(v)\}\,u, \quad (51)$$

where $P(v)$ means a certain function of v, namely,

$$P(v) = \zeta(v) - \eta v/\omega, \quad (51^*)$$

by putting

$$GH^2 = \rho^2 = \frac{k^2}{M^2}(\sigma-s) = \frac{k^2}{M^2}(\wp v - \wp u). \quad (52)$$

Here k and M are certain homogeneity factors, to be defined more precisely hereafter; while v is the parameter of the Elliptic Integral of the Third Kind, of the form

$$v = \omega_1 + (p+r)\,\omega_3, \quad \text{or} \quad \omega_1 + f\omega_3; \quad (53)$$

and u is a variable, which grows uniformly with the time.

Now $$\frac{A-D.B-D.C-D}{ABC}\delta^3=-\tfrac{1}{2}\frac{k^3}{M^3}\sqrt{(-\Sigma)}, \qquad (54)$$

Σ being the value of S when we put $s=\sigma$; also

$$\sqrt{R}=\frac{k^3}{M^3}\sqrt{S}, \qquad d\rho^2=\frac{k^2}{M^2}ds; \qquad (55)$$

so that $$d\varpi=M\frac{\delta}{k}\frac{ds}{\sqrt{S}}-\frac{1}{2}\frac{\sqrt{(-\Sigma)}}{\sigma-s}\frac{ds}{\sqrt{S}}$$

$$=\left\{M\frac{\delta}{k}-\tfrac{1}{2}P(v)\right\}du+dI(v),$$

or $$\varpi=\left\{M\frac{\delta}{k}-\tfrac{1}{2}P(v)\right\}\frac{nt}{M}+I(v), \qquad (56)$$

where n is a constant factor, to be determined in the sequel.

Thus, when v and $P(v)$ are chosen so as to make $I(v)$ pseudo-elliptic, the herpolhode of H is made an algebraical curve by taking

$$M\frac{\delta}{k}=\tfrac{1}{2}P(v); \qquad (57)$$

and we shall find that this makes

$$\delta=HY_1=\text{arc }BP_1-(1-p-r)\text{ arc }BA. \qquad (58)$$

6. Moving H to a new position H' on the generating line HY_1 (Fig. 1) gives a new rolling quadric which is confocal not with the original quadric (32), but with a homothetic quadric, altered in the linear scale of the ratio $\sqrt{(\delta'/\delta)}$, if

$$H'Y_1=\delta'.$$

Hence Sylvester's theorem; for, if OK is the geometric mean of OG and OG', the invariable plane of K will touch a confocal to the original quadric at K', where OH' produced meets it; also

$$\frac{KK'}{GH}=\frac{KK'}{G'H'}=\frac{OK}{OG'}=\frac{OG}{OK},$$

so that $$OK.KK'=OG.GH; \qquad (59)$$

and K' therefore lies on the hyperbola through H, having OG and OY_1 as asymptotes.

7. It will be interesting at this stage to give the geometrical interpretation of the various relations obtained by Darboux, in his Notes to Despeyrous' *Mécanique*, with the additional relations given on p. 242 of the *Proc. Lond. Math. Soc.*, Vol. XXVI.

As the number of different quantities requiring consideration is large, it is advisable to make a slight change in Darboux's and Routh's notation, and to distinguish by means of the suffixes 1 and 2 the quantities associated with the two momental quadrics, instead of by means of an accent.

Thus we write Darboux's equations for the motion of the first or second momental quadric, in his notation,

$$\frac{p^2}{a}+\frac{q^2}{b}+\frac{r^2}{c}=h, \tag{60}$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=1, \tag{61}$$

and, taking his p, q, r to represent the component angular velocities about the axes, we must take a, b, c as representing constant components of angular velocity, and h as the constant component of the angular velocity of the momental quadric about the invariable line, either OG or OC, according as the suffix 1 or 2 is supplied; and now, in Dr. Routh's notation,

$$T=Dh^2, \quad G=Dh. \tag{62}$$

Introducing a homogeneity factor n, of the dimensions of angular velocity, and the factor k previously employed, of the dimensions of length, Darboux's notation is connected up with our notation by putting

$$\frac{p}{n}=\frac{x}{k}, \quad \frac{q}{n}=\frac{y}{k}, \quad \frac{r}{n}=\frac{z}{k}, \quad \frac{h}{n}=\frac{\delta}{k}; \tag{63}$$

$$\left.\begin{aligned}\frac{a}{n}&=\frac{D}{A}\,\frac{\delta}{k}=\frac{HV}{k}(\pm),\\ \frac{b}{n}&=\frac{D}{B}\,\frac{\delta}{k}=\frac{HT}{k}(\pm),\\ \frac{c}{n}&=\frac{D}{C}\,\frac{\delta}{k}=\frac{HP}{k}(\pm).\end{aligned}\right\} \tag{64}$$

Thus, as the points are placed in Fig. 1,

$$\frac{a_1,\ b_1,\ c_1,\ h_1}{n} = \frac{HV_1,\ -HT_1,\ -HP_1,\ HY_1}{k}, \tag{65}$$

$$\frac{a_2,\ b_2,\ c_2,\ h_2}{n} = \frac{HV_2,\ -HT_2,\ \ HP_2,\ HY_2}{k}; \tag{66}$$

and thence

$$\left.\begin{aligned} \frac{a_1-h_1}{n} &= \frac{Y_1V_1}{k}, & \frac{b_1-h_1}{n} &= -\frac{Y_1T_1}{k}, & \frac{c_1-h_1}{n} &= -\frac{Y_1P_1}{k}; \\ \frac{a_2-h_2}{n} &= \frac{Y_2V_2}{k}, & \frac{b_2-h_2}{n} &= -\frac{Y_2T_2}{k}, & \frac{c_2-h_2}{n} &= -\frac{Y_2P_2}{k}; \end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned} \frac{b_1-c_1}{n} &= -\frac{P_1T_1}{k}, & \frac{c_1-a_1}{n} &= -\frac{P_1V_1}{k}, & \frac{a_1-b_1}{n} &= \frac{V_1T_1}{k}; \\ \frac{b_2-c_2}{n} &= -\frac{P_2T_2}{k}, & \frac{c_2-a_2}{n} &= -\frac{P_2V_2}{k}, & \frac{a_2-b_2}{n} &= \frac{V_2T_2}{k}. \end{aligned}\right\} \tag{68}$$

Thus, from (42),

$$\left.\begin{aligned} -\frac{\rho_a^2}{k^2} &= \frac{b_1-h_1 . c_1-h_1}{n^2} = \frac{Y_1T_1 . Y_1P_1}{k^2} = \frac{a^2}{k^2}\kappa'^2 \operatorname{sn}^2(p+r), \\ -\frac{\rho_b^2}{k^2} &= \frac{c_1-h_1 . a_1-h_1}{n^2} = -\frac{Y_1P_1 . Y_1V_1}{k^2} = -\frac{a^2}{k^2}\kappa'^2 \operatorname{cn}^2(p+r), \\ -\frac{\rho_c^2}{k^2} &= \frac{a_1-h_1 . b_1-h_1}{n^2} = -\frac{Y_1V_1 . Y_1T_1}{k^2} = -\frac{a^2}{k^2} \operatorname{dn}^2(p+r). \end{aligned}\right\} \tag{69}$$

An important relation connecting the quantities above is

$$\frac{A_1T_1-G_1^2}{B_1C_1} + \frac{A_2T_2-G_2^2}{B_2C_2} = 0, \tag{70}$$

and this may be written

$$\frac{D_1(A_1-D_1)}{B_1C_1}h_1^2 + \frac{D_2(A_2-D_2)}{B_2C_2}h_2^2 = 0,$$

or

$$\frac{b_1c_1}{a_1}(h_1-a_1) + \frac{b_2c_2}{a_2}(h_2-a_2) = 0,$$

or

$$\frac{HT_1 . HP_1}{HV_1} Y_1V_1 = \frac{HT_2 . HP_2}{HV_2} Y_2V_2, \tag{71}$$

a geometrical relation which can be interpreted; and similarly for the other two relations of the same nature.

The relations

$$(a_1-h_1)(b_1-c_1) = (a_2-h_2)(b_2-c_2), \text{ \&c.},$$

are equivalent to the geometrical relations of equations (23),

$$\left.\begin{aligned} Y_1V_1.P_1T_1 &= Y_2V_2.P_2T_2 = \beta^2, \\ Y_1T_1.P_1V_1 &= Y_2T_2.P_2V_2 = \alpha^2, \\ Y_1P_1.V_1T_1 &= Y_2P_2.T_2V_2 = \alpha^2-\beta^2. \end{aligned}\right\} \qquad (23^*)$$

The relations

$$h_1^2-(h_1-b_1)(h_1-c_1) = h_2^2-(h_2-b_2)(h_2-c_2), \text{ \&c.},$$

are equivalent to

$$\left.\begin{aligned} HY_1^2-Y_1T_1.Y_1P_1 &= HY_2^2-Y_2T_2.Y_2P_2 = OH^2-\alpha^2, \\ HY_1^2+Y_1P_1.Y_1V_1 &= HY_2^2+Y_2P_2.Y_2V_2 = OH^2-\beta^2, \\ HY_1^2+Y_1V_1.Y_1T_1 &= HY_2^2+Y_2V_2.Y_2T_2 = OH^2. \end{aligned}\right\} \qquad (72)$$

The relations

$$\left(1-\frac{A_1}{B_1}\right)\left(1-\frac{A_1}{C_1}\right) = \left(1-\frac{A_2}{B_2}\right)\left(1-\frac{A_2}{C_2}\right), \text{ \&c.},$$

or

$$\left(1-\frac{b_1}{a_1}\right)\left(1-\frac{c_1}{a_1}\right) = \left(1-\frac{b_2}{a_2}\right)\left(1-\frac{c_2}{a_2}\right), \text{ \&c.},$$

are equivalent to

$$\frac{T_1V_1.P_1V_1}{HV_1^2} = \frac{T_2V_2.P_2V_2}{HV_2^2},$$

or

$$\frac{OT_1.OM_1}{OM^2} = \frac{OT_2.OM_2}{OM^2} = \frac{\alpha^2}{OM^2}, \qquad (73)$$

$$\frac{P_1T_1.V_1T_1}{HT_1^2} = \frac{P_2T_2.V_2T_2}{HT_2^2},$$

or

$$\frac{ON_1.OT_1}{ON^2} = \frac{ON_2.OT_2}{ON^2} = \frac{\beta^2}{ON^2}; \qquad (74)$$

$$\frac{P_1V_1.P_1T_1}{HP_1^2} = \frac{P_2V_2.P_2T_2}{HP^2},$$

or

$$\frac{OM_1.M_1T_1}{MM_1^2} = \frac{OM_2.M_2T_2}{MM_2^2}; \qquad (75)$$

a pole and polar property.

From the last set of relations it follows that

$$\frac{B_1-C_1}{A_1} = -\frac{B_2-C_2}{A_2};\ \&c.,$$

or

$$\frac{a_1}{b_1} - \frac{a_1}{c_1} = -\frac{a_2}{b_2} + \frac{a_2}{c_2},\ \&c.,$$

equivalent to the geometrical relations

$$\left.\begin{aligned} \frac{HV_1.P_1T_1}{HT_1.HP_1} &= \frac{HV_2.P_2T_2}{HT_2.HP_2}, \\ \frac{HT_1.P_1V_1}{HV_1.HP_1} &= \frac{HT_2.P_2V_2}{HV_2.HP_2}, \\ \frac{HP_1.T_1V_1}{HV_1.HT_1} &= \frac{HP_2.T_2V_2}{HV_2.HT_2}. \end{aligned}\right\} \tag{76}$$

We can also write

$$\begin{aligned} \frac{k}{n}(-a_1+b_1+c_1) &= -HV_1-HT_1-HP_1 \\ &= HY_1-Y_1V_1-Y_1T_1-Y_1P_1 \\ &= HY_1 - a\,\frac{1-\kappa'^2\,\mathrm{sn}^4(p+r)}{\mathrm{sn}\,(p+r)\,\mathrm{cn}\,(p+r)\,\mathrm{dn}\,(p+r)} \\ &= HY_1 - \frac{2a}{\mathrm{sn}\,2\,(p+r)}; \end{aligned} \tag{77}$$

and similarly

$$\frac{k}{n}(a_1-b_1+c_1) = HY_1 - 2a\,\frac{\mathrm{dn}\,2\,(p+r)}{\mathrm{sn}\,2\,(p+r)}, \tag{78}$$

$$\frac{k}{n}(a_1+b_1-c_1) = HY_1 + 2a\,\frac{\mathrm{cn}\,2\,(p+r)}{\mathrm{sn}\,2\,(p+r)}. \tag{79}$$

Also

$$\left.\begin{aligned} \left(\frac{A-D}{A}\right)^2\delta^2 &= -\frac{\rho_b^2\rho_c^2}{\rho_a^2} = a^2\kappa'^4\,\frac{\mathrm{sn}^2(p+r)\,\mathrm{cn}^2(p+r)}{\mathrm{dn}^2\,(p+r)} = a^2\kappa'^2\,\frac{1-\mathrm{dn}\,2(p+r)}{1+\mathrm{dn}\,2(p+r)}, \\ \left(\frac{B-D}{B}\right)^2\delta^2 &= -\frac{\rho_c^2\rho_a^2}{\rho_b^2} = a^2\,\frac{\mathrm{sn}^2(p+r)\,\mathrm{dn}^2(p+r)}{\mathrm{cn}^2\,(p+r)} = a^2\,\frac{1-\mathrm{cn}\,2(p+r)}{1+\mathrm{cn}\,2(p+r)}, \\ \left(\frac{C-D}{C}\right)^2\delta^2 &= -\frac{\rho_a^2\rho_b^2}{\rho_c^2} = a^2\,\frac{\mathrm{cn}^2(p+r)\,\mathrm{dn}^2(p+r)}{\mathrm{sn}^2\,(p+r)} = a^2\kappa^2\,\frac{\mathrm{dn}\,2(p+r)+\mathrm{cn}\,2(p+r)}{\mathrm{dn}\,2(p+r)-\mathrm{cn}\,2(p+r)}. \end{aligned}\right\} \tag{80}$$

8. After leaving the plane of the focal ellipse, the coordinates of P_1 and Y_1 change to

$$a\frac{\operatorname{cn}(p+r)}{\operatorname{dn}(p+r)}\operatorname{dn} q, \quad a\kappa^2\frac{\operatorname{sn}(p+r)}{\operatorname{dn}(p+r)}\operatorname{cn} q, \quad 0; \tag{81}$$

and $$a\operatorname{cn}(p+r)\operatorname{dn}(p+r)\operatorname{dn} q, \quad a\operatorname{dn}(p+r)\operatorname{sn}(p+r)\operatorname{cn} q,$$

$$a\kappa'^2\operatorname{sn}(p+r)\operatorname{cn}(p+r)\operatorname{sn} q; \tag{82}$$

writing q for qK, and remembering that the elliptic functions of q are to the modulus κ.

The direction cosines of the generating line Y_1P_1, or of the parallel line OG, are thence

$$\operatorname{sn}(p+r)\operatorname{dn} q, \quad -\operatorname{cn}(p+r)\operatorname{cn} q, \quad -\operatorname{dn}(p+r)\operatorname{sn} q; \tag{83}$$

and a change of $p+r$ into $p-r$ will give the direction cosines of the other generating line Y_2P_2, or of the parallel line OC.

If θ denotes the angle between the generating lines, or between OG and OC,

$$\begin{aligned}\cos\theta = {} & \operatorname{sn}(p+r)\operatorname{sn}(p-r)\operatorname{dn}^2 q \\ & +\operatorname{cn}(p+r)\operatorname{cn}(p-r)\operatorname{cn}^2 q \\ & +\operatorname{dn}(p+r)\operatorname{dn}(p-r)\operatorname{sn}^2 q \end{aligned} \tag{84}$$

$$= \frac{1-2\operatorname{sn}^2 p\operatorname{dn}^2 q+\kappa'^2\operatorname{sn}^2 p\operatorname{sn}^2 r}{1-\kappa'^2\operatorname{sn}^2 p\operatorname{sn}^2 r}. \tag{85}$$

This can be determined otherwise by noticing that θ is the angle between the asymptotes of the central section of the hyperboloid made by the plane OGC, while $\mu-\lambda$ and $\mu-\nu$ are the squares of the semi-axes of the section; so that, from (10), (11), (25),

$$\tan^2\tfrac{1}{2}\theta = \frac{\lambda-\mu}{\mu-\nu},$$

$$\cos\theta = \frac{\lambda-2\mu+\nu}{\lambda-\nu}$$

$$= \frac{a^2\dfrac{\operatorname{cn}^2 p}{\operatorname{sn}^2 p}+2a^2\kappa^2\operatorname{sn}^2 q-a^2\operatorname{dn}^2 r}{a^2\dfrac{\operatorname{cn}^2 p}{\operatorname{sn}^2 p}+a^2\operatorname{dn}^2 r}$$

$$= \frac{1+2\kappa^2\operatorname{sn}^2 p\operatorname{sn}^2 q-\operatorname{sn}^2 p\operatorname{dn}^2 r}{1-\kappa'^2\operatorname{sn}^2 p\operatorname{sn}^2 r}. \tag{86}$$

Also, in the plane of the focal ellipse (Fig. 1),

$$HS.HS' = \alpha^2+\lambda+\beta^2+\lambda-OH^2$$

$$= \alpha^2+\lambda+\beta^2+\lambda-\alpha^2-\lambda-\beta^2-\nu$$

$$= \lambda-\nu$$

$$= \alpha^2 \frac{1-\kappa'^2 \operatorname{sn}^2 p \operatorname{sn}^2 r}{\operatorname{sn}^2 p}; \tag{87}$$

so that, putting
$$HS.HS' = k^2, \tag{88}$$

and denoting OY_1 or GH by ρ_1, then, from (29),

$$\frac{\rho_1^2}{k^2} = \frac{\operatorname{sn}^2 p \operatorname{dn}^2(p+r)-\kappa^2 \operatorname{sn}^2 p \operatorname{sn}^2 q}{1-\kappa^2 \operatorname{sn}^2 p \operatorname{sn}^2 q}; \tag{89}$$

and therefore

$$2\frac{\rho_1^2}{k^2}+\cos\theta = \frac{1-2\operatorname{sn}^2 p+\kappa'^2 \operatorname{sn}^2 p \operatorname{sn}^2 r+2\operatorname{sn}^2 p \operatorname{dn}^2(p+r)}{1-\kappa'^2 \operatorname{sn}^2 p \operatorname{sn}^2 r}$$

$$= \frac{1-2\operatorname{sn}^2 p+\kappa'^2 \operatorname{sn}^2 p \operatorname{sn}^2 r}{1-\kappa'^2 \operatorname{sn}^2 p \operatorname{sn}^2 r}+2\frac{\alpha^2}{k^2}\operatorname{dn}^2(p+r)$$

$$= \cos\theta_3+2\frac{\alpha^2}{k^2}\operatorname{dn}^2(p+r), \tag{90}$$

a constant, denoted by E in the sequel (§ 12); thus

$$\rho_1^2 = \tfrac{1}{2}k^2(E-\cos\theta). \tag{91}$$

Making $\operatorname{cn} q = 0$, $\operatorname{sn} q = 1$, brings the hyperboloid into the plane of the focal hyperbola (Fig. 3); the examination of this case will be useful in showing the way of treating the hyperbola in proper analogy with the ellipse, when elliptic functions are employed.

Retaining the same lettering as in Fig. 1, the focus of the focal hyperbola is at A, and the vertex at S; the point T_1 becomes the point of contact of the generating line $T_1P_1V_1$ with the hyperbola, and P_1 now lies on OA.

If the tangent L_1t_1 is drawn to the auxiliary circle of the hyperbola from L_1, the foot of the ordinate of T_1, the angle

$$zOt_1 = \operatorname{am}\{(1-p-r)K', \kappa'\}, \tag{92}$$

and the modular angle of the modulus κ' is zOD, where OD is the asymptote of the hyperbola.

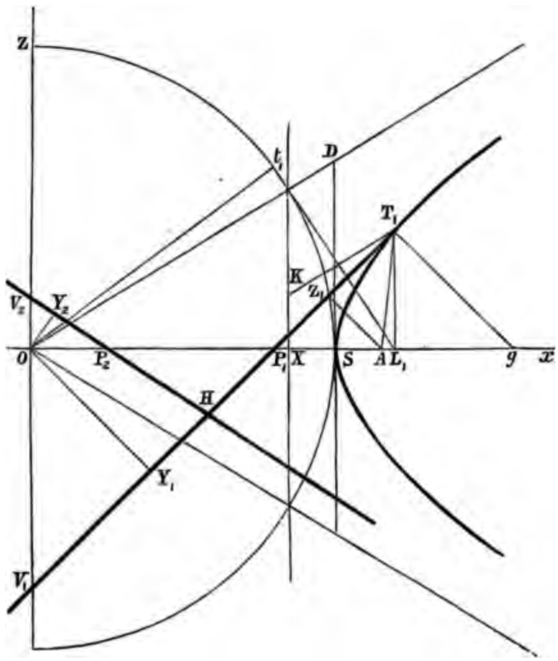


FIG. 3.

If AZ_1 is the perpendicular drawn from the focus A on the tangent at T_1, then Z_1 lies on the auxiliary circle; and we find that the angle

$$Z_1AT_1 = \text{am}\,(p+r)\,\text{K}'. \tag{93}$$

If the normal at T_1 meets OA in g,

$$\kappa' = \frac{AT_1}{Ag} = \frac{\sin AgT_1}{\sin AT_1g}; \tag{94}$$

so that, denoting the angle AOY_1 by ω_1,

$$\sin\omega_1 = \kappa'\,\text{sn}\,(p+r)\,\text{K}',$$

$$\omega_1 = \text{am}\left\{(p+r)\text{K}'\kappa',\ \frac{1}{\kappa'}\right\}, \tag{95}$$

$$\theta_2 = \omega_1 - \omega_2 = \text{am}\left\{(p+r)\text{K}'\kappa',\ \frac{1}{\kappa'}\right\} - \text{am}\left\{(p-r)\text{K}'\kappa',\ \frac{1}{\kappa'}\right\}. \tag{96}$$

9. The tangent line HK of the polhode, and therefore also of the herpolhode, of H in the invariable plane of G is normal to the hyperboloid (§ 3), or the plane OGC is perpendicular to HK (Fig. 2).

If ψ denotes the azimuthal angle of the plane OGC, OG being held fixed in a vertical position, the angle

$$KGH = \varpi_1 - \psi,$$

and

$$GK = \rho_1 \cos(\varpi_1 - \psi), \tag{97}$$

while

$$GK \sin\theta = OC - OG\cos\theta$$
$$= \delta_2 - \delta_1 \cos\theta; \tag{98}$$

so that

$$\cos(\varpi_1 - \psi) = \frac{\delta_2 - \delta_1 \cos\theta}{\rho_1 \sin\theta}$$

$$= \sqrt{2}\,\frac{\dfrac{\delta_2}{k} - \dfrac{\delta_1}{k}\cos\theta}{\sin\theta\sqrt{(E-\cos\theta)}} \tag{99}$$

$$\sin(\varpi_1 - \psi) = \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(E-\cos\theta)}}, \tag{100}$$

where

$$\Theta = (1-\cos^2\theta)(E-\cos\theta) - 2\left(\frac{\delta_2}{k} - \frac{\delta_1}{k}\cos\theta\right)^2$$
$$= \cos\theta - \operatorname{ch}\theta_1 \,.\, \cos\theta - \cos\theta_2 \,.\, \cos\theta - \cos\theta_3, \tag{101}$$

when resolved into factors.

Dropping for the moment the suffix from ρ_1,

$$\cos\theta - \cos\theta_3 = 2\,\frac{a^2\,\mathrm{dn}^2(p+r) - \rho^2}{k^2}, \tag{102}$$

and denoting, as before in (69), the values of ρ corresponding to θ_3, θ_2, θ_1 by ρ_a, ρ_b, ρ_c, then, as in (14),

$$\rho_a^2 = a^2\,\mathrm{dn}^2(p+r), \tag{103}$$

$$\rho_a^2 - \rho^2 = \tfrac{1}{2}k^2(\cos\theta - \cos\theta_3). \tag{104}$$

Similarly

$$\rho_b^2 - \rho^2 = \tfrac{1}{2}k^2(\cos\theta - \cos\theta_2), \tag{105}$$

$$\rho_c^2 - \rho^2 = \tfrac{1}{2}k^2(\cos\theta - \operatorname{ch}\theta_1), \tag{106}$$

where

$$\rho_b^2 = a^2\kappa'^2\,\mathrm{cn}^2(p+r), \tag{107}$$

$$\rho_c^2 = -a^2\kappa'^2\,\mathrm{sn}^2(p+r); \tag{108}$$

so that
$$\rho_a^2-\rho_b^2 = \alpha^2\kappa^2 = \beta^2, \tag{109}$$
$$\rho_b^2-\rho_c^2 = \alpha^2\kappa'^2 = \alpha^2-\beta^2, \tag{110}$$
$$\rho_a^2-\rho_c^2 = \alpha^2. \tag{111}$$

Also (41)
$$R = 4\,.\,\rho_a^2-\rho^2\,.\,\rho_b^2-\rho^2\,.\,\rho_c^2-\rho^2$$
$$= \tfrac{1}{2}k^6\Theta\,; \tag{112}$$

so that
$$\frac{\sin\theta\, d\theta}{\surd(2\Theta)} = \frac{k\, d\rho^2}{\surd R}; \tag{113}$$

and we can put
$$k\frac{d\rho^2}{dt} = n\surd R, \tag{114}$$
$$\sin\theta\frac{d\theta}{dt} = n\surd(2\Theta), \tag{115}$$

where n denotes some constant angular velocity.

Then, from the differential equation (40) of the herpolhode of H in the invariable plane of G,

$$\frac{d\varpi_1}{dt} = \frac{\delta_1}{\surd R}\frac{d\rho^2}{dt} + \frac{\surd(-\rho_a^2\rho_b^2\rho_c^2)}{\rho^2\surd R}\frac{d\rho^2}{dt}$$
$$= n\frac{\delta_1}{k} + n\frac{\surd(-\rho_a^2\rho_b^2\rho_c^2)}{k\rho^2}. \tag{116}$$

When
$$\rho^2 = 0,$$
$$\cos\theta = E,$$

and
$$\rho_a^2\rho_b^2\rho_c^2 = -\tfrac{1}{4}k^4(\delta_2-\delta_1 E)^2\,; \tag{117}$$

so that
$$\frac{d\varpi_1}{dt} = n\frac{\delta_1}{k} + n\frac{\dfrac{\delta_2}{k}-\dfrac{\delta_1}{k}E}{E-\cos\theta}$$
$$= n\frac{\dfrac{\delta_2}{k}-\dfrac{\delta_1}{k}\cos\theta}{E-\cos\theta}. \tag{118}$$

Now, from (99), (100),
$$\varpi_1-\psi = \cos^{-1}\surd 2\,\frac{\dfrac{\delta_2}{k}-\dfrac{\delta_1}{k}\cos\theta}{\sin\theta\surd(E-\cos\theta)}$$
$$= \sin^{-1}\frac{\surd\Theta}{\sin\theta\surd(E-\cos\theta)}. \tag{119}$$

[redacted]

Therefore

$$\frac{d\varpi_1}{dt} - \frac{d\psi}{dt}$$

$$= -\frac{\sqrt{2}\,\dfrac{\dfrac{\delta_2}{k} - \dfrac{\delta_1}{k}\cos\theta}{\sin\theta\sqrt{(E-\cos\theta)}}\left\{\dfrac{\dfrac{\delta_1}{k}\sin\theta}{\dfrac{\delta_2}{k}-\dfrac{\delta_1}{k}\cos\theta} - \dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{2(E-\cos\theta)}\right\}}{\dfrac{\sqrt{\Theta}}{\sin\theta\sqrt{(E-\cos\theta)}}}\,\frac{d\theta}{dt}$$

$$= -2n\frac{\delta_1}{k} + 2n\frac{\dfrac{\delta_2}{k}\cos\theta - \dfrac{\delta_1}{k}\cos^2\theta}{\sin^2\theta} + n\frac{\dfrac{\delta_2}{k} - \dfrac{\delta_1}{k}\cos\theta}{E-\cos\theta}$$

$$= \quad 2n\frac{-\dfrac{\delta_1}{k} + \dfrac{\delta_2}{k}\cos\theta}{\sin^2\theta} + n\frac{\dfrac{\delta_2}{k} - \dfrac{\delta_1}{k}\cos\theta}{E-\cos\theta}; \tag{120}$$

so that, subtracting (118) and (120),

$$\frac{d\psi}{dt} = 2n\frac{\dfrac{\delta_1}{k} - \dfrac{\delta_2}{k}\cos\theta}{\sin^2\theta}. \tag{121}$$

Equations (115) and (121) define the motion of the axis of a top, if, with our notation,

$$n^2 = \frac{Wgh}{A},$$

$$\frac{G}{A} = 2n\frac{\delta_1}{k}, \quad \frac{Cr}{A} = 2n\frac{\delta_2}{k}; \tag{122}$$

we have thus derived the motion of the axis of a top OC from the motion in the herpolhode of H.

We shall find, from (117) and (91),

$$\tfrac{1}{2}\left(\frac{\delta_2}{\delta_1} - E\right) = \frac{A-D\,.\,B-D\,.\,C-D}{ABC}\,\frac{\delta_1^2}{k^2}, \tag{123}$$

$$\tfrac{1}{2}\left(\frac{\delta_2}{\delta_1} - \cos\theta\right) = \frac{\rho^2}{k^2} + \frac{A-D\,.\,B-D\,.\,C-D}{ABC}\,\frac{\delta_1^2}{k^2}; \tag{124}$$

and as, in the herpolhode of H in the invariable plane of G,

$$p = \rho\cos(\varpi_1 - \psi) = \frac{\delta_2 - \delta_1\cos\theta}{\sin\theta},$$

$$\frac{p^2}{\delta_1^2} = \frac{\left(\dfrac{\rho^2}{k^2} + \dfrac{A-D\,.\,B-D\,.\,C-D}{ABC}\dfrac{\delta_1^2}{k^2}\right)^2}{\left\{\left(\dfrac{D}{A}+\dfrac{D}{B}+\dfrac{D}{C}-1\right)\dfrac{\delta_1^2}{k^2}-\dfrac{\rho^2}{k^2}\right\}\left(\dfrac{\rho^2}{k^2}+\dfrac{A-D\,.\,B-D\,.\,C-D}{ABC}\dfrac{\delta_1^2}{k^2}\right)-\dfrac{D^3}{ABC}\dfrac{\delta_1^2}{k^2}\dfrac{\rho^2}{k^2}}, \tag{50}$$

therefore this denominator must be the same as $\frac{1}{4}\sin^2\theta$; so that

$$\begin{aligned}\tfrac{1}{4}\sin^2\theta = &\left\{\left(\frac{D}{A}+\frac{D}{B}+\frac{D}{C}-1\right)\frac{\delta_1^2}{k^2}\right.\\ &\left.+\frac{A-D\,.\,B-D\,.\,C-D}{ABC}\frac{\delta_1^2}{k^2}-\tfrac{1}{2}\left(\frac{\delta_2}{\delta_1}-\cos\theta\right)\right\}\tfrac{1}{2}\left(\frac{\delta_2}{\delta_1}-\cos\theta\right)\\ &-\frac{D^3}{ABC}\frac{\delta_1^2}{k^2}\left\{\tfrac{1}{2}\left(\frac{\delta_2}{\delta_1}-\cos\theta\right)-\frac{A-D\,.\,B-D\,.\,C-D}{ABC}\frac{\delta_1^2}{k^2}\right\},\end{aligned} \tag{125}$$

and, equating to zero the coefficient of $\cos\theta$,

$$\frac{\delta_2}{\delta_1} = \frac{(A+B+C)\,D^2-2D^3}{ABC}\frac{\delta_1^2}{k^2}. \tag{126}$$

Provided we take Darboux's Ω as the equivalent of our n^2, this relation (126) reduces, by (63) and (64), to his relation

$$\Omega\frac{\delta_2}{\delta_1} = Q-2\frac{R}{h}; \tag{127}$$

P, Q, R, in Darboux's notation representing respectively

$$a+b+c, \quad bc+ca+ab, \quad abc;$$

so that a, b, c are the roots of the cubic

$$x^3-Px^2+Qx-R=0.$$

Equating the constant terms leads to the relation

$$1 = \left\{\frac{D^4(A+B+C-2D)^2}{A^2B^2C^2}+\frac{4D^3\,.\,A-D\,.\,B-D\,.\,C-D}{A^2B^2C^2}\right\}\frac{\delta^4}{k^4}, \tag{128}$$

reducing, as above, to Darboux's relation

$$\Omega^2 = Q^2-4R(P-h).$$

From these relations we can deduce

$$\frac{\delta_2-\delta_1}{\delta_2+\delta_1} = \frac{\wp' v_1}{\wp' v_2}, \tag{129}$$

(§ 13), and a variety of other similar relations, all of which can now receive a geometrical interpretation.

10. With the Weierstrassian notation, we put

$$\left.\begin{aligned} \alpha^2+\lambda &= m^2(e_1-\wp v_2), & \beta^2+\lambda &= m^2(e_2-\wp v_2), & \lambda &= m^2(e_3-\wp v_2);\\ \alpha^2+\mu &= m^2(e_1-\wp u), & \beta^2+\mu &= m^2(e_2-\wp u), & \mu &= m^2(e_3-\wp u);\\ \alpha^2+\nu &= m^2(e_1-\wp v_1), & \beta^2+\nu &= m^2(e_2-\wp v_1), & \nu &= m^2(e_3-\wp v_1); \end{aligned}\right\} \quad (130)$$

where $v_1 = \omega_1+p\omega_3,\quad v_2 = r\omega_3,$

and $$v = v_1+v_2, \qquad (131)$$

as in § 5; and now the homogeneity factors M^2 and k^2 are chosen, so that

$$M^2 = \wp v_1 - \wp v_2; \qquad (132)$$

and, as in (88),

$$\begin{aligned} k^2 &= HS.HS' \\ &= \lambda-\nu \\ &= m^2(\wp v_1-\wp v_2); \end{aligned} \qquad (133)$$

and $$\alpha^2 = m^2(e_1-e_3), \qquad (134)$$

$$\frac{k^2}{\alpha^2} = \frac{\wp v_1-\wp v_2}{e_1-e_3} = \frac{M^2}{s_1-s_3}; \qquad (135)$$

and, to agree with the definition of Darboux's L, we put

$$L = M\frac{\delta_1}{k} = \surd(e_1-e_3)\frac{\delta_1}{\alpha} = \frac{\delta_1}{m}. \qquad (136)$$

But, from (19) and (151),

$$\begin{aligned} \frac{Y_1V_1}{\alpha} &= \frac{\operatorname{cn}(p+r)\operatorname{dn}(p+r)}{\operatorname{sn}(p+r)} \\ &= \sqrt{\left(\frac{\sigma-s_2\,.\,\sigma-s_3}{s_1-s_3\,.\,s_1-\sigma}\right)}, \end{aligned}$$

$$\frac{Y_1V_1}{m} = \tfrac{1}{2}\frac{\surd(-\Sigma)}{s_1-\sigma};$$

so that

$$\left.\begin{aligned} \frac{HV_1}{m} &= \tfrac{1}{2}\frac{\surd(-\Sigma)}{s_1-\sigma}+L,\\ \text{and similarly}\quad \frac{HT_1}{m} &= \tfrac{1}{2}\frac{\surd(-\Sigma)}{\sigma-s_2}-L,\\ \frac{HP_1}{m} &= \tfrac{1}{2}\frac{\surd(-\Sigma)}{\sigma-s_3}-L. \end{aligned}\right\} \quad (137)$$

It is readily proved, by means of the transformation

$$s(u)-s_a = \frac{s_a-s_b \,.\, s_a-s_c}{s(u-\omega_a)-s_a}, \tag{138}$$

in the standard integral (51), that

$$P(v)+P(\omega_a-v) = \frac{\sqrt{(-\Sigma)}}{\sigma-s_a}; \tag{139}$$

so that, when we cancel the secular term in (56) by putting

$$\frac{HY_1}{m} = \tfrac{1}{2}P(v) = L, \tag{135}$$

then, at the same time,

$$\frac{HV_1}{m} = \tfrac{1}{2}P(v-\omega_1), \tag{140}$$

$$\frac{HT_1}{m} = \tfrac{1}{2}P(\omega_2-v), \tag{141}$$

$$\frac{HP_1}{m} = \tfrac{1}{2}P(\omega_3-v); \tag{142}$$

so that the corresponding rolling quadric is given by

$$\left.\begin{aligned}
\frac{D_1}{A_1} &= \frac{a_1}{h_1} = \frac{HV_1}{HY_1} = \frac{P(v-\omega_1)}{P(v)}, \\
\frac{D_1}{B_1} &= \frac{b_1}{h_1} = -\frac{HT_1}{HY_1} = \frac{P(v-\omega_2)}{P(v)}, \\
\frac{D_1}{C_1} &= \frac{c_1}{h_1} = -\frac{HP_1}{HY_1} = \frac{P(v-\omega_3)}{P(v)}.
\end{aligned}\right\} \tag{143}$$

If x_1, y_1 denote the coordinates of P_1 on the focal ellipse, and ϕ_1 its excentric angle measured from the minor axis,

$$\left.\begin{aligned}
\frac{x_1^2}{\alpha^2} &= \sin^2\phi_1 = \text{sn}^2(1-p-r)\,\text{K}' = \frac{s_1-s_3}{s_1-s_2}\,\frac{\sigma-s_2}{\sigma-s_3}, \\
\frac{y_1^2}{\beta^2} &= \cos^2\phi_1 = \text{cn}^2(1-p-r)\,\text{K}' = \frac{s_2-s_3}{s_1-s_2}\,\frac{s_1-\sigma}{\sigma-s_3}.
\end{aligned}\right\} \tag{144}$$

11. The curve σ_a described in the invariable plane of G by the projection of a point A fixed in a principal axis Ox, as well as the curve σ'_a described by the point in which Ox cuts the invariable plane (Fig. 2), have been examined by Poinsot in his *Théorie nouvelle de la rotation des corps*, as these six associated curves have the same analytical character as the herpolhode of H, when all restrictions of sign and magnitude have been removed from A, B, C, D.

Denoting the polar coordinates of σ_a in the invariable plane by ρ_a, ϖ_a, then $\varpi_a-\varpi_1$ is the angle between the planes OGH and OGx.

The equation of the plane OGH is found to be

$$\kappa^2\frac{\text{sn}\,(p+r)}{\text{dn}\,q}x-\frac{\text{cn}\,(p+r)}{\text{cn}\,q}y+\frac{\text{dn}\,(p+r)}{\text{sn}\,q}z=0. \qquad (145)$$

The direction cosines of OG being

$$\text{sn}\,(p+r)\,\text{dn}\,q,\quad -\text{cn}\,(p+r)\,\text{cn}\,q,\quad -\text{dn}\,(p+r)\,\text{sn}\,q, \qquad (83)$$

the equation of the plane xOG is

$$0\,.\,x-y\,\text{dn}\,(p+r)\,\text{sn}\,q+z\,\text{cn}\,(p+r)\,\text{cn}\,q=0. \qquad (146)$$

Denoting for a moment the two planes (145) and (146) by

$$Px+Qy+Rz=0,$$

$$P'x+Q'y+R'z=0,$$

the tangent of the angle between them is

$$\frac{\sqrt{\{(QR'-Q'R)^2+(RP'-R'P)^2+(PQ'-P'Q)^2\}}}{PP'+QQ'+RR'}; \qquad (147)$$

and $$PP'+QQ'+RR'=\frac{\text{cn}\,(p+r)\,\text{dn}\,(p+r)}{\text{sn}\,q\,\text{cn}\,q}, \qquad (148)$$

$$QR'-Q'R=\kappa^2\,\text{sn}^2(p+r),$$

$$RP'-R'P=-\kappa^2\,\text{sn}\,(p+r)\,\text{cn}\,(p+r)\frac{\text{cn}\,q}{\text{dn}\,q},$$

$$PQ'-P'Q=-\kappa^2\,\text{sn}\,(p+r)\,\text{dn}(p+r)\frac{\text{sn}\,q}{\text{dn}\,q},$$

$$(QR'-Q'R)^2+(RP'-R'P)^2+(PQ'-P'Q)^2$$

$$=\frac{\kappa^4\,\text{sn}^2\,(p+r)}{\text{dn}^2\,q};$$

so that $$\tan\,(\varpi_a-\varpi_1)=\frac{\kappa^2\,\text{sn}\,(p+r)\,\text{sn}\,q\,\text{cn}\,q}{\text{cn}\,(p+r)\,\text{dn}\,(p+r)\,\text{dn}\,q}. \qquad (149)$$

Now, in (51),

$$s-s_3 = \frac{s_1-s_3}{\operatorname{sn}^2(q\mathrm{K}+\mathrm{K}'i)} = (s_1-s_3)\,\kappa^2\operatorname{sn}^2 q\mathrm{K}$$

$$= (s_2-s_3)\operatorname{sn}^2 q\mathrm{K},$$

$$\operatorname{sn}^2 q = \frac{s-s_3}{s_2-s_3},\quad \operatorname{cn}^2 q = \frac{s_2-s}{s_2-s_3},\quad \operatorname{dn}^2 q = \frac{s_1-s}{s_1-s_3}; \tag{150}$$

and

$$\sigma-s_3 = \frac{s_1-s_3}{\operatorname{sn}^2\{\mathrm{K}+(p+r)\mathrm{K}'i,\ \kappa\}}$$

$$= (s_1-s_3)\frac{\operatorname{dn}^2\{(p+r)\,\mathrm{K}'i,\ \kappa\}}{\operatorname{cn}^2\{(p+r)\,\mathrm{K}'i,\ \kappa\}}$$

$$= (s_1-s_3)\operatorname{dn}^2\{(p+r)\,\mathrm{K}',\ \kappa'\},$$

$$\left.\begin{aligned} \operatorname{dn}^2(p+r) &= \frac{\sigma-s_3}{s_1-s_3},\\ \kappa'^2\operatorname{sn}^2(p+r) &= \frac{s_1-\sigma}{s_1-s_3},\\ \operatorname{sn}^2(p+r) &= \frac{s_1-\sigma}{s_1-s_2},\\ \operatorname{cn}^2(p+r) &= \frac{\sigma-s_2}{s_1-s_2}; \end{aligned}\right\} \tag{151}$$

so that, otherwise expressed,

$$\tan(\varpi_a-\varpi_1) = \sqrt{\left(\frac{s_1-\sigma\,.\,s_2-s\,.\,s-s_3}{s_1-s\,.\,\sigma-s_2\,.\,\sigma-s_3}\right)}$$

$$= \frac{s_1-\sigma}{\sqrt{(-\Sigma)}}\cdot\frac{\sqrt{S}}{s_1-s}, \tag{152}$$

and, interpreted geometrically, if SZ is the perpendicular from S on OY_1 (Fig. 1),

$$\left.\begin{aligned} \sqrt{(s_1-\sigma)} &= \frac{\kappa' a}{m}\operatorname{sn}(p+r) = \frac{OS}{m}\sin AOY_1 = \frac{SZ}{m},\\ \sqrt{(\sigma-s_2)} &= \frac{\kappa' a}{m}\operatorname{cn}(p+r) = \frac{OS}{m}\cos AOY_1 = \frac{OZ}{m},\\ \sqrt{(\sigma-s_3)} &= \frac{a}{m}\operatorname{dn}(p+r) = \frac{OY_1}{m}. \end{aligned}\right\} \tag{153}$$

[redacted]

Next, by the theorem for the addition of the elliptic integrals of the third kind with parameters v and $\omega_a - v$, writing $s(v)$ for σ, and $S(v)$ for Σ,

$$I(v) + I(\omega_a - v)$$

$$= \tfrac{1}{2}\int \frac{P(v)\{s(v)-s\} - \surd\{-S(v)\}}{s(v)-s}\,\frac{ds}{\surd S}$$

$$+\tfrac{1}{2}\int \frac{P(\omega_a - v)\{s(\omega_a - v)-s\} - \surd\{-S(\omega_a - v)\}}{s(\omega_a - v)-s}\,\frac{ds}{\surd S}$$

$$= \tan^{-1}\frac{s(v)-s_a}{\surd\{-S(v)\}}\,\frac{\surd S}{s-s_a}; \tag{154}$$

this is readily verified by a differentiation.

Thence
$$\varpi_a = \varpi_1 + \tan^{-1}\frac{s_1-\sigma}{\surd(-\Sigma)}\,\frac{\surd S}{s_1-s}$$

$$= \left\{M\frac{\delta_1}{k} - \tfrac{1}{2}P(v)\right\}\frac{nt}{M} + I(\omega_a - v). \tag{155}$$

Also, putting
$$OA = k_a,$$

$$\left(\frac{\rho_a}{k_a}\right)^2 = \sin^2 xOG = 1 - \mathrm{sn}^2(p+r)\,\mathrm{dn}^2 q$$

$$= 1 - \frac{s_1-\sigma}{s_1-s_2}\,\frac{s_1-s}{s_1-s_3} = 1 - \frac{s_a - s}{s_a - s(\omega_a - v)}$$

$$= \frac{s - s(\omega_a - v)}{s_a - s(\omega_a - v)}, \tag{156}$$

so that the polar coordinates ρ_a, ϖ_a of the curve σ_a satisfy relations similar to those for ρ, ϖ of the herpolhode of H.

So also, if the length $G\sigma'_a$ is denoted by ρ'_a,

$$\left(\frac{\rho'_a}{\delta}\right)^2 = \tan^2 xOG$$

$$= \frac{\rho_a^2}{k_a^2 - \rho_a^2}$$

$$= \frac{s - s(\omega_a - v)}{s_a - s}$$

$$= \frac{s(\omega_a - u) - s(v)}{s_a - s(v)}, \tag{157}$$

so that the curve σ'_a is of a similar analytical character.

If $I(v)$ is pseudo-elliptic, and the secular term is cancelled, so as to make the herpolhode of H an algebraical curve, then the six associated curves described by

$$\sigma_a, \quad \sigma_b, \quad \sigma_c, \quad \sigma'_a, \quad \sigma'_b, \quad \sigma'_c$$

are also algebraical.

In the article "Sulla funzione caratteristica del moto di rotazione di un corpo non sollecitato da forze" (*Atti della R. Accademia di Napoli*, December, 1893), Colonel Siacci has proved that the points in which the invariable plane EOF (Fig. 2) cuts the principal sections of the rolling quadric describe curves as if under a central force to O composed of terms proportional to the first, third, and fifth powers of the distance; these and other similar theorems in this article can be proved in the manner above.

12. The motion of the axis of the top is given by the two equations for the Conservation of Energy and Momentum,

$$\tfrac{1}{2}A\frac{d\theta^2}{dt^2}+\tfrac{1}{2}A\sin^2\theta\frac{d\psi^2}{dt^2}=Wgh\,(D-\cos\theta), \tag{158}$$

$$A\sin^2\theta\frac{d\psi}{dt}+Cr\cos\theta=G, \tag{159}$$

connecting θ, the angle between the axis OC of the top and its highest vertical position, and ψ, the azimuth of the axis (the suffix 1 employed by Dr. Routh for A, C, and G is now omitted).

In accordance with Dr. Routh's results (*Quarterly Journal of Mathematics*, XXIII, p. 38), we put

$$\left.\begin{aligned}\frac{G}{A}&=2\frac{T_1}{G_1}=2h_1=2n\frac{\delta_1}{k},\\ \frac{Cr}{A}&=2\frac{T_2}{G_2}=2h_2=2n\frac{\delta_2}{k}.\end{aligned}\right\} \tag{160}$$

The homogeneity factor M, and Darboux's L and B, now must satisfy the relations

$$\frac{L}{M}=\frac{G}{\surd(4AWgh)}=\frac{h_1}{n}=\frac{\delta_1}{k}, \tag{161}$$

$$\frac{B}{M}=\frac{Cr}{\surd(4AWgh)}=\frac{h_2}{n}=\frac{\delta_2}{k}. \tag{162}$$

From (158) and (159), we obtain

$$\sin^2\theta\frac{d\theta^2}{dt^2} = 2\frac{Wgh}{A}(D-\cos\theta)(1-\cos^2\theta)-\left(\frac{G-Cr\cos\theta}{A}\right)^2$$

$$= 2\frac{Wgh}{A}(E-\cos\theta)(1-\cos^2\theta)-\left(\frac{Cr-G\cos\theta}{A}\right)^2$$

$$= 2\frac{Wgh}{A}\Theta,$$

where $$\Theta = (\cos\theta-\text{ch}\,\theta_1)(\cos\theta-\cos\theta_2)(\cos\theta-\cos\theta_3), \qquad (163)$$

as in (101), when resolved into factors; the angle θ oscillating between θ_2 and θ_3, so that

$$\theta_2 < \theta < \theta_3. \qquad (164)$$

We have chosen the homogeneity factor n in (122), such that

$$n^2 = \frac{Wgh}{A}, \qquad (165)$$

and at the same time taken (Fig. 1)

$$k^2 = SH\,.\,S'H, \qquad (166)$$

as we shall find that this makes Darboux's

$$\Omega = n^2. \qquad (167)$$

Also, in accordance with Dr. Routh's results, on Fig. 1,

$$\left.\begin{aligned}\text{ch}\,\theta_1 &= \frac{OH^2+OS^2}{OD^2},\\ \cos\theta_2 &= \frac{OH^2-OS^2}{OD^2},\\ \cos\theta_3 &= \frac{OH^2-AB^2}{OD^2};\end{aligned}\right\} \qquad (168)$$

$$\left.\begin{aligned}\text{sh}\,\theta_1 &= 2\frac{OS.OM}{OD^2} = 2\frac{SZ.HV_1}{OD^2},\\ \sin\theta_2 &= 2\frac{OS.ON}{OD^2} = 2\frac{OZ.HT_1}{OD^2},\\ \sin\theta_3 &= 2\frac{OY_1.HP_1}{OD^2} = 2\frac{OY_2.HP_2}{OD^2};\end{aligned}\right\} \qquad (169)$$

if OD is conjugate to OH in the confocal ellipse through H; so that

$$OD^2 = HS\,.\,HS' = k^2; \qquad (170)$$

and $$\frac{G}{\sqrt{(AWgh)}} = 2\frac{HY_1}{OD}, \quad \frac{Cr}{\sqrt{(AWgh)}} = 2\frac{HY_2}{OD}. \qquad (171)$$

We now see that the relations in (72) are equivalent to

$$\left.\begin{aligned} \frac{h_1^2-(h_1-b_1)(h_1-c_1)}{n^2} &= \frac{OH^2-OA^2}{k^2} \\ &= \tfrac{1}{2}(\cos\theta_2+\cos\theta_3), \\ \frac{h_1^2-(h_1-c_1)(h_1-a_1)}{n^2} &= \tfrac{1}{2}(\operatorname{ch}\theta_1+\cos\theta_3), \\ \frac{h_1^2-(h_1-a_1)(h_1-b_1)}{n^2} &= \tfrac{1}{2}(\operatorname{ch}\theta_1+\cos\theta_2); \end{aligned}\right\} \qquad (172)$$

and thus the expressions are unchanged when the suffix 1 is changed into 2.

We have also to prove that

$$\frac{\Omega}{n^2} = \frac{HS\,.\,HS'}{k^2}, \qquad (173)$$

so that $$\Omega = n^2, \qquad (174)$$

when we take $$HS\,.\,HS' = k^2. \qquad (175)$$

If Ω is defined by Darboux's relation

$$\Omega^2 = Q^2-4R\,(P-h), \qquad (176)$$

given at the end of § 9, where P, Q, R are also defined, we should have to prove that, in Fig. 1,

$$\begin{aligned} &(HT_1.HP_1-HV_1.HP_1-HV_1.HT_1)^2 \\ &\quad -4HV_1.HT_1.HP_1\,(HV_1-HT_1-HP_1-HY_1)= HS^2\,.\,HS'^2, \end{aligned} \qquad (177)$$

a geometrical relation that is not very obvious.

So also the other definitions of Darboux,

$$\Omega h_1 h_2 = Q_1 h_1^2-2R_1 h_1 = Q_2 h_2^2-2R_2 h_2, \ \text{\&c.}, \qquad (178)$$

are capable of a geometrical interpretation.

Adding equations (168) gives

$$\begin{aligned} \operatorname{ch}\theta_1+\cos\theta_2+\cos\theta_3 &= \frac{3.OH^2-AB^2}{k^2} \\ &= \frac{2P_1h_1-Q_1}{n^2} = \frac{2P_2h_2-Q_2}{n^2}. \end{aligned} \qquad (179)$$

The geometrical theorems upon which (179) can be made to depend are the following: dropping the suffix 1, and remembering that, in Fig. 1,

$$\cos V_1HV_2 = \frac{AB^2-OH^2}{OD^2},$$

$$\sin V_1HV_2 = \frac{2OY.HP}{OD^2},$$

where $$OD^2 = HS.HS';$$

then $$HY_2 = HY_1\cos V_1HV_2 + OY_1 \sin V_1HV_2,$$

$$OD^2.HY_2 = HY(AB^2-HY^2-OY^2)+2OY^2.HP$$

$$= -HY^3+HY(AB^2-3OY^2)+2OY^2.YP; \quad (180)$$

while, putting

$$a-h=a', \quad b-h=b', \quad c-h=c',$$

so that $$a' = \frac{n}{k}YV, \quad b' = -\frac{n}{k}YT, \quad c' = -\frac{n}{k}YP; \quad (181)$$

$$\frac{k^3}{n^3}(2R-Qh) = \frac{k^3}{n^3}\{-h^3+(b'c'+c'a'+a'b')h+2a'b'c'\}$$

$$= -HY^3+(YT.YP-YP.YV-YV.YT)$$

$$+2YV.VT.YP, \quad (182)$$

and, from Fig. 1 and equations (23),

$$\left.\begin{aligned} YV.YT &= OY^2, \\ YT.YP &= PV.YT-YV.YT = OA^2-OY^2, \\ YP.YV &= YV.YT-PT.YV = OY^2-OB^2; \end{aligned}\right\} \quad (183)$$

so that $$YT.YP-YP.YV-YV.YT$$

$$= OA^2+OB^2-3OY^2 = AB^2-3OY^2,$$

$$YV.YT.YP = OY^2.YP; \quad (184)$$

so that $$OD^2.HY_2 = \frac{k^3}{n^3}(2R-Qh),$$

or $$\frac{\Omega}{n^2} = \frac{OD^2}{k^2}. \quad (185)$$

Equation (177) can be proved by noticing that, if we put

$$\left.\begin{aligned} a'+b'+c' &= P', \\ b'c'+c'a'+a'b' &= Q', \\ a'b'c' &= R', \end{aligned}\right\} \tag{186}$$

then

$$\left.\begin{aligned} P' &= \frac{n}{k}(-YP-YT+YV), \\ Q' &= \frac{n^2}{k^2}(AB^2-3OY^2), \\ R' &= \frac{n^3}{k^3}\,OY^2.YP; \end{aligned}\right\} \tag{187}$$

and

$$\begin{aligned} \Omega^2 &= Q^2-4R(P-h) \\ &= (h^2-Q')^2-4R'(2h+P'). \end{aligned} \tag{188}$$

But

$$\begin{aligned} \frac{k^2}{n^2}(h^2-Q') &= HY^2+3OY^2-AB^2 \\ &= OH^2+2OY^2-AB^2, \end{aligned} \tag{189}$$

$$\frac{k}{n}(2h+P') = -YP+HV-HT; \tag{190}$$

so that

$$\begin{aligned} \frac{k^4}{n^4}\Omega^2 &= (OH^2+2OY^2-AB^2)^2-4OY^2.YP.(-YP+HV-HT) \\ &= OD^4, \end{aligned} \tag{191}$$

after reduction.

13. The solution of equation (163) by elliptic functions may be written, in accordance with the preceding notation,

$$\left.\begin{aligned} \wp u-e_1 &= s-s_1 = \tfrac{1}{2}M^2(\cos\theta-\text{ch}\,\theta_1), \\ \wp u-e_2 &= s-s_2 = \tfrac{1}{2}M^2(\cos\theta-\cos\theta_2), \\ \wp u-e_3 &= s-s_3 = \tfrac{1}{2}M^2(\cos\theta-\cos\theta_3), \\ -\wp' u &= \surd S = \tfrac{1}{2}M^3\surd(2\Theta); \end{aligned}\right\} \tag{192}$$

and, now

$$\frac{du}{dt} = \frac{n}{M},$$

and

$$u = \frac{nt}{M}+\omega_2, \quad \text{or} \quad \frac{nt}{M}+\omega_3, \tag{193}$$

for $\cos\theta$ to oscillate between $\cos\theta_2$ and $\cos\theta_3$.

Putting $u = v_1$ and v_2, when $\cos\theta = +1$ and -1, then

$$\wp v_1 - \wp v_2 = M^2, \tag{194}$$

and

$$v_1 = \omega_1 + p\omega_3, \quad v_2 = r\omega_3.$$

Now

$$1 - \cos\theta = 2\frac{\wp v_1 - \wp u}{\wp v_1 - \wp v_2},$$

$$1 + \cos\theta = 2\frac{\wp u - \wp v_2}{\wp v_1 - \wp v_2},$$

$$\tan^2\tfrac{1}{2}\theta = \frac{\wp v_1 - \wp u}{\wp u - \wp v_2}$$

$$= \left(\frac{\sigma v_2}{\sigma v_1}\right)^2 \frac{\sigma(v_1 - u)\,\sigma(v_1 + u)}{\sigma(u - v_2)\,\sigma(u + v_2)}.$$

Equation (159) can be transformed into

$$\frac{d\psi i}{du} = \frac{-\frac{1}{2}\wp' v_1}{\wp v_1 - \wp u} + \frac{\frac{1}{2}\wp' v_2}{\wp u - \wp v_2},$$

$$\psi i = \tfrac{1}{2}\log\frac{\sigma(v_1 - u)}{\sigma(v_1 + u)}e^{2u\zeta v_1} + \tfrac{1}{2}\log\frac{\sigma(u - v_2)}{\sigma(u + v_2)}e^{2u\zeta v_2};$$

so that

$$\tan\tfrac{1}{2}\theta\, e^{\psi i} = \frac{\sigma v_2}{\sigma v_1}\,\frac{\sigma(v_1 - u)}{\sigma(u + v_2)}\,e^{u(\zeta v_1 + \zeta v_2)}, \tag{195}$$

which is Klein's expression for the stereographic projection on the invariable plane of G of the path of a point on the axis of the Top ("Ueber die Bewegung des Kreisels," *Gött. Nach.*, 1896).

When working with the Jacobian notation,

$$\frac{u}{\sqrt{(s_1 - s_3)}} = \frac{nt}{\sqrt{\{\frac{1}{2}(\text{ch}\,\theta_1 - \cos\theta_3)\}}} + \text{K}'i$$

$$= mt + \text{K}'i, \text{ suppose}; \tag{196}$$

$$\cos\theta - \cos\theta_3 = \frac{\wp u - e_3}{s_1 - s_3}(\text{ch}\,\theta_1 - \cos\theta_3)$$

$$= \frac{\text{ch}\,\theta_1 - \cos\theta_3}{\text{sn}^2(npt + \text{K}'i)}$$

$$= (\cos\theta_2 - \cos\theta_3)\,\text{sn}^2 pt,$$

$$\cos\theta = \cos\theta_2\,\text{sn}^2 mt + \cos\theta_3\,\text{cn}^2 mt. \tag{197}$$

The values v_1-v_2 and v_1+v_2 of u make $\cos\theta = D$, or E, where, in (163),

$$E = D - \frac{G^2 - C^2r^2}{2AWgh} = D - 2\frac{\delta_1^2-\delta_2^2}{k^2},$$

and then $$E+2\frac{\delta_1^2}{k^2} = D+2\frac{\delta_2^2}{k^2} = \operatorname{ch}\theta_1+\cos\theta_2+\cos\theta_3, \tag{198}$$

and E is the quantity which Dr. Routh denotes by r, while his

$$\frac{e^2}{f^2} = 2\frac{\delta_1^2}{k^2} = \frac{G^2}{2AWgh}.$$

Also

$$\operatorname{ch}\theta_1 - E = 2\frac{\delta_1^2}{k^2} - \cos\theta_2 - \cos\theta_3$$

$$= 2\frac{HY_1^2 - OH + OA^2}{k^2}$$

$$= 2\frac{OA^2 - OY_1^2}{k^2}$$

$$= 2\frac{a^2}{k^2}\kappa'^2\operatorname{sn}^2(p+r)K', \tag{199}$$

$$E - \cos\theta_2 = \operatorname{ch}\theta_1 + \cos\theta_3 - 2\frac{\delta_1^2}{k^2}$$

$$= 2\frac{OH^2 - OB^2 - HY_1^2}{k^2}$$

$$= 2\frac{OY_1^2 - OB^2}{k^2}$$

$$= 2\frac{a^2}{k^2}\kappa'^2\operatorname{cn}^2(p+r)K', \tag{200}$$

$$E - \cos\theta_3 = 2\frac{OH^2 - HY_1^2}{k^2}$$

$$= 2\frac{OY_1^2}{k^2}$$

$$= 2\frac{a^2}{k^2}\operatorname{dn}^2(p+r)K', \tag{201}$$

and thus E or r lies between $\operatorname{ch}\theta_1$ and $\cos\theta_2$.

Similarly, D is the same function of the point P_2, or of the elliptic functions of $(p-r)K'$; and D also lies between $\operatorname{ch}\theta_1$ and $\cos\theta_2$; D being L/f^2, in Dr. Routh's notation (*Q. J. M.*, XXIII, p. 42).

Putting $$\frac{G}{\sqrt{(AWgh)}} = 2\frac{L}{M}, \quad \frac{Cr}{\sqrt{(AWgh)}} = 2\frac{B}{M}, \tag{202}$$

to agree with Darboux's notation, we find as before (*Proc. Lond. Math. Soc*, XXVI, p. 220)

$$L^2 = -\wp v_1 - \wp v_2 - \wp(v_1+v_2), \tag{203}$$

$$B^2 = -\wp v_1 - \wp v_2 - \wp(v_1 - v_2), \tag{204}$$

$$L^2 - B^2 = \wp(v_1 - v_2) - \wp(v_1+v_2), \tag{205}$$

$$\text{ch}\,\theta_1 + \cos\theta_2 + \cos\theta_3 = E + 2\frac{L^2}{M^2} = D + 2\frac{B^2}{M^2}, \tag{206}$$

and, with $$v = v_1 + v_2, \quad w = v_1 - v_2, \tag{207}$$

$$3\wp v = M^2 E - L^2, \tag{208}$$

$$M^2 \cos\theta = L^2 + \wp v + 2\wp u, \tag{209}$$

$$\left.\begin{aligned} M^2\,\text{ch}\,\theta_1 &= L^2 + \wp v + 2e_1,\\ M^2 \cos\theta_2 &= L^2 + \wp v + 2e_2,\\ M^2\cos\theta_3 &= L^2 + \wp v + 2e_3.\end{aligned}\right\} \tag{210}$$

Putting $\cos\theta = E$ in (163) gives

$$i\wp' v = -\sqrt{(-\Sigma)} = \tfrac{1}{2}M^3\frac{Cr - GE}{\sqrt{(AWgh)}}$$

$$= M^2 B - (L^2 + 3\wp v)L, \tag{211}$$

so that $$M^2 B = L^3 + 3L\wp v + i\wp' v. \tag{212}$$

Equating coefficients of $\cos\theta$ in Θ, in (163),

$$\frac{GCr}{AWgh} = 1 + \cos\theta_2\cos\theta_3 + \cos\theta_3\,\text{ch}\,\theta_1 + \text{ch}\,\theta_1\cos\theta_2,$$

or $$4\frac{BL}{M^2} = 1 + \frac{3L^4 + 6L^2\wp v + 3\wp^2 v - g_2}{M^4}; \tag{213}$$

and, therefore, eliminating B, by means of (212),

$$4L(L^3 + 3L\wp v + i\wp' v)$$

$$= M^4 + 3L^4 + 6L^2\wp v + 3\wp^2 v - g_2,$$

or
$$M^4 = L^4+6L^2\wp v+4Li\wp' v-3\wp' v-g_2$$
$$= (L^2+3\wp v)^2+4Li\wp' v-2\wp'' v, \qquad (214)$$

and now
$$\frac{Cr}{\sqrt{(AWgh)}} = 2\frac{B}{M} = 2\frac{L^3+3L\wp v+i\wp' v}{M^3}$$
$$= 2\frac{M^4+3(L^2+\wp v)^2-g_2}{LM^3}. \qquad (215)$$

We now find that

$$\left.\begin{aligned} M^2 \operatorname{sh}\theta_1 &= 2\{\sqrt{(\wp v-e_2,\ \wp v-e_3)}+L\sqrt{(e_1-\wp v)}\},\\ M^2 \sin\theta_2 &= 2\{\sqrt{(e_1-\wp v,\ \wp v-e_3)}-L\sqrt{(\wp v-e_2)}\},\\ M^2 \sin\theta_3 &= 2\{\sqrt{(e_1-\wp v,\ \wp v-e_2)}-L\sqrt{(\wp v-e_3)}\}.\end{aligned}\right\} \qquad (216)$$

Equations (56) and (117) can now be written

$$\varpi_1 = \frac{L-\frac{1}{2}P(v)}{M}nt+I(v), \qquad (217)$$

$$\psi = \frac{L-\frac{1}{2}P(v)}{M}nt+I(v)-\chi, \qquad (218)$$

where
$$\sin\chi = \frac{\sqrt{\Theta}}{\sin\theta\sqrt{(E-\cos\theta)}}. \qquad (219)$$

When the integral $I(v)$ in (51) is made pseudo-elliptic by taking $\mu(p+r)$ an integer in (53), and at the same time assigning an appropriate value to $P(v)$, then, putting $z=\cos\theta$, (218) becomes of the form

$$\psi-pt = \frac{1}{\mu}\sin^{-1}\frac{z^{\mu-1}+C'z^{\mu-2}+D'z^{\mu-3}+\dots}{(1-z^2)^{\frac{1}{2}\mu}}\sqrt{(z_\alpha-z.z\sim z_\beta)}$$
$$= \frac{1}{\mu}\cos^{-1}\frac{Pz^{\mu-1}+Qz^{\mu-2}+Rz^{\mu-3}+\dots}{(1-z^2)^{\frac{1}{2}\mu}}\sqrt{(2.z_\gamma\sim z)}, \qquad (220)$$

where
$$\frac{p}{n} = \frac{P}{\mu} = \frac{L-\frac{1}{2}P(v)}{M}, \qquad (221)$$

leading by differentiation to

$$\frac{d\psi}{dz} = \frac{\dfrac{G-Crz}{\sqrt{(AWgh)}}}{(1-z^2)\sqrt{(2Z)}}. \qquad (222)$$

Here
$$Z = z - z_1 \,.\, z - z_2 \,.\, z - z_3,$$
$$z = \cos\theta, \quad z_1 = \operatorname{ch}\theta_1, \quad z_2 = \cos\theta_2, \quad z_3 = \cos\theta_3, \tag{223}$$
and (222) is derivable from the dynamical equations (163) and (159),
$$\frac{dz}{dt} = n\sqrt{(2Z)}, \tag{224}$$
$$\frac{d\psi}{dt} = \frac{G - Crz}{A\,(1-z^2)}. \tag{225}$$

Knowing the leading coefficient P, the identities derived from these differentiations enable us to determine the remaining unknown coefficients; thus
$$C' = \tfrac{1}{2}\,(P^2 - z_\alpha - z_\beta), \quad \&c. \tag{226}$$

The curve described by a point on the axis of the top becomes a purely algebraical curve when the secular term pt, associated with the azimuth ψ, is cancelled by making $p = 0$; and, therefore, by putting, as in (57),
$$L = \tfrac{1}{2}P\,(v), \tag{57}$$

and this makes (216), from (140), (141), (142),
$$\left.\begin{aligned} M^2 \operatorname{sh}\theta_1 &= \sqrt{(s_1 - \sigma)}\; P\,(v - \omega_1), \\ M^2 \sin\theta_2 &= \sqrt{(\sigma - s_2)}\; P(\omega_2 - v), \\ M^2 \sin\theta_3 &= \sqrt{(\sigma - s_3)}\; P(\omega_3 - v). \end{aligned}\right\} \tag{227}$$

These algebraical gyrostat curves are the ones that are worked out in the sequel, and represented stereoscopically.

14. Curves described by a point on the axis of the Top will have cusps when P_2 is placed on the focal ellipse at B; and then
$$z_2 = \cos\theta_2 = \frac{G}{Cr} = \frac{L}{B}, \tag{228}$$
or, from (210), (214), (215),
$$\frac{L^2 + \wp v + 2e_2}{M_2} = \frac{LM^2}{L^3 + 3L\wp v + i\wp v},$$
$$(L^2 + \wp v + 2e_2)(L^3 + 3L\wp v + i\wp' v)$$
$$= L\,(L^2 + 3\wp v)^2 + 4L^2 i\wp' v + 2L\wp'' v, \tag{229}$$

an equation breaking up into the factors

$$\left(L+\tfrac{1}{2}\frac{i\wp' v}{\wp v-e_2}\right)\left\{\left(L+\tfrac{1}{2}\frac{i\wp' v}{\wp v-e_2}\right)^2-\frac{e_1-e_2\,.\,e_3-e_2}{\wp v-e_2}\right\}=0; \quad (230)$$

the second factor being immediately obtainable from another form of the cusp condition

$$z_2=\frac{1+z_1 z_3}{z_1+z_3}. \quad (231)$$

Thence, from (139),

$$L-\tfrac{1}{2}P(v)-\tfrac{1}{2}P(\omega_2-v)=0, \quad \text{or} \quad \sqrt{\{e_2-\wp(\omega_2-v)\}}. \quad (232)$$

When the secular term in ψ is cancelled by putting

$$L=\tfrac{1}{2}P(v),$$

the first relation

$$P(\omega_2-v)=0 \quad (233)$$

would imply a negative discriminant, which is excluded in this dynamical problem; but the second relation gives

$$\tfrac{1}{2}P(v-\omega_2)=\sqrt{\{e_2-\wp(v-\omega_2)\}}. \quad (234)$$

15. If H denotes for a moment the resultant angular momentum of the top,

$$H^2=A^2\left(\frac{d\theta^2}{dt^2}+\sin^2\theta\,\frac{d\theta^2}{dt^2}\right)+C^2r^2$$

$$=2AWgh\,(D-\cos\theta)+C^2r^2$$

$$=2AWgh\,(E-\cos\theta). \quad (235)$$

If σ denotes the value of s corresponding to v_1+v_2 of u, we may put

$$\sigma-s=\tfrac{1}{2}M^2(E-\cos\theta), \quad (236)$$

and also, from (52),

$$\sigma-s=M^2\frac{\rho_1^2}{k^2}, \quad (237)$$

so that

$$\tfrac{1}{2}(E-\cos\theta)=\frac{\rho_1^2}{k^2}, \quad (238)$$

and

$$\frac{H}{G}=\frac{\rho_1}{\delta_1}, \quad (239)$$

so that the vector OH represents to scale the resultant angular momentum H.

If OG is held fixed in a vertical position while H describes its herpolhode in the fixed horizontal invariable plane of G, OC will imitate the associated motion of the axis of a top, and OH will represent the resultant angular momentum.

If the momental spheroid of the top at the fixed point O is a sphere, then OH will also represent the resultant angular velocity; but, in the general case, the resultant angular velocity will be represented by the vector OI to a point I fixed in the generator HP_2.

We have now connected together the motion of a top with the two associated states of motion *à la Poinsot*, in accordance with the statement of Jacobi (*Werke* II, p. 480).

According to the investigations of M. le Colonel Mannheim, in his *Géométrie cinématique*, p. 203, the osculating plane of the polhode of H, on the rolling quadric in Fig. 2, has its pole P with respect to this quadric on GH produced, such that

$$\frac{HP}{GP} = \frac{OE^2}{OG^2}\sin^2 EOF$$

$$= \frac{D^2}{ABC}\,\frac{\delta^2}{OF^2}; \tag{240}$$

and, from equations (48) and (50),

$$OF^2 = \frac{\dfrac{D^2}{ABC}\delta^2\rho^2}{\rho^2 + \dfrac{A-D.B-D.C-D}{ABC}\delta^2}; \tag{241}$$

so that we find

$$GP = -\frac{ABC}{A-D.B-D.C-D}\,\frac{\rho^2}{\delta^2}. \tag{241*}$$

The centre of curvature R of the herpolhode of H is obtained by M. Mannheim by drawing the axis of curvature of the polhode to meet the plane OHK in Q, and drawing QR perpendicular to the plane GHK.

If the osculating plane of the polhode is perpendicular to the plane OHK, the points Q and R are at an infinite distance, and the herpolhode has a point of inflexion; in this way M. Mannheim analyses the theorems of de Sparre and Hess, concerning the undulations of the herpolhode.

At a point of inflexion of the herpolhode, the tangent HK and the plane OGC are stationary, and C coincides with K; this will be the case when the axis OC of the top describes looped curves.

16. As the first application, take the case of Halphen's algebraical herpolhode (*F. E.*, II, p. 279), corresponding to the parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3.$$

Then P_1 is Fagnano's point on the focal ellipse (Fig. 1): and we take

$$s_1 = (1+c)^2, \quad s_2 = c^2, \quad s_3 = 0,$$

$$\sigma = c + c^2, \quad \sqrt{-\Sigma} = 2(c+c^2), \quad P(v) = 1;$$

so to build up the pseudo-elliptic integral

$$I(\omega_1 + \tfrac{1}{2}\omega_3) = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\{(1+c)^2 - s \cdot c^2 - s\}}}{c + c^2 - s}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{\sqrt{s}}{c+c^2-s}. \tag{242}$$

Then we find, from (139),

$$\left.\begin{aligned} P(v-\omega_1) &= 1+2c, \\ P(v-\omega_2) &= -1-2c, \\ P(v-\omega_3) &= -1; \end{aligned}\right\} \tag{243}$$

so that the rolling quadric is given by

$$\left.\begin{aligned} \frac{D_1}{A_1} &= \frac{a_1}{h_1} = \frac{HV_1}{HY_1} = 1+2c, \\ \frac{D_1}{B_1} &= \frac{b_1}{h_1} = \frac{HT_1}{HY_1} = -1-2c, \\ \frac{D_1}{C_1} &= \frac{c_1}{h_1} = \frac{HP_1}{HY_1} = -1. \end{aligned}\right\} \tag{244}$$

Then H is the mid-point of P_1Y_1 or T_1V_1, as drawn in Fig. 1; and the rolling quadric is a hyperboloid of two sheets, whose equation may be written

$$\frac{x^2}{a^2} + \frac{y^2}{-a^2} + \frac{z^2}{-b^2} = 1,$$

where $$a^2 = (1+2c)\,\delta_1^2, \quad b^2 = \delta_1^2, \tag{245}$$

which rolls on the fixed invariable plane of G at a distance b from its centre, and traces out Halphen's algebraical herpolhode

$$(\xi^2 + b^2)(\eta^2 + b^2) = a^4. \tag{246}$$

This is obtainable by writing ϖ_1 for the pseudo-elliptic integral

$$I(\omega_1+\tfrac{1}{2}\omega_3),$$

and $$\rho^2/k^2 \text{ for } c+c^2-s;$$

afterwards changing to Cartesian coordinates ξ and η.

Also, at P_1, $$\frac{x_1^2}{a^2}=\mathrm{sn}^2\tfrac{1}{2}K'=\frac{1+c}{1+2c},$$

$$\frac{y_1^2}{\beta^2}=\mathrm{cn}^2\tfrac{1}{2}K'=\frac{c}{1+2c},$$

and $$\frac{\beta}{a}=\kappa=\frac{c}{1+c}. \qquad (247)$$

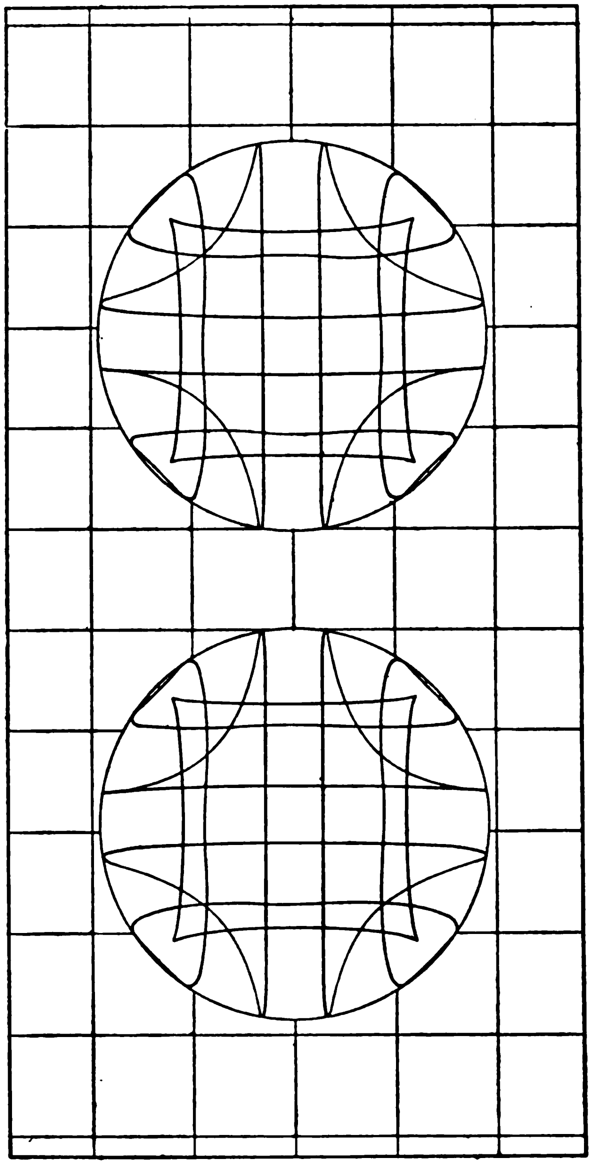
At H, $$\left.\begin{aligned}\frac{x^2}{a^2}&=\tfrac{1}{4}\frac{a^2}{x_1^2}=\frac{1+2c}{4+4c},\\ \frac{y^2}{\beta^2}&=\tfrac{1}{4}\frac{\beta^2}{y_1^2}=\frac{1+2c}{4c}.\end{aligned}\right\} \qquad (248)$$

At P_2, $$\left.\begin{aligned}\frac{x_2^2}{a^2}&=\frac{(1+c)(1-2c)^2}{(1+2c)^3},\\ \frac{y_2^2}{\beta^2}&=\frac{c(3+2c)^2}{(1+2c)^3}.\end{aligned}\right\} \qquad (249)$$

In the associated movement of the top, putting $z=\cos\theta$,

$$\begin{aligned}\psi&=\int\frac{\dfrac{G-Crz}{\sqrt{(AWgh)}}}{1-z^2}\frac{dz}{\sqrt{(2Z)}}\\ &=\tfrac{1}{2}\cos^{-1}\frac{Q}{1-z^2}\sqrt{(2.z-z_3)}\\ &=\tfrac{1}{2}\sin^{-1}\frac{z+C'}{1-z^2}\sqrt{(z_1-z\,.\,z_2-z)},\end{aligned} \qquad (250)$$

where $$\left.\begin{aligned}z_1&=\frac{5+2c}{\sqrt{(9+4c+4c^2)}},\\ z_2&=\frac{-3+2c}{\sqrt{(9+4c+4c^2)}},\\ z_3&=-\frac{3+4c+4c^2}{(1+2c)\sqrt{(9+4c+4c^2)}},\end{aligned}\right\} \qquad (251)$$



$$\left.\begin{aligned}\frac{G}{\surd(AWgh)} &= \frac{2}{(1+2c)^{\frac{1}{2}}\,(9+4c+4c^2)^{\frac{1}{2}}},\\ \frac{Cr}{\surd(AWgh)} &= \tfrac{2}{3}Q,\\ Q &= -4\,\frac{(1+2c)^{\frac{1}{2}}}{(9+4c+4c^2)^{\frac{1}{2}}},\\ C' &= \frac{1+2c}{\surd(9+4c+4c^2)}.\end{aligned}\right\} \qquad (252)$$

The spherical curve described by a point on the axis of the top has four cusps when

$$c = \tfrac{1}{2}.$$

The annexed stereoscopic diagram shows the nature of these curves in general, for different values of c, namely,

$$c = \tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{15}{2};$$

the values of c less than $\frac{1}{2}$ give undulating, not looped, curves, lying inside the four-cusped curve; the curves were calculated and drawn by Mr. T. I. Dewar.

In these figures the describing point has been taken on the axis, on the side of O remote from the centre of gravity.

It is easily proved that, if P_1 is Fagnano's point, its coordinates are

$$\sqrt{\left(\frac{\alpha^3}{\alpha+\beta}\right)},\quad \sqrt{\left(\frac{\beta^3}{\alpha+\beta}\right)};$$

$$V_1P_1 = Y_1T_1 = \alpha,\quad T_1P_1 = Y_1V_1 = \beta,\quad P_1Y_1 = \alpha-\beta;$$

$$OY_1 = \surd(\alpha\beta),\quad OT_1 = \surd(\alpha\,.\,\alpha+\beta),\quad OV_1 = \surd(\beta\,.\,\alpha+\beta),\ \&c., \qquad (253)$$

whence the point P_1 can be determined, and the quarter period of the elliptic functions can be bisected geometrically.

17. With algebraical cases of the motion for the parameter

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \qquad (254)$$

we take $\qquad s_1 = (1-c)^2,\quad s_2 = c^2,\quad s_3 = (c-c^2)^2,$

with

$$\left.\begin{aligned}\sigma &= 2c\,(1-c)^2,\\ \surd(-\Sigma) &= 2\,(1-c)^2\,(1-2c)(2c-c^2),\\ P\,(v) &= \tfrac{2}{3}\,(2-c)(1-2c).\end{aligned}\right\} \qquad (255)$$

Thence we find, by (139),

$$\left.\begin{aligned} P(v-\omega_1) &= \tfrac{2}{3}(2-c)(1+c), \\ P(v-\omega_2) &= -\tfrac{2}{3}(1-c+c^2), \\ P(v-\omega_3) &= -\tfrac{2}{3}(1-2c)(1+c), \end{aligned}\right\} \tag{256}$$

so that the rolling quadric is given by

$$\left.\begin{aligned} \frac{D}{A} &= \frac{a}{h} = \frac{1+c}{1-2c}, \\ \frac{D}{B} &= \frac{b}{h} = -\frac{1-c+c^2}{(2-c)(1-2c)}, \\ \frac{D}{C} &= \frac{c}{h} = -\frac{1+c}{2-c}. \end{aligned}\right\} \tag{257}$$

The corresponding herpolhode is obtained by writing

$$\frac{\rho^2}{k^2} \text{ for } 2c(1-c)^2-s,$$

and ϖ for the pseudo-elliptic integral

$$\begin{aligned} I(\omega_1+\tfrac{1}{3}\omega_3) &= \tfrac{1}{3}\sin^{-1}\frac{\{s-(1-c)^2(2-3c+2c^2)\}\sqrt{(c^2-s)}}{\{2c(1-c)^2-s\}^{\frac{3}{2}}} \\ &= \tfrac{1}{3}\cos^{-1}\frac{(2-c)(1-2c)\sqrt{(1-c)^2-s\,.\,s-(c-c^2)^2\}}}{\{2c(1-c)^2-s\}^{\frac{3}{2}}}. \end{aligned} \tag{258}$$

Also, at P_1,

$$\left.\begin{aligned} \frac{x_1^2}{a^2} &= \text{sn}^2\,\tfrac{2}{3}K' = 1-c^2, \\ \frac{y_1^2}{\beta^2} &= \text{cn}^2\,\tfrac{2}{3}K' = c^2. \end{aligned}\right\} \tag{259}$$

With the parameter $v = \omega_1+\tfrac{2}{3}\omega_3$, (260)

we take

$$\left.\begin{aligned} \sigma &= 2c^2(1-c), \\ \sqrt{(-\Sigma)} &= 2c^3(1+c)(1-c)(1-2c), \\ P(v) &= \tfrac{2}{3}(1+c)(1-2c), \end{aligned}\right\} \tag{261}$$

and now, at P_1,

$$\frac{x_1}{a} = \text{sn}\,\tfrac{1}{3}K' = 1-c; \tag{262}$$

and thus

$$\text{sn}\,\tfrac{1}{3}K' + \text{cn}\,\tfrac{2}{3}K' = 1. \tag{263}$$

[redacted]

This leads immediately to the poristic relation for triangles, inscribed and circumscribed to two circles of radii R and r, their centres being a distance a apart,

$$\frac{r}{R-a}+\frac{r}{R+a}=1. \tag{264}$$

It is curious that the corresponding poristic relation for triangles circumscribed to the ellipse (the focal ellipse) and inscribed in a confocal ellipse can be written in a similar form,

$$\frac{\alpha}{\sqrt{(\alpha^2+\lambda)}}+\frac{\beta}{\sqrt{(\beta^2+\lambda)}}=1, \tag{265}$$

while the poristic relation for quadrilaterals inscribed and circumscribed to two circles,

$$\left(\frac{r}{R-a}\right)^2+\left(\frac{r}{R+a}\right)^2=1, \tag{266}$$

assumes an analogous form for confocals,

$$\frac{\alpha^2}{\alpha^2+\lambda}+\frac{\beta^2}{\beta^2+\lambda}=1. \tag{267}$$

The second of these pseudo-elliptic integrals, $I(\omega_1+\frac{2}{3}\omega_3)$, can be derived from the first, $I(\omega_1+\frac{1}{3}\omega_3)$, by writing $1-c$ for c, so that the cases are not essentially distinct.

Putting $z=\cos\theta$, the corresponding motion of the axis of the top can be written

$$\psi=\int\frac{\dfrac{G-Crz}{\sqrt{(AWgh)}}}{(1-z^2)\sqrt{\{2.z_1-z.z_2-z.z-z_3\}}}\,dz$$

$$=\tfrac{1}{3}\cos^{-1}\frac{(Qz+R)\sqrt{(2.z_2-z)}}{(1-z^2)^{\frac{3}{2}}}$$

$$=\tfrac{1}{3}\sin^{-1}\frac{(z^2+C'z+D')\sqrt{\{z_1-z.z-z_3\}}}{(1-z^2)^{\frac{3}{2}}}, \tag{268}$$

where

$$\left.\begin{aligned}(3M)^2z_1&=(1+c)(13-33c+21c^2-5c^3),\\(3M)^2z_2&=-(5-16c+12c^2-16c^3+5c^4),\\(3M)^2z_3&=-(1+c)(5-21c+33c^2-13c^3),\\(3M)^4&=(1+c)^2\{27(1-c)^6-2(1-4c+c^2)^3\},\end{aligned}\right\} \tag{269}$$

$$(3M)^2 C' = 2(1+c)^2(2-c)(1-2c),$$

$$(3M)^4 D' = (1+c)^2(19-84c+141c^2-160c^3+141c^4-84c^5+19c^6),$$

$$\frac{G}{\sqrt{(AWgh)}} = \frac{2(2-c)(1-2c)}{3M},$$

$$\frac{Cr}{\sqrt{(AWgh)}} = Q = -\frac{2(1+c)^2(2-c)(1-2c)(5-8c+5c^2)}{(3M)^3},$$

$$R = \frac{2(1+c)^2(2-c)(1-2c)(7-12c-3c^2+32c^3-3c^4-12c^5+7c^6)}{(3M)^5}. \tag{270}$$

As c ranges from 0 to $\frac{1}{2}$, the curves have six undulations or loops, with cusps when

$$c = 2-\sqrt{3}.$$

But, as c ranges from $\frac{1}{2}$ to 1, z_1 and z_2 cross and change places, and the curves have three loops or undulations; with cusps when

$$c = 1-\sqrt[3]{4}+\sqrt[3]{2}.$$

The curves are shown in the annexed stereoscopic views, drawn by Mr. T. I. Dewar, for the values

$$c = 0{\cdot}268,\ 0{\cdot}37,\ 0{\cdot}42,\ 0{\cdot}47; \quad \text{and } 0{\cdot}52,\ 0{\cdot}5763,\ 0{\cdot}6725.$$

18. According to the results worked out in the *Proc. L. M. S.*, Vol. xxv, p. 288, when

$$v = \omega_1 + \tfrac{1}{4}\omega_3,$$

we may put

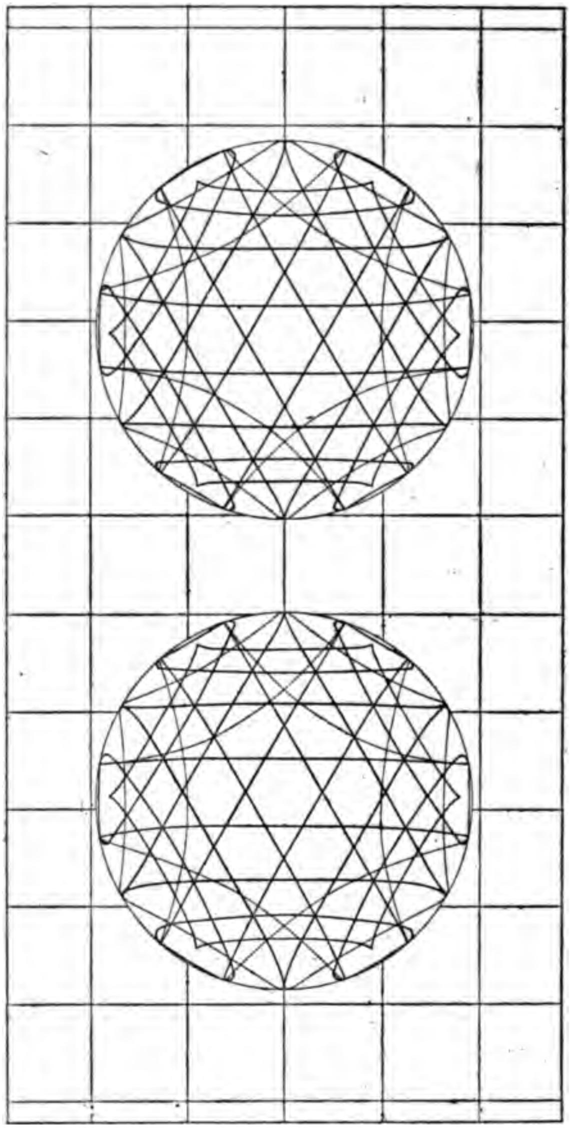
$$\left.\begin{aligned} s_1 &= \tfrac{1}{4}(1-2c)^2(1-2c+2c)^2, \\ s_2 &= c^2(1-c)^2(1-2c+2c^2)^2, \\ s_3 &= c^2(1-c)^2(1-2c)^2, \end{aligned}\right\} \tag{271}$$

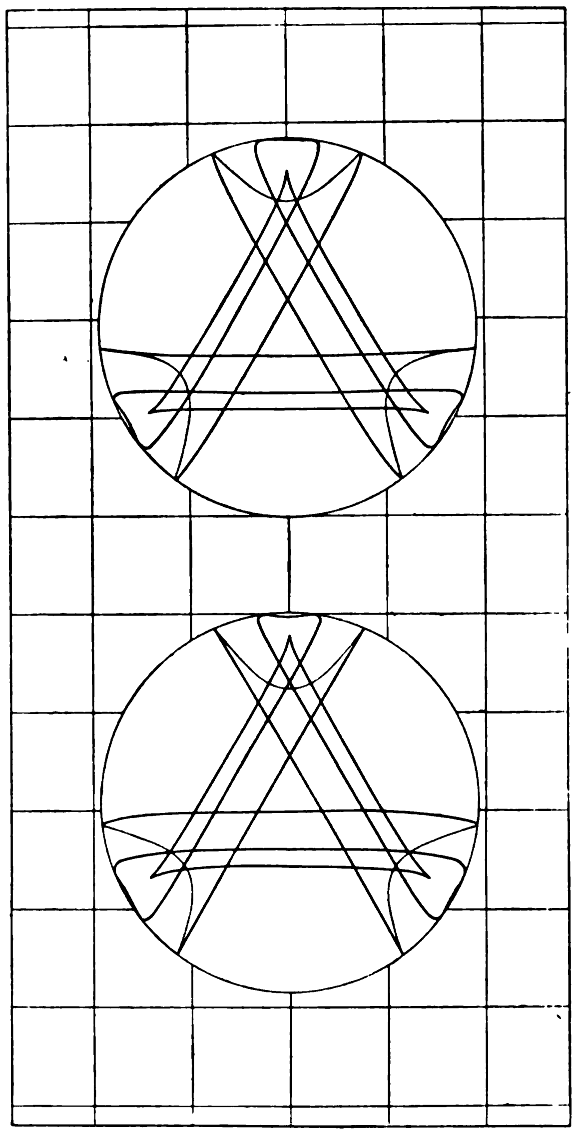
and

$$\left.\begin{aligned} \sigma &= c(1-c)^2(1-2c)^2(1-2c+2c^2), \\ \sqrt{(-\Sigma)} &= c(1-c)^3(1-2c)^3(1-2c+2c^2)(1-4c+2c^2), \\ P(v) &= \tfrac{1}{4}(3-8c+6c^2)(1-4c+2c^2), \end{aligned}\right\} \tag{272}$$

so that, for an algebraical case

$$\frac{HY_1}{a} = \frac{(3-8c+6c^2)(1-4c+2c^2)}{4(1-2c)^2}. \tag{273}$$





Since $$p+r = \tfrac{1}{2},$$

$$\left.\begin{aligned} \frac{x_1^2}{a^2} &= \mathrm{sn}^2\,\tfrac{3}{4}\mathrm{K}' = \frac{1-2c}{(1-2c+2c^2)(1-2c^2)}, \\ \frac{y_1^2}{\beta^2} &= \mathrm{cn}^2\,\tfrac{3}{4}\mathrm{K}' = \frac{4c^3\,(1-c)}{(1-2c+2c^2)(1-2c^2)}, \end{aligned}\right\} \qquad (274)$$

and $$\mathrm{dn}^2\,\tfrac{3}{4}\mathrm{K}' = \frac{4c^3\,(1-c)}{(1-2c)^3}.$$

So also, with

$$v = \omega_1 + \tfrac{3}{4}\omega_3,$$

$$\sigma = c^2\,(1-c)(1-2c)(1-2c+2c^2),$$

$$\sqrt{(-\Sigma)} = c^3\,(1-c)(1-2c)(1-2c+2c^2)(1-4c+2c^2),$$

$$P\,(v) = \tfrac{1}{4}\,(1+2c^2)(1-4c+2c^2),$$

$$\frac{HY_1}{a} = \frac{(1+2c^2)(1-4c+2c^2)}{4\,(1-2c)^2}, \qquad (275)$$

and $$\frac{x_1^2}{a^2} = \mathrm{sn}^2\,\tfrac{1}{4}\mathrm{K}' = \frac{(1-2c)^3}{(1-2c+2c^2)(1-2c^2)}, \qquad (276)$$

$$\frac{y_1^2}{\beta^2} = \mathrm{cn}^2\,\tfrac{1}{4}\mathrm{K}' = \frac{4c\,(1-c)^3}{(1-2c+2c^2)(1-2c^2)}, \qquad (277)$$

$$\mathrm{dn}^2\,\tfrac{1}{4}\mathrm{K}' = \frac{4c\,(1-c)^3}{1-2c}. \qquad (278)$$

But this second case is derivable from the first by changing

$$c \quad \text{into} \quad \frac{1-c}{1-2c};$$

so that, confining our attention to the first case, the pseudo-elliptic integral is

$$I\,(v) = \tfrac{1}{2}\int \frac{P\,(v)(\sigma-s)-\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{s-\gamma}{(\sigma-s)^2}\sqrt{(s_1-s\,.\,s_2-s)}$$

$$= \tfrac{1}{4}\sin^{-1}\frac{2\rho s-\phi}{(\sigma-s)^2}\sqrt{(s-s_3)}, \qquad (279)$$

where $$\gamma = (1-c)^2\,(1-2c)^2\,(1-6c+14c^2-12c^3+4c^4),$$

$$\phi = \tfrac{1}{2}\,(1-c)^2\,(1-2c)^2\,(1-2c+2c^2)^2\,(1-4c+2c^2). \qquad (280)$$

The corresponding algebraical herpolhode is obtained by writing

$$\varpi \quad \text{for} \quad I(v),$$

and

$$\frac{\rho^2}{k^2} \quad \text{for} \quad \sigma - s,$$

and the rolling quadric which will produce this herpolhode is given by

$$\left.\begin{aligned}
\frac{D_1}{A_1} = \frac{a_1}{h_1} &= \frac{HV_1}{HY_1} = \frac{P(v-\omega_1)}{P(v)} = \frac{(1+0-2c^2)(3-4c+2c^2)}{(3-8c+6c^2)(1-4c+2c^2)}, \\
\frac{D_1}{B_1} = \frac{b_1}{h_1} &= -\frac{HT_1}{HY_1} = \frac{P(v-\omega_2)}{P(v)} = -\frac{(1+0-2c^2)(1-4c+6c^2)}{(3-8c+6c^2)(1-4c+2c^2)}, \\
\frac{D_1}{C_1} = \frac{c_1}{h_1} &= -\frac{HP_1}{HY_1} = \frac{P(v-\omega_3)}{P(v)} = -\frac{1+0+2c^2}{3-8c+6c^2}.
\end{aligned}\right\} \tag{281}$$

In the associated motion of the top, the position of the axis is given by the equation

$$\begin{aligned}
\psi &= \int \frac{\dfrac{G-Crz}{\sqrt{(AWgh)}}}{1-z^2}\,\frac{dz}{\sqrt{(2Z)}} \\
&= \tfrac{1}{4}\cos^{-1}\frac{Qz^2+Rz+S}{(1-z^2)^2}\sqrt{(2.z-z_3)} \\
&= \tfrac{1}{4}\sin^{-1}\frac{z^3+C'z^2+D'z+E'}{(1-z^2)^2}\sqrt{(z_1-z\,.\,z_2-z)}.
\end{aligned} \tag{282}$$

The secular term is cancelled by putting

$$L = \tfrac{1}{2}P(v) = \tfrac{1}{8}(3-8c+6c^2)(1-4c+2c^2),$$

and now

$$\left.\begin{aligned}
(8M)^4 &= (1-2c^2)^2(49-752c+5172c^2-21344c^3+59772c^4-122240c^5 \\
&\qquad +193888c^6-244480c^7+239088c^8-170752c^9 \\
&\qquad +82752c^{10}-24064c^{11}+3136c^{12}), \\
(8M)^2 z_1 &= (1-2c^2)(25-184c+522c^2-768c^3+620c^4-288c^5+56c^6), \\
(8M)^2 z_2 &= (1-2c^2)(-7+72c-310c^2+768c^3-1044c^4+736c^5-200c^6), \\
(8M)^2 z_3 &= -7+72c-296c^2+624c^3-936c^4+1248c^5-1184c^6+576c^7 \\
&\qquad -112c^8,
\end{aligned}\right\} \tag{283}$$

$$\left.\begin{aligned}\frac{G}{\sqrt{(AWgh)}} &= 2\frac{L}{M} = \frac{P(v)}{M} = \frac{(3-8c+6c^2)(1-4c+2c^2)}{4M},\\ \frac{Cr}{\sqrt{(AWgh)}} &= \tfrac{3}{4}Q\\ &= -6\,\frac{(1-4c+2c^2)(1-2c^2)^2(7-48c+146c^2-256c^3+292c^4-192c^5+56c^6)}{(8M)^3},\end{aligned}\right\} \tag{284}$$

and the remaining coefficients, R, S, C', D', E', can be found by differentiation, or squaring and adding in (282); thus

$$\left.\begin{aligned} C' &= \tfrac{1}{2}(z_1+z_2),\\ \text{or}\quad (8M)^2 C' &= (1-2c^2)^2(9-56c+124c^2-112c^3+36c^4),\\ D' &= \tfrac{1}{2}(3C'^2-4-z_1z_2),\\ E' &= C'(C'^2+D'-z_1z_2)-Q^2,\ \&c.\end{aligned}\right\} \tag{285}$$

From (227), we find

$$\left.\begin{aligned} M^2\,\mathrm{sh}\,\theta_1 &= \tfrac{1}{8}(1+0-2c^2)(3-4c+2c^2)\\ &\qquad\sqrt{\{(1-2c)^2(1-2c+2c^2)(1-4c+2c^2)\}},\\ M^2\sin\theta_2 &= \tfrac{1}{4}(1+0-2c^2)(1-4c+6c^2)\\ &\qquad\sqrt{\{c(1-c)^2(1-2c+2c^2)(1-4c+2c^2)\}},\\ M^2\sin\theta_3 &= \tfrac{1}{4}(1+0+2c^2)(1-4c+2c^2)\\ &\qquad\sqrt{\{c(1-c)^2(1-2c)^2\}}.\end{aligned}\right\} \tag{286}$$

It will be noticed that some of the expressions above are reciprocal in $c\sqrt{2}$; thus, putting

$$c\sqrt{2}+\frac{1}{c\sqrt{2}} = b\sqrt{2},$$

$$(8M)^4 = c^6(b^2-2)(49b^6-376b^5+1146b^4-1728b^3+1260b^2-288b-72),$$

$$(8M)^2 z_2 = c^4(-7b^4+36b^3-60b^2+24b+12),$$

$$\left.\begin{aligned}\frac{G}{\sqrt{(AWgh)}} &= \frac{c^2(3b-4)(b-2)}{M},\\ \frac{Cr}{\sqrt{(AWgh)}} &= \frac{3c^6(b-2)(b^2-2)(7b^3-24b^2+26b-8)}{4M^3},\\ &\&c.\end{aligned}\right\} \tag{287}$$

The cusp condition (234), with

$$P(\omega_2 - v) = \tfrac{1}{4}(2-4c+6c^2)(1+0-2c^2),$$

$$\sigma(\omega_2 - v) = c^3(1-c)(1-2c)^2(1-2c+2c^2),$$

$$s_2 - \sigma(\omega_2 - v) = c^3(1-c)(1-2c+2c^2)(1+0-2c^2),$$

becomes

$$(1-4c+6c^2)^2(1+0-2c^2) = 64c^3(1-c)(1-2c+2c^2)(1+0-2c^2),$$

and thus $$1-2c^2 = 0,$$

which makes $$s_1 = s_2,$$

or $$(1-4c+6c^2)^2(1-2c^2) = 64c^3(1-c)(1-2c+2c^2),$$

$$1-8c+26c^2-96c^3+172c^4-160c^5+56c^6 = 0. \qquad (288)$$

Mr. Dewar finds that this equation has two real roots,

$$c = 0{\cdot}1788928 \quad \text{and} \quad 1{\cdot}4061845.$$

The value $$c = 0{\cdot}1788928 \qquad (289)$$

makes
$$\left.\begin{aligned}
z_1 &= 3{\cdot}75316,\\
z_2 &= -0{\cdot}440942 = \cos 116^\circ 10',\\
z_3 &= -0{\cdot}63301 \;\;= \cos 125^\circ 16',\\
C' &= 1{\cdot}65611,\\
D' &= 2{\cdot}94151,\\
E' &= 1{\cdot}06068,\\
Q &= 3{\cdot}33072,\\
R &= 2{\cdot}50412,\\
S &= 1{\cdot}50331;
\end{aligned}\right\} \qquad (290)$$

and now $$Qz^2 + Rz + S = 0$$

has imaginary roots, while

$$z^3 + C'z^2 + D'z + E' = 0$$

has a pair of imaginary roots, and a real root

$$-0{\cdot}44094 = z_2.$$

A point on the axis thus describes a curve having eight cusps, with apsidal angle

$$\tfrac{1}{4}\pi.$$

The value $c = 1{\cdot}4061845$

makes

$$\left.\begin{aligned} z_1 &= 1{\cdot}14762, \\ z_2 &= -0{\cdot}63543 = \cos 129^\circ 27', \\ z_3 &= -0{\cdot}96948 = \cos 165^\circ 49', \\ C' &= 0{\cdot}25609, \\ D' &= -1{\cdot}53701, \\ E' &= -0{\cdot}82345, \\ Q &= -0{\cdot}79585, \\ R &= 0{\cdot}19151, \\ S &= 0{\cdot}87787. \end{aligned}\right\} \quad (291)$$

Now

$$Qz^2 + Rz + S = 0$$

has real roots $-0{\cdot}93681$ and $1{\cdot}17745$; while

$$z^3 + C'z^2 + D'z + E' = 0$$

has roots

$$-0{\cdot}63543 = z_2,$$

$$-0{\cdot}96467, \text{ and } 1{\cdot}34401;$$

and the curve has eight cusps, at apsidal distance $\frac{3}{4}\pi$.

Both cusped curves are shown stereoscopically in the figure annexed.

19. For the case of quinquisection, we take (*Proc. L. M. S.*, Vol. xxv, p. 289)

$$\left.\begin{aligned} s_1 &= 4\,(c^2 + \surd C)^2, \\ s_2 &= (c+1)^2\,(c-1)^4, \\ s_3 &= 4\,(c^2 - \surd C)^2, \end{aligned}\right\} \quad (292)$$

where

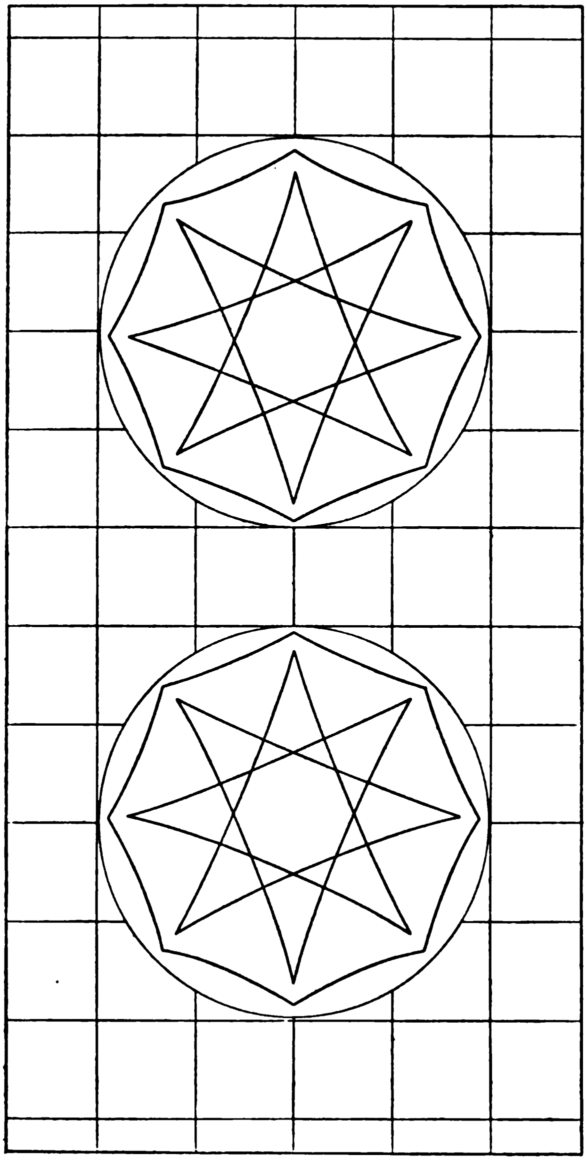
$$C_1 = c^3 + c^2 - c,$$

and, with

$$\left.\begin{aligned} v &= \omega_1 + \tfrac{1}{5}\omega_3, \\ \sigma &= 8c\,(c+1)^2\,(c-1), \\ \surd(-\Sigma) &= 8c\,(c+1)^3\,(c-1)(c^2-4c-1), \\ P\,(v) &= \tfrac{2}{5}\,(c+3)(c^2-4c-1), \end{aligned}\right\} \quad (293)$$

so that

$$\frac{HY_1}{a} = \frac{(c+3)(c^2-4c-1)}{20c\,\sqrt[4]{C_1}}. \quad (294)$$



With
$$\left.\begin{aligned} v &= \omega_1 + \tfrac{2}{5}\omega_3, \\ \sigma &= 4c\,(c+1)(c-1)^3, \\ \surd(-\Sigma) &= 8c^3\,(c+1)(c-1)^3\,(c^2-4c-1), \\ P\,(v) &= \tfrac{2}{5}\,(3c-1)(c^2-4c-1), \end{aligned}\right\} \tag{295}$$

so that
$$\frac{HY_1}{a} = \frac{(3c-1)(c^2-4c-1)}{20c\sqrt[4]{C_1}}, \tag{296}$$

obtained from the preceding by a change of c into $-1/c$.

The coordinates of P_1 on the focal ellipse being denoted by x_1, y_1,

$$\frac{y_1^2}{\beta^2} = \text{cn}^2\,\tfrac{2}{5}\text{K}' = \frac{s_2-s_3}{s_1-\sigma}\,\frac{s_1-\sigma}{\sigma-s_3}, \tag{297}$$

and this reduces to
$$\frac{y_1}{\beta} = \text{cn}\;\tfrac{2}{5}\text{K}' = \frac{\surd C_1-1}{\surd C_1+1}, \tag{298}$$

so that
$$\frac{x_1}{a} = \text{sn}\;\tfrac{2}{5}\text{K}' = \frac{2\sqrt[4]{C_1}}{\surd C_1+1} \tag{299}$$

(*Proc. L. M. S.*, Vol. xxv, p. 302), and then

$$\text{dn}\,\tfrac{2}{5}\text{K}' = \frac{\surd C_1-1}{2c}. \tag{300}$$

With
$$\left.\begin{aligned} v &= \omega_1 + \tfrac{3}{5}\omega_3, \\ \sigma &= 4\,(c+1)(c-1)^2\,(\surd C + c), \\ \surd(-\Sigma) &= 8(c+1)(c-1)^2(c^3-c^2-5c+1+4\surd C_1)\surd C_1, \\ P\,(v) &= \tfrac{4}{5}\,(c^3-c^2-3c+2-5\surd C_1), \end{aligned}\right\} \tag{301}$$

and
$$\frac{HY_1}{a} = \frac{(c-2)(c^2+c-1)-5c\surd C_1}{10c\sqrt[4]{C_1}}. \tag{302}$$

So also for the parameter

$$v = \omega_1 + \tfrac{4}{5}\omega_3. \tag{303}$$

In these last two cases the herpolhode and the spherical curve described by a point on the axis of the top are composed of five symmetrical loops.

But, as before, we can make one standard form serve for all the different cases, by examining the different shapes corresponding to the regions of c, the boundaries of these regions being places where the values of s_1, s_2, or s_3 change places.

Starting then with the pseudo-elliptic integral for

$$\left.\begin{aligned} \sigma &= 8c(c+1)^3(c-1), \\ \sqrt{(-\Sigma)} &= 8c(c+1)^3(c-1)(c^2-4c-1), \\ P(v) &= \tfrac{3}{8}(c+3)(c^2-4c-1), \end{aligned}\right\} \qquad (304)$$

we find

$$I(v) = \tfrac{1}{2}\int \frac{P(v)(\sigma-s)-\sqrt{(-\Sigma)}}{\sigma-s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{8}\cos^{-1}\frac{\frac{8}{3}P(v)\,s-\phi}{(\sigma-s)^{\frac{5}{2}}}\sqrt{(s_1-s\,.\,s-s_3)}$$

$$= \tfrac{1}{8}\sin^{-1}\frac{s^2-\gamma s+\delta}{(\sigma-s)^{\frac{5}{2}}}\sqrt{(s_2-s)}, \qquad (305)$$

where

$$\left.\begin{aligned} \phi &= 4(c+1)^3(c^3+7c^2-c+1)(c^2-4c-1), \\ \gamma &= 4(2c^4+7c^3+17c^2+5c+1), \\ \delta &= 16c(c+1)^3(c^5+3c^4+34c^3-2c^2-3c-1). \end{aligned}\right\} \qquad (306)$$

From (139),

$$\left.\begin{aligned} P(\omega_1-v) &= \tfrac{3}{8}\left\{(2c+1)(c^2+c-1)+5c\sqrt{C_1}\right\}, \\ P(\omega_2-v) &= \tfrac{3}{8}(c^3-c^2+7c-3), \\ P(\omega_3-v) &= \tfrac{3}{8}\left\{(2c+1)(c^2+c-1)-5c\sqrt{C_1}\right\}. \end{aligned}\right\} \qquad (307)$$

Thence algebraical herpolhodes are obtained by writing ϖ for $I(v)$, and $\frac{\rho^2}{k^2}$ for $\sigma-s$; and, in the associated motion of the top,

$$\psi = \int \frac{\dfrac{G-Crz}{\sqrt{(AWgh)}}}{1-z^2}\,\frac{dz}{\sqrt{(2Z)}}$$

$$= \tfrac{1}{8}\cos^{-1}\frac{Qz^3+Rz^2+Sz+T}{(1-z^2)^{\frac{5}{2}}}\sqrt{(2.z_2-z)}$$

$$= \tfrac{1}{8}\sin^{-1}\frac{z^4+C'z^3+D'z^2+E'z+F'}{(1-z^2)^{\frac{5}{2}}}\sqrt{(z_1-z.z-z_3)}, \qquad (308)$$

$$\left.\begin{aligned} (5M)^2 z_1 &= -8(c^2+c-1)(3c^4-9c^3-13c^2-11c-2)+400c^3\sqrt{C}, \\ (5M)^2 z_2 &= (13c^6-26c^5-25c^4+60c^3-125c^2+14c+17), \\ (5M)^2 z_3 &= -8(c^2+c-1)(3c^4-9c^3-13c^2-11c-2)-400c^3\sqrt{C}, \\ (5M)^4 &= 64(c^2+c-1)(9c^{10}-45c^9+40c^8+150c^7-430c^6 \\ &\qquad +884c^5-360c^4+70c^3+25c^2-15c-4), \end{aligned}\right\} \qquad (309)$$

$$\left.\begin{aligned} \frac{G}{\surd(AWgh)} &= 2\frac{L}{M} = \frac{P(v)}{M} = \frac{2(c+3)(c^2-4c-1)}{5M}, \\ \frac{Cr}{\surd(AWgh)} &= \tfrac{3}{5}Q \\ Q &= -\frac{16(c^2-4c-1)(c^2+c-1)(c^5+0+20c^3+0+5c-2)}{25M^3}, \\ C' = \tfrac{1}{2}(z_1+z_3) &= -\frac{8(c^2+c-1)(3c^4-9c^3-13c^2-11c-2)}{25M^2}. \end{aligned}\right\} \quad (310)$$

Writing the conditions for cusps

$$\tfrac{1}{2}P(v-\omega_2) = \surd\{e_2 - \wp(v-\omega_2)\}$$
$$= \surd\{s_2 - s(\tfrac{4}{5}\omega_2)\},$$

or $$\tfrac{1}{5}(c^3-c^2+7c-3) = \pm(c+1)(c-1)^2, \quad (311)$$

we obtain $$(c-2)(c^2+c-1) = 0,$$

and $$3c^3-3c^2+c+1 = 0,$$

or $$(3c-1)^3+10 = 0,$$

so that, rejecting the roots of

$$c^2+c-1 = 0,$$

which give imaginary solutions,

$$c = 2 \quad \text{and} \quad \tfrac{1}{3}\{1-\sqrt[3]{(10)}\} \quad (312)$$

give ten-cusped solutions, with apsidal angles $\frac{1}{5}\pi$ and $\frac{3}{5}\pi$.

The special numerical case of

$$c = 2$$

was worked out independently, to serve as a check upon the algebraical calculations; this makes

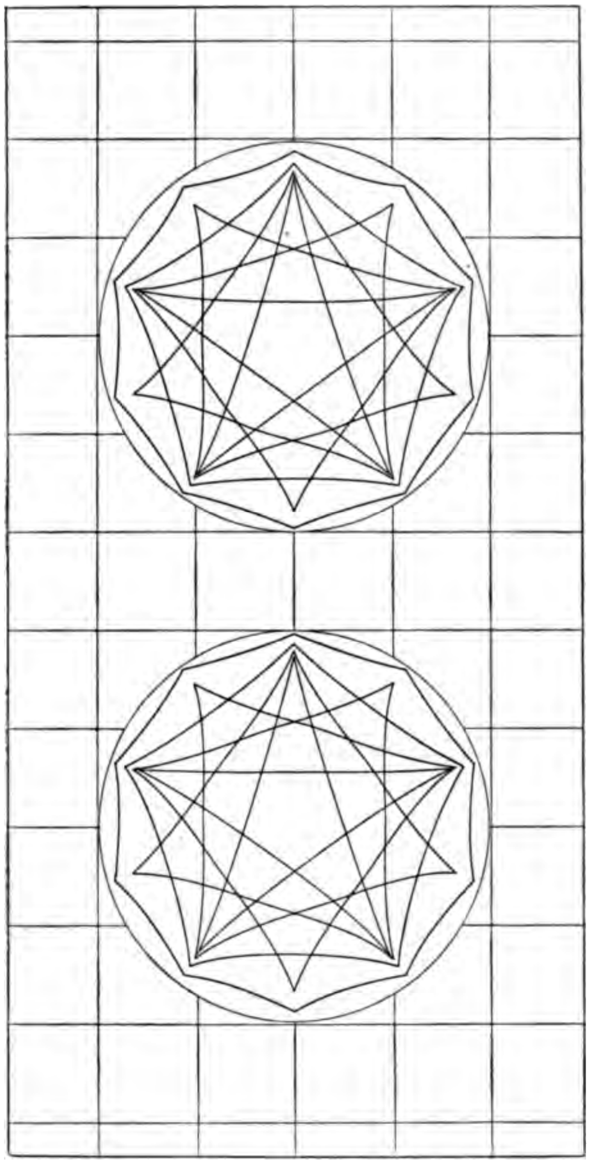
$$z_1 = \frac{8+2\surd(10)}{3}, \quad z_2 = -\frac{\surd(10)}{8}, \quad z_3 = \frac{-8+2\surd(10)}{3},$$

$$\frac{G}{\surd(AWgh)} = -\frac{2^{\frac{1}{2}}.5^{\frac{3}{2}}}{2\surd 3}, \quad \frac{Cr}{\surd(AWgh)} = \frac{2.2^{\frac{1}{2}}.5^{\frac{3}{2}}}{\surd 3};$$

$$C' = \tfrac{2}{3}\surd(10), \quad D' = \frac{11}{2}, \quad E' = \surd(10), \quad F' = \frac{29}{4};$$

$$Q = \frac{2.2^{\frac{2}{3}}.5^{\frac{5}{6}}}{3\surd 3}, \quad R = \frac{2^{\frac{4}{3}}.5^{\frac{7}{6}}}{4\surd 3}, \quad S = \frac{7.2^{\frac{2}{3}}.5^{\frac{5}{6}}}{8\surd 3}, \quad T = \frac{2^{\frac{1}{3}}.5^{\frac{7}{6}}}{12\surd 3}.$$

The spherical curve has ten cusps at apsidal angular interval $\frac{1}{5}\pi$.



The other ten-cusped figure, obtained by putting

$$c = \frac{1-\sqrt[3]{(10)}}{3},$$

is also shown, having an apsidal angle $\frac{3}{5}\pi$, and here Mr. Dewar finds

$$z_1 = 1{\cdot}4205413,$$
$$z_2 = -0{\cdot}6069240 = \cos 127^\circ 22',$$
$$z_3 = -0{\cdot}9184686 = \cos 156^\circ 42'.$$

As c increases from 2, the ten-cusped figure changes into ten-looped figures, until

$$c^2-4c-1=0,$$

or

$$c=\sqrt{5}+2=4{\cdot}236068,$$

and now

$$z_1 = z_2.$$

For greater values of c, the quantities z_1 and z_2 change places, and the curves have loops, until these loops degenerate into cusps; and this will be the case when

$$\tfrac{1}{2}P(v-\omega_1) = \sqrt{\{e_1 - \wp(v-\omega_1)\}}$$
$$= \sqrt{\{s_1 - s(\tfrac{1}{5}\omega_3)\}},$$

or $\frac{2}{5}\{(2c+1)(c^2+c-1)+5c\sqrt{C_1}\}$

$$= \sqrt{\{4(c^2+\sqrt{C_1})^2+4(c^2+\sqrt{C_1})^2(\sqrt{C_1}-1)\}}$$
$$= 2(c^2+\sqrt{C_1})\sqrt[4]{C_1}, \qquad (313)$$

and dividing out $\sqrt[4]{(c^2+c-1)}$,

$$\sqrt[4]{(c^2+c-1)}\{(2c+1)\sqrt{(c^2+c-1)}+5c\sqrt{c}\} = 5(c^2+\sqrt{C_1})\sqrt[4]{c},$$

leading, on rationalization, to the equation

$$16c^{10}-145c^9+195c^8+0+240c^6-1194c^5+990c^4-240c^3+0-5c-1=0; \qquad (314)$$

and Mr. Dewar finds that this equation has two real roots,

$$c = 7{\cdot}40092 \quad \text{and} \quad -0{\cdot}1082623.$$

The first value of c gives five cusps at intervals $\frac{4}{5}\pi$ in azimuth, and makes

$$\left.\begin{aligned} z_1 &= 1{\cdot}0925520, \\ z_2 &= -0{\cdot}6416767 = \cos 129^\circ 55', \\ z_3 &= -0{\cdot}9808773 = \cos 168^\circ 47'. \end{aligned}\right\} \qquad (315)$$

As c decreases from 2, the curves have ten undulating branches, until c passes the value 1, when the curves become imaginary.

After c passes 0, the curves become real again, consisting of five undulations, until we reach the root

$$c = -0{\cdot}1082623,$$

when cusps appear, at an interval $\frac{2}{5}\pi$ in azimuth, and now

$$z_1 = 2{\cdot}22605, \quad z_2 = -0{\cdot}53617 = \cos 122^\circ 25',$$

$$z_3 = -0{\cdot}79412 = \cos 142^\circ 34'.$$

For smaller values of c, the curves consist of five loops, until

$$c = -\sqrt{5}+2 = -0{\cdot}236068,$$

and in passing through this value the five loops change into ten loops; and these become cusps, at an interval $\frac{2}{5}\pi$ in azimuth, when

$$c = \tfrac{1}{3}\{1-\sqrt[3]{(10)}\} = -0{\cdot}3848117,$$

as already described on p. 604.

Afterwards the curves have ten undulations, but these become imaginary when c crosses the value -1.

The four cases where cusps occur are shown in the preceding stereoscopic diagram, drawn by Mr. Dewar.

20. Twelve branched gyrostat curves and the associated herpolhodes can be built up by means of the pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{P(v)(\sigma-s)-\sqrt{(-\Sigma)}}{(\sigma-s)\sqrt{S}}\,ds$$

$$= \tfrac{1}{6}\cos^{-1}\frac{3P(v)\,s^2-\theta s+\phi}{(\sigma-s)^3}\sqrt{(s-s_3)}$$

$$= \tfrac{1}{6}\sin^{-1}\frac{s^2-\gamma s+\delta}{(\sigma-s)^3}\sqrt{(s-s_1\,.\,s-s_2)}, \qquad (316)$$

where $\quad v = \omega_1+\tfrac{1}{6}\omega_3,$

$$P(v) = \tfrac{1}{6}(1-a)(5+3a+3a^2+a^3),$$

$$\sigma = a(1+a)(1+a+a^2)(1+0+a^2),$$

$$s_1 = \tfrac{1}{16}(1+a)^2(1+0+a^2)^2\{(1-a)+(1+a)\sqrt{A}\}^2,$$

$$s_2 = \tfrac{1}{16}(1+a)^2(1+0+a^2)^2\{(1-a)-(1+a)\sqrt{A}\}^2,$$

$$s_3 = a^2(1+a+a^2)^2,$$

$$A = \frac{1+4a+a^2}{1+a^2},$$

$$\gamma = 3+0+5a^2+4a^3+6a^4+4a^5+2a^6,$$

$$\delta = (1+0+a^2)(1+a+a^2)^2(1+0+2a^2+0+2a^4+2a^5+a^6),$$

$$\theta = \tfrac{1}{2}(1-a)(1+0+a^2)(5+11a+25a^2+35a^3+34a^4 \\ +22a^5+10a^6+2a^7),$$

$$\phi = \tfrac{1}{2}(1-a)(1+a)^2(1+0+a^2)^2(1+a+a^2)^3(1+0+2a^2+a^3),$$

$$\surd(-\Sigma) = a(1-a)^4(1+a+a^2),$$

$$\sigma_1 = \sigma - s_1 = -\tfrac{1}{8}(1-a^4)\{1-4a-4a^2-4a^3-a^4 \\ +(1+a)^2(1+0+a^2)\surd A\},$$

$$\sigma_2 = \sigma - s_2 = -\tfrac{1}{8}(1-a^4)\{1-4a-4a^2-4a^3-a^4 \\ -(1+a)^2(1+0+a^2)\surd A\},$$

$$\sigma_3 = \sigma - s_3 = a(1+a+a^2),$$

$$P(v-\omega_1) = \tfrac{1}{12}\{13-16a-12a^2-14a^3-5a^4+3(1+a)^2(1+0+a^2)\surd A\},$$

$$P(v-\omega_2) = \tfrac{1}{12}\{13-16a-12a^2-14a^3-5a^4-3(1+a)^2(1+0+a^2)\surd A\},$$

$$P(v-\omega_3) = \tfrac{1}{3}(1-a)(4+3a+3a^2+2a^3).$$

If the secular term is cancelled in the gyrostat curves, we can put

$$\psi = \int \frac{\dfrac{G-Crz}{\surd(AWgh)}}{(1-z^2)\surd(2.z-z_1.z-z_2.z-z_3)}\,dz$$

$$= \tfrac{1}{6}\cos^{-1}\frac{Qz^4+Rz^3+Sz^2+Tz+V}{(1-z^2)^3}\surd(2.z-z_3)$$

$$= \tfrac{1}{6}\sin^{-1}\frac{z^5+C'z^4+D'z^3+E'z^2+F'z+G'}{(1-z^2)^3}\surd(z_1-z.z_2-z), \qquad (317)$$

and $\quad L = \tfrac{1}{2}P(v) = \tfrac{1}{12}(1-a)(5+3a+3a^2+a^3),$

$$\sigma_1+\sigma_2+\sigma_3 = -\tfrac{1}{4}(1-8a-8a^2-8a^3-2a^4+4a^5+4a^6+4a^7+a^8),$$

$$L^2+\sigma_1+\sigma_2+\sigma_3$$
$$= \tfrac{1}{144}(-11+298+232+298+70-140-140-140-35),$$

$$4L\surd(-\Sigma) = \tfrac{1}{3}a(1-a)(1-a^4)(1+a+a^2)(5+3a+3a^2+a^3),$$

$$2\wp''v = 4(\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2)$$
$$= -a(1+a)(1+0+a^2)(1+a+a^2)(2-6a-3a^2-3a^3+2a^4+2a^5),$$

$$(12M)^4 = (-11+298+232+298+70-140-140-140-35)^2 \\ -6912a(1-a)(1-a)^4(1+a+a^2)(5+3a+3a^2+a^3) \\ +20736a(1+a)(1+0+a^2)(1+a+a^2)(2-6-3-3+2+5),$$

$$\left.\begin{aligned} &\begin{matrix}(12M)^2 z_1 \\ (12M)^2 z_2\end{matrix} = \left\{\begin{aligned} &25-154+88+154-2+4+4+4+1 \\ &\qquad\qquad \pm 36(1+a)^2(1+0+a^2)\surd A,\end{aligned}\right. \\ &(12M)^2 z_3 = -11+10-56+410+70-140-140-140-35.\end{aligned}\right\} \quad (318)$$

21. In the case of the spherical pendulum, it is not possible to cancel the secular term, as previously; so we must return to the general case, and take, with

$$z = -\cos\theta,$$

$$\psi - pt = \int \frac{\dfrac{G+Crz}{\sqrt{(AWgh)}} - \dfrac{p}{n}(1-z^2)}{1-z^2}\,\frac{dz}{\sqrt{(2Z)}},$$

where $$Z = z_3 - z\,.\,z - z_2\,.\,z - z_1. \qquad (319)$$

In the spherical pendulum, the condition

$$Cr = 0$$

leads to $$L^3 + 3L\wp v + i\wp' v = 0,$$

or $$L^3 + L(\sigma_1+\sigma_2+\sigma_3) - \sqrt{(-\Sigma)} = 0, \qquad (320)$$

a cubic equation for L, with one real root; the same equation is derivable from the relation

$$1 + z_2 z_3 + z_3 z_1 + z_1 z_2 = 0, \qquad (321)$$

or $$\text{th}\,\tfrac{1}{2}\theta_1 \tan\tfrac{1}{2}\theta_2 \tan\tfrac{1}{2}\theta_3 = 1,$$

which exists between z_1, z_2, z_3, or θ_1, θ_2, θ_3, for a spherical pendulum.

Thence, from (166) and (168),

$$HS^2\,.\,HS'^2 + OH^4 - OS^4 + 2OH^2(OH^2 - AB^2) = 0, \qquad (322)$$

if H is on the pedal of the ellipse.

Also (206) $$z_1 + z_2 + z_3 = -D. \qquad (323)$$

We also find $$\frac{G}{\sqrt{(AWgh)}} = 4\,\text{sh}\,\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2 \sin\tfrac{1}{2}\theta_3$$

$$= 4\,\text{ch}\,\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 \cos\tfrac{1}{2}\theta_3$$

$$= \sqrt{(2\,\text{sh}\,\theta_1 \sin\theta_2 \sin\theta_3)}. \qquad (324)$$

In the spherical pendulum, OH and OC are always at right angles, and H must be placed at Y_2 in Fig. 1; then OY_2 and OC are always at right angles, and OY_2 may be supposed a rod, pivoted at O and passing through a ring fixed at Y_2 in the generator Y_2P_2.

Then, with $$Cr = 0,$$

$$\wp' v_1 + \wp' v_2 = 0, \qquad (325)$$

but $$\wp' v_1 - \wp' v_2 = 0, \qquad (326)$$

when $G = 0$, and H moves up to Y_1.

If OC meets the invariable plane of G in C', then

$$KG\,.\,GC' = \delta_1^2, \tag{327}$$

since OC is at right angles to the plane KOH in the spherical pendulum; the curve described by C' is thus the polar reciprocal of the herpolhode of H, a special herpolhode in which

$$D_1 = \tfrac{1}{2}(A_1 + B_1 + C_1),$$

or

$$\frac{2}{h_1} = \frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1}. \tag{328}$$

22. In the general case again, by means of equation (124),

$$GK.GC' = p\delta_1 \tan\theta = (\delta_2 \sec\theta - \delta_1)\,\delta_1$$

$$= \delta^2 \frac{\dfrac{\rho^2}{k^2} + \dfrac{A-D.B-D.C-D}{ABC}\,\dfrac{\delta^2}{k^2}}{\tfrac{1}{2}E - \dfrac{\rho^2}{k^2}}, \tag{329}$$

whence the curve of C' can be derived from the herpolhode of H.

The vector OH in Fig. 2 representing to scale the resultant angular momentum of the Top, the velocity of H in the invariable plane of G is proportional to the impressed couple $Wgh \sin\theta$; and, to the linear scale employed above, the velocity of H,

$$\frac{ds}{dt} = \tfrac{1}{2}nk\sin\theta\,; \tag{330}$$

so that, employing equation (121), the radius of curvature of the herpolhode

$$\frac{ds}{d\psi} = \frac{\tfrac{1}{4}k^2 \sin^3\theta}{\delta_1 - \delta_2\cos\theta}, \tag{331}$$

which can be expressed in terms of ρ by means of equation (91).

Also, by means of (115),

$$\frac{ds}{d\theta} = \tfrac{1}{2}k\,\frac{\sin^2\theta}{\sqrt{(2\Theta)}}, \tag{332}$$

which gives the rectification of the herpolhode of H; but this leads to hyperelliptic integrals.

23. With $$v = \omega_1 + \omega_3, \tag{333}$$

in the spherical pendulum, P_1 is at B, and H is at the point where the tangent at B cuts the pedal.

Then we find that $$OH = OS, \tag{334}$$

so that $$\cos\theta_3 = 0,$$

and the bob of the pendulum must be projected horizontally from the level of O.

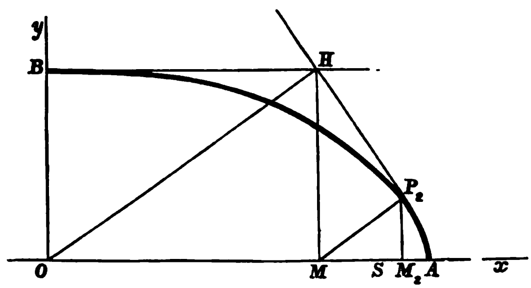


FIG. 4.

If this velocity of projection is $2pl$, where l denotes the length of the thread, the motion is given by

$$\psi - pt = \sin^{-1}\frac{p\sqrt{(-2\cos\theta)}}{n\sin\theta}$$

$$= \cos^{-1}\frac{\sqrt{(n^2+2p^2\cos\theta - n^2\cos^2\theta)}}{n\sin\theta}. \tag{335}$$

Then $$\cos\theta_3 = -\frac{OB}{OS}, \tag{336}$$

and θ_3 is the supplement of BOH; while

$$\text{ch}\,\theta_1 = \frac{OS}{OB}; \tag{337}$$

but the eccentricity of the focal ellipse must be greater than $\frac{1}{2}\sqrt{2}$, to give a real case of this motion.

24. With the parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3,$$

and $\sigma - s_1 = \sigma_1 = -1 - c, \quad \sigma - s_2 = \sigma_2 = c, \quad \sigma - s_3 = \sigma_3 = c + c^2,$

$$\sqrt{(-\Sigma)} = 2(c + c^2),$$

the condition that $\qquad Cr = 0,$

or $$L^3 + L(\sigma_1 + \sigma_2 + \sigma_3) - \sqrt{(-\Sigma)} = 0,$$

becomes $$L^3 - L(1 - c - c^2) - 2(c + c^2) = 0,$$

or
$$\left.\begin{aligned} c + c^2 &= -\frac{L - L^3}{2 - L}, \\ 1 + 2c &= \frac{(1-2L)^{\frac{1}{2}}(2 - L - 2L^2)^{\frac{1}{2}}}{(2-L)^{\frac{1}{2}}}, \end{aligned}\right\} \qquad (338)$$

and, from (214),

$$\left.\begin{aligned} M^4 &= (L^2 - 1 + c + c^2)^2 - 8L(c + c^2) + 8(c + c^2) \\ &= \frac{4(1-L)^3(1+L)(1-2L)}{2-L}, \\ M^2 z_1 &= -L^2 - 1 - 3c - c^2 \\ &= -L^2 + \frac{L - L^3}{2 - L} - \frac{(1-2L)^{\frac{1}{2}}(2 - L - 2L^2)^{\frac{1}{2}}}{(2-L)^{\frac{1}{2}}} \\ &= \left(\frac{1-2L}{2-L}\right)^{\frac{1}{2}} \{L(1-2L)^{\frac{1}{2}} - (2-L)^{\frac{1}{2}}(2 - L - 2L^2)^{\frac{1}{2}}\}, \\ M^2 z_2 &= \left(\frac{1-2L}{2-L}\right)^{\frac{1}{2}} \{L(1-2L)^{\frac{1}{2}} + (2-L)^{\frac{1}{2}}(2 - L - 2L^2)^{\frac{1}{2}}\}, \\ M^2 z_3 &= 2\,\frac{(1-L)^2(1+L)}{2-L}. \end{aligned}\right\} \qquad (339)$$

We now find that the curve described by the bob of the spherical pendulum can be written with

$$z = -\cos\theta,$$

$$Z = z_3 - z \,.\, z - z_2 \,.\, z - z_1,$$

$$\frac{dz}{dt} = n\sqrt{(2Z)},$$

$$\psi - pt = \int \frac{2\frac{L}{M} - \frac{p}{n}(1-z^2)}{1-z^2} \frac{dz}{\sqrt{(2Z)}}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{Pz+Q}{1-z^2}\sqrt{(2.z_3 - z)}$$

$$= \tfrac{1}{2}\sin^{-1}\frac{z+C'}{1-z^2}\sqrt{(z-z_2.z-z_1)}, \qquad (340)$$

where
$$z_1 = \frac{L(1-2L)^{\frac{1}{2}} - (2-L)^{\frac{1}{2}}(2-L-2L^2)^{\frac{1}{2}}}{2(1-L)^{\frac{3}{2}}(1+L)^{\frac{1}{2}}},$$

$$z_2 = \frac{L(1-2L)^{\frac{1}{2}} + (2-L)^{\frac{1}{2}}(2-L-2L^2)^{\frac{1}{2}}}{2(1-L)^{\frac{3}{2}}(1+L)^{\frac{1}{2}}},$$

$$z_3 = \frac{(1-L)^{\frac{1}{2}}(1+L)^{\frac{1}{2}}}{(1-2L)^{\frac{1}{2}}},$$

$$\sin\theta_3 = \frac{(-2L+L^2)^{\frac{1}{2}}}{(1-2L)^{\frac{1}{2}}},$$

$$z_1 z_2 = -\frac{1}{1-L},$$

and
$$z_3 = -\frac{1+z_1 z_2}{z_1+z_2},$$

as required for the spherical pendulum.

$$\left.\begin{aligned} 2\frac{p}{n} = P &= \frac{(1-2L)^{\frac{3}{2}}(2-L)^{\frac{1}{2}}}{(1-L)^{\frac{3}{2}}(1+L)^{\frac{1}{2}}} \\ Q &= \frac{(1-2L)^{\frac{1}{2}}(2-L)^{\frac{1}{2}}}{(1-L)^{\frac{1}{2}}(1+L)^{\frac{3}{2}}}, \\ C' &= \frac{(1-2L)^{\frac{1}{2}}(1-L)^{\frac{1}{2}}}{(1+L)^{\frac{1}{2}}}. \end{aligned}\right\} \qquad (341)$$

Fig. 1 may be taken for the construction of this case of the spherical pendulum, by moving H to H', where tangent Y_1P_1 cuts the pedal again.

25. With the parameter

$$v = \omega_1 + \tfrac{1}{3}\omega_3$$

in the spherical pendulum, the cubic equation for L becomes

$$L^3 - (1-8c+15c^2-8c^3+c^4)\,L - 2c\,(1-c)^2\,(1-2c)(2-c) = 0. \quad (342)$$

Put $$L = cx, \quad \text{and} \quad c + \frac{1}{c} = a,$$

and then $$x^3 - (a^2-8a+13)\,x - 2\,(a-2)(2a-5) = 0, \quad (343)$$

a cubic in x, but a quadratic in a; and thence

$$a = \frac{9+4x+\sqrt{(1+3x^2+4x^3+x^4)}}{4+x}, \quad (344)$$

so that numerical cases can be constructed by assigning special values to x.

With $$v = \omega_1 + \tfrac{1}{4}\omega_3,$$

$$L^3 - \tfrac{1}{4}(1-20c+132c^2-428c^3+788c^4-856c^5+528c^6-160c^7+16c^8)\,L$$
$$-\,C\,(1-c)^3\,(1-2c)^3\,(1-2c+2c^2)(1-4c+2c^2) = 0, \quad (345)$$

which becomes a quartic in b, when we put

$$L = c^2x,$$

$$c\sqrt{2} + \frac{1}{c\sqrt{2}} = b\sqrt{2}.$$

As writers on this dynamical subject may be mentioned

Jonquières (M. l'amiral de), (*Comptes Rendus*, 12 July, 1886, "Théorie élémentaire, d'après Poinsot, de la toupie");

E. Padova (*Ven. Inst. Atti*, III);

J. E. Campbell (*Messenger of Mathematics*, Vol. XXIII);

A. MacAulay ("Octonions," *Proc. R. S.*, 1896);

A. de St. Germain ("Note sur le pendule sphérique," *Darboux's Bulletin*, May, 1896).

[*Note to p.* 552.—The quadrics (32) and (36) will be momental quadrics of real positive matter only when P_1 and P_2 in Fig. 1 are placed on opposite quadrants of the focal ellipse; and now the herpolhodes cannot have points of inflexion.]

On a Twofold Generalization of Stieltjes' Theorem. By HENRY TABER, Worcester, Mass. Received June 8th, 1896. Read June 11th, 1896.

In a paper "Sur une propriété des déterminants symétriques gauche" which appeared in Volume XVII., Second Series (1892), of the *Mémoires de la Société Royale des Sciences de Liège*, M. François Deruyts gave the following very interesting theorem :—

If the minors of order 2k of a skew symmetric determinant are all zero, the minors of order 2k−1 are all zero also.

As an immediate consequence of this theorem follow certain theorems of some interest relating to orthogonal substitutions, among which is included a two-fold generalization of Stieltjes' theorem.

Let the transformation A defined by the system of equations

$$x'_r = a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \quad (r = 1, 2, \ldots n)$$

be any orthogonal substitution in n variables. Let $A_{(+1)}$ denote the linear transformation defined by the system of equations

$$x'_r = a_{r1}x_1 + \ldots + a_{r,r-1}x_{r-1} + (a_{rr} - \rho)\,x_r + a_{r,r+1}x_{r+1} + \ldots + a_{rn}x_n \qquad (r = 1, 2, \ldots n),$$

for $\rho = +1$, and let $A_{(-1)}$ denote the linear transformation defined by this system of equations for $\rho = -1$.

Further, let Det. $\left[A^m_{(+1)}\right]$, m being any positive integer, denote the determinant of the transformation $A^m_{(+1)}$, the m^{th} power of $A_{(+1)}$, obtained by m applications (*i.e.*, $m-1$ repetitions) of the transformation $A_{(+1)}$. Similarly, let Det. $\left[A^m_{(-1)}\right]$ denote the determinant of $A^m_{(-1)}$, the m^{th} power of $A_{(-1)}$. Then we have at once the following theorems :—

I. *If the orthogonal substitution A is proper, and if the minors of order $2k$ of the determinant Det.* $\left[A^{2m+1}_{(+1)}\right]$ *are all zero, the minors of order $2k-1$ are all zero also.*

II. *If the determinant of the orthogonal substitution A is equal to* -1, *and the minors of order* $2k+1$ *of Det.* $\left[A^{2m+1}_{(+1)}\right]$ *are all zero, the minors of order* $2k$ *are all zero also.*

III. *If the determinant of the orthogonal substitution* A *is equal to* $+1$, *and if the* $(2\kappa)^{\text{th}}$* *minors of Det.* $\left[A^{2m+1}_{(-1)}\right]$ *are all zero, the* $(2\kappa+1)^{\text{th}}$ *minors are all zero also.*

IV. *If the determinant of the orthogonal substitution* A *is equal to* -1, *and if the* $(2\kappa-1)^{\text{th}}$ *minors of Det.* $\left[A^{2m+1}_{(-1)}\right]$ *are all zero, the* $(2\kappa)^{\text{th}}$ *minors are all zero also.*

Theorem III. is the two-fold generalization of Stieltjes' theorem above referred to. For $m=0$, we have

$$\text{Det.}\left[A_{(-1)}\right] = \begin{vmatrix} a_{11}+1, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}+1, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots & a_{nn}+1 \end{vmatrix};$$

and the theorem is that, if the $(2\kappa)^{\text{th}}$ minors of this determinant are all zero, the $(2\kappa+1)^{\text{th}}$ minors are all zero also. For $\kappa=0$, this is Stieltjes' theorem.†

Deruyts' theorem gives at once the conditions that must be satisfied by the numbers belonging to the roots ± 1 of the characteristic equation of an orthogonal substitution. Thus let $+1$ be a root of multiplicity p of the characteristic equation of the orthogonal substitution A. Then the nullity of the transformation $A_{(+1)}$ is at least one,‡ and the nullity of $A^2_{(+1)}$, &c., the successive powers of $A_{(+1)}$, in-

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* That is, the minors of Det. $\left[A^{2m+1}_{-1)}\right]$ of order $n-2\kappa$.

† See Netto, *Acta Mathematica*, Vol. IX., p. 295.

‡ The nullity of the linear transformation defined by the system of equations

$$x'_r = a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n \quad (r = 1, 2, \dots n)$$

is p if all the $(p-1)^{\text{th}}$ minors of the matrix

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{matrix}$$

are zero, but not all the p^{th} minors.

creases until a power of exponent μ is attained whose nullity is equal to p. The nullity of the $(\mu+1)^{\text{th}}$ and of higher powers of $A_{(+1)}$ is then also p. Let

$$p_1,\ p_2,\ \dots\ p_{\mu-1},\ p_\mu = p,$$

designate respectively the nullities of

$$A_{(+1)},\ A^2_{(+1)},\ \dots\ A^{\mu-1}_{(+1)},\ A^{\mu}_{(+1)};$$

I term the numbers p_1, p_2, &c., the numbers *belonging* to the root $+1$ of the characteristic equation of A. For any linear transformation A, we must have

$$p_1 \geqq p_2 - p_1 \geqq \dots \geqq p_\mu - p_{\mu-1} \geqq 1;\text{*}$$

and, since A is orthogonal, it follows from Deruyts' theorem that p_1, p_3, &c., the numbers with odd suffixes belonging to the root $+1$ of the characteristic equation of A, are all even or all odd according as $p = p_\mu$ is even or odd.

Similarly, if $q_1, q_2, \dots q_\nu$ are the numbers belonging to the root -1 of the characteristic equation of A,

$$q_1 \geqq q_2 - q_1 \geqq \dots \geqq q_\nu - q_{\nu-1} \geqq 1,$$

and q_1, q_3, &c., the numbers with odd suffixes, are all even or all odd according as $q = q_\nu$ is even or odd.†

These theorems may be proved as follows:—If $+1$ is a root of multiplicity p, and -1 a root of multiplicity q of the characteristic equation of the orthogonal substitution A, an orthogonal substitution E can always be found, such that

$$A = EBE^{-1},$$

where B is an orthogonal substitution in n variables defined by the

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* In Volume XXVI. of these *Proceedings*, page 368, line 19, the conditions for the numbers belonging to a root other than ± 1, should read $m_1 \geqq m_2 - m_1 \geqq$ &c., as there given.

† The conditions given above are the only conditions for the numbers belonging to the root $+1$ or -1.

three orthogonal substitutions

$$X'_r \quad = b^{(1)}_{r1} X_1 \quad + b^{(1)}_{r2} X_2 \quad + \dots + b^{(1)}_{rp} X_p \qquad (r = 1, 2, \dots p),$$

$$X'_{p+r} = b^{(2)}_{r1} X_{p+1} + b^{(2)}_{r2} X_{p+2} + \dots + b^{(2)}_{rq} X_{p+q} \qquad (r = 1, 2, \dots q),$$

$$X'_{p+q+r} = b^{(3)}_{r1} X_{p+q+1} + b^{(3)}_{r2} X_{p+q+2} + \dots + b^{(3)}_{r,\,n-p-q} X_n \quad (r = 1, 2, \dots \overline{n-p-q}),$$

$+1$ being a root of multiplicity p of the characteristic equation of the transformation B_1 defined by the first p equation, -1 being a root of multiplicity q of the characteristic equation of the transformation B_2 defined by the second set of equations (q in number), while the roots of the characteristic equation of B_3, the transformation defined by the third system of equations, are the roots other than ± 1 of the characteristic equation of A.

Let the transformation I_1 be defined by the system of equations

$$X'_r = X_r \qquad (r = 1, 2, \dots p);$$

let I_2 be defined by the equations

$$X'_{p+r} = X_{p+r} \quad (r = 1, 2, \dots q);$$

and let I be defined by the equations

$$X'_r = X_r \qquad (r = 1, 2, \dots n).$$

Then I_1 is the identical transformation for the p-way extension $(X_1, X_2, \dots X_p)$; I_2 is the identical transformation for the q-way extension $(X_{p+1}, X_{p+2}, \dots X_{p+q})$; and I is the identical transformation for the n-way extension $(X_1, X_2, \dots X_n)$.

Since -1 is not a root of the characteristic equation of the linear transformation B_1 in p variables, the determinant of $I_1 + B_1$* is not zero; and so we may put

$$C_1 = (I_1 - B_1)(I_1 + B_1)^{-1}.$$

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* Following Cayley, I regard the operations of addition and subtraction as capable of extension to linear transformations; if the linear transformations A and B are defined respectively by the two systems of equations

$$X'_r = a_{r1} X_1 + a_{r2} X_2 + \dots + a_{rn} X_n \quad (r = 1, 2, \dots n),$$

$$X'_r = b_{r1} X_1 + b_{r2} X_2 + \dots + b_{rn} X_n \quad (r = 1, 2, \dots n),$$

the transformation $A \pm B$ is defined by the system of equations

$$X'_r = (a_{r1} \pm b_{r1}) X_1 + (a_{r2} \pm b_{r2}) X_2 + \dots + (a_{rn} \pm b_{rn}) X_n \quad (r = 1, 2, \dots n).$$

Whence, denoting by $\breve{C}_1$ the transverse or conjugate of C_1,* we have

$$\breve{C}_1 = (\breve{I}_1+\breve{B}_1)^{-1}(\breve{I}_1-\breve{B}_1)$$
$$= (I_1+B_1^{-1})^{-1}(I_1-B_1^{-1})$$
$$= -(I_1+B_1)^{-1}(I_1-B_1) = -C_1,$$

that is, C_1 is skew symmetric. Moreover,

$$(I_1+C_1)(I_1+B_1) = 2I_1;$$

therefore the determinant of I_1+C_1 is not zero; and, consequently,

$$B_1 = (I_1-C_1)(I_1+C_1)^{-1}.$$

Whence we obtain $\quad B_1-I_1 = -2C_1(I_1+C_1)^{-1},$

$$(B_1-I_1)^{2m+1} = (-2)^{2m+1}C_1^{2m+1}(I_1+C_1)^{-2m-1}.$$

Since the determinant of I_1+C_1 is not zero, the nullity of $(B_1-I_1)^{2m+1}$ is equal to the nullity of C_1^{2m+1}; therefore, if the minors of order r of the determinant of $(B_1-I_1)^{2m+1}$ are all zero, the minors of order r of the determinant of C_1^{2m+1} are all zero, and conversely. But C_1^{2m+1} is skew symmetric, being an odd power of a skew symmetric linear transformation. Therefore, by Deruyts' theorem, if the minors of order $2k$ of the determinant of $(B_1-I_1)^{2m+1}$ are all zero, the minors of order $2k-1$ are all zero also. That is, if the nullity of $(B_1-I_1)^{2m+1}$ is as great as $p-2k+1$, it is as great as $p-2k+2$.

Furthermore, since $+1$ is not a root of the characteristic equation of B_2, nor of the characteristic equation of B_3, the nullity of $(B-I)^{2m+1}$ is equal to the nullity of $(B_1-I_1)^{2m+1}$. Again, the nullity of

$$A_{(+1)}^{2m+1} = (A-I)^{2m+1} = (EBE^{-1}-I)^{2m+1} = E(B-I)^{2m+1}E^{-1}$$

is equal to the nullity of $(B-I)^{2m+1}$. Therefore, if the nullity of $A_{(+1)}^{2m+1}$ is as great as $p-2k+1$, it is as great as $p-2k+2$.

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* If the transformation C is defined by the equations

$$X_r' = C_{r1}X_1 + C_{r2}X_2 + \dots + C_{rn}X_r \quad (r = 1, 2, \dots n),$$

the transformation $\breve{C}$, the transverse of C, is defined by the equations

$$X_r' = C_{1r}X_1 + C_{2r}X_2 + \dots + C_{nr}X_r \quad (r = 1, 2, \dots n).$$

We have $\quad \breve{CC'} = \breve{C'}\breve{C}, \quad (\breve{C^{-1}}) = (\breve{C})^{-1}.$

Similarly, since $+1$ is not a root of the characteristic equation of B_2, we may put

$$C_2 = (I_2+B_2)(I_2-B_2)^{-1}.$$

Whence we derive, as before,

$$\breve{C}_2 = (\breve{I}_2-\breve{B}_2)^{-1}(\breve{I}_2+\breve{B}_2)$$
$$= (I_2-B_2^{-1})^{-1}(I_2+B_2^{-1})$$
$$= -(I_2-B_2)^{-1}(I_2+B_2) = -C_2.$$

We also have $$(I_2+C_2)(I_2-B_2) = 2I_2;$$

consequently, the determinant of I_2+C_2 is not zero, and therefore we have

$$B_2 = -(I_2-C_2)(I_2+C_2)^{-1}.$$

Whence we obtain

$$(B_2+I_2)^{2m+1} = 2^{2m+1}C_2^{2m+1}(I_2+C_2)^{-2m-1}.$$

By the same reasoning as that employed above, since the determinant of I_2+C_2 is not zero, and since C_2^{2m+1} is skew symmetric, it follows from Deruyts' theorem that, if the nullity of $(B_2+I_2)^{2m+1}$ is as great as $q-2k+1$, it is as great as $q-2k+2$. But, since -1 is not a root of the characteristic equation of either B_1 or B_2, the nullity of $(B_2+I_2)^{2m+1}$ is equal to the nullity of $(B+I)^{2m+1}$. Again, the nullity of

$$A_{(-1)}^{2m+1} = (A+I)^{2m+1} = (EBE^{-1}+I)^{2m+1} = E(B+I)^{2m+1}E^{-1}$$

is equal to the nullity of $(B+I)^{2m+1}$. Therefore, if the nullity of $A_{(-1)}^{2m+1}$ is as great as $q-2k+1$, it is as great as $q-2k+2$.

If the determinant of A is equal to $+1$, p, the multiplicity of the root $+1$ of the characteristic equation of A, is even or odd according as n is even or odd. Therefore, *if A is a proper orthogonal substitution, and if n is $\frac{\text{even}}{\text{odd}}$, the nullity of $A_{(+1)}^{2m+1}$ is $\frac{\text{even}}{\text{odd}}$.*

On the other hand, if A is improper, p is odd if n is even, and is even if n is odd. Therefore in this case, *if n is $\frac{\text{even}}{\text{odd}}$, the nullity of $A_{(+1)}^{2m+1}$ is $\frac{\text{odd}}{\text{even}}$.* These two theorems are equivalent respectively to I. and II.

Further, irrespective of the value of n, q is even if A is proper, and is odd if A is improper. Therefore, *if A is $\frac{proper}{improper}$, the nullity of $A^{2m+1}_{(-1)}$ is $\frac{even}{odd}$*. This theorem is equivalent to III. and IV.

If $p_1, p_2, \dots p_\mu$ are the numbers belonging to the root $+1$ of multiplicity p of the characteristic equation of A, then, since the nullity of $A^{2m+1}_{(+1)}$ is even or odd according as p is even or odd, it follows that p_1, p_3, &c., the numbers with odd suffices, are all even if $p = p_\mu$ is even, or all odd if p is odd.

Similarly, if $q_1, q_2, \dots q_\nu$ are the numbers belonging to the root -1 of multiplicity q of the characteristic equation of A, then, since the nullity of $A^{2m+1}_{(-1)}$ is even or odd according as q is even or odd, it follows that q_1, q_3, &c., are all even or all odd according as $q = q_\nu$ is even or odd.

[*Postscript, December 25th*, 1896.—Employing the notation of the preceding paper, Stieltjes' theorem, as originally enunciated, is

If A and B are two proper orthogonal substitutions, the determinant of $A+B$ vanishes only if the first minors of this determinant all vanish.

The theorem was given by Stieltjes for n (the number of variables) equal to two, or equal to three. Stieltjes stated that he believed the theorem to hold for $n = 4$; and he suggested the inquiry whether it held for any value of n.*

If B is the identical substitution, this theorem becomes the theorem given in the preceding paper as Stieltjes' theorem, namely:

If A is a proper orthogonal substitution, the determinant of $A+I$ (where I is the identical substitution) vanishes only if the first minors of this determinant all vanish.

This theorem is formally included in Stieltjes' theorem, and is apparently only a special case of Stieltjes' theorem. But the latter also follows from this theorem. For, if A and B are any two proper orthogonal substitutions, AB^{-1} (where B^{-1} denotes the reciprocal of B, which, since B is orthogonal, is thus the transverse or conjugate of B) is also a proper orthogonal substitution; and, since the determinant of B is not zero, the nullity of $A+B = (AB^{-1}+I)\,B$ is equal to the nullity of $AB^{-1}+I$. By the preceding theorem, the determinant of $AB^{-1}+I$ vanishes only if the first minors of this determinant all vanish. That is, if the nullity of $AB_{-1}+I$ is as

* *Acta Mathematica*, Vol. VI., p. 319.

great as 1, it is as great as 2. Therefore, if the nullity of $A+B$ is as great as 1, it is as great as 2, which is Stieltjes' theorem.

Since the content of both theorems is the same, the latter may also be termed Stieltjes' theorem.

I have recently learned that the most interesting case of Theorem III., also of Theorem IV., of the preceding paper, namely, for $m=1$, were given by A. Voss, in the *Mathematische Annalen*, Vol. XIII., p. 330. For $m=1$, Theorems III. and IV. are:

If the substitution

$$X_r = a_{r1}y_1 + a_{r2}y_2 + \dots + a_{rn}y_n \qquad (r = 1, 2, \dots n)$$

is orthogonal, and the r^{th} minors of

$$\begin{vmatrix} a_{11}+1, & a_{12}, & \dots \\ a_{21}, & a_{22}+1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

are all zero, but not all the $(r+1)^{th}$ minors, r is an odd number if the substitution is proper, and is even if the substitution is improper.

Voss states that this theorem is true of the determinant

$$\text{Det.}\left[A_{(+1)}\right] = \begin{vmatrix} a_{11}-1, & a_{12}, & \dots \\ a_{21}, & a_{22}-1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

This is the case if n is even, but not otherwise. Thus, for $n=3$, if $a_{11}=+1$, $a_{22}=a_{33}=-1$, and $a_{rs}=0$ for $r \neq s$, the substitution is proper, and Det. $\left[A_{(+1)}\right]$ vanishes, but not its first minors. The proper statement, for n either odd or even, of the theorem relating to the Det. $\left[A_{(+1)}\right]$ corresponding to Voss' theorem, just given, regarding the Det. $\left[A_{(-1)}\right]$, is obtained from Theorems I. and II., by putting $m=1$. The theorem thus obtained may be shown to follow immediately from Voss' investigations. But I do not know that it has anywhere been given.

To Voss is due, I believe, the first complete proof of Stieltjes' theorem. Netto's proof does not extend to every case.

I have also found that Theorems I., II., III., and IV. may be derived from theorems relating to the elementary divisors $(\rho \pm 1)^{e_r}$ of the characteristic function of an orthogonal substitution, given by Frobenius in *Crelle's Journal*, Vol. LXXXIV.

Let ρ be a variable parameter not contained in the coefficients of the linear substitution A; and let ρ_0 be a root of multiplicity l of the characteristic equation of A, viz.,

$$\text{Det.}\left[A-\rho I\right]=0,$$

where I is the identical substitution. Then $(\rho_0-\rho)^l$ is a divisor of the left-hand member of this equation, the characteristic function of A.

The minors of all orders of this determinant are polynomials in ρ, and the minors of order $n-r$ may all contain $\rho_0-\rho$ if $l>r$. Let l_r be the highest power of $\rho_0-\rho$ contained in all the minors of order $n-r$. If l_ω is the last of the series l_1, l_2, &c., that is not zero,

$$l > l_1 > l_2 \dots > l_\omega > 0;$$

and, if $\quad e=l-l_1, \quad e_1=l_1-l_2, \; \dots \; e_{\omega-1}=l_{\omega-1}-l_\omega, \quad e_\omega=l_\omega,$

then $$e \geqq e_1 \geqq e_2 \geqq \dots \geqq e_\nu \geqq 1,$$

$$(\rho_0-\rho)^l=(\rho_0-\rho)^e(\rho_0-\rho)^{e_1}(\rho_0-\rho)^{e_2}\dots(\rho_0-\rho)^{e_\omega}.$$

The several divisors $(\rho_0-\rho)^e$, $(\rho_0-\rho)^{e_1}$, &c., of $(\rho_0-\rho)^l$ are elementary divisors (*elementar theiler*) of the characteristic function of A. These divisors all vanish for $\rho=\rho_0$. Corresponding to each root of the characteristic equation of A is a system of elementary divisors that vanish if ρ be equal to the root in question.

Let now $m_1, m_2, \dots m_\mu = l$ be the numbers belonging to the root ρ_0. Form a diagram corresponding to ρ_0 by arranging l dots in rows and columns, so that there shall be μ rows respectively of

$$m_1, \; m_2-m_1, \; \dots \; m_\mu-m_{\mu-1}$$

equidistant dots, and so that the first dot in each row falls in the same (left-hand) column. The number of columns will then be m_1, which will be found to be equal to $\omega+1$; and the number of dots in the successive columns (counting from the left) will be equal severally and respectively, to $e, e_1, e_2, \dots e_\omega$.

For an orthogonal substitution Frobenius has shown that the elementary divisors $(\rho\pm 1)^{2k}$ occur in pairs with equal exponent. It will be found that this is equivalent to the theorem given above, that the numbers with odd subscripts belonging to the root ± 1 of the characteristic equation of an orthogonal substitution are all even or all odd according as the multiplicity of the roots ± 1 is even or odd. From this theorem, Theorems I., II., III., and IV. may be derived.]

Waves in Canals and on a Sloping Bank. By H. M. MACDONALD.
Read June 11th, 1896. Received in revised form December 3rd, 1896.

1. In his treatise on *Hydrodynamics* (1895), Ch. IX., Professor Lamb has alluded to a former paper on the above subject in the following terms:—"There is some divergence here, and elsewhere in the text, from the views maintained by Macdonald in the paper cited" (Footnote, p. 435). The first statement in my paper alluded to, occurs on p. 109, Vol. XXV. of the *Proceedings*, where it is stated that a long wave with infinite velocity of propagation would not be generated; this statement was due to the idea that such a system required to be set up all along its length simultaneously, and would need an infinity of energy to do it. There is nothing in this statement which conflicts with Professor Lamb's interpretation of the motion (with which I entirely agree), imagining it to be set up. The other statement alluded to occurs on p. 111, where it is stated that straight-crested waves in a canal of triangular cross section are only possible in certain cases.

Professor Lamb, § 239, assuming the existence of free oscillations of the liquid contained between two transverse partitions in a canal, deduces the existence of straight-crested waves in a canal of unlimited length. There is a serious objection to the use of a physical assumption as the basis of such a proof, that it assumes that the mathematical theory of the physical phenomena is correct; in the present case it assumes the existence of an irrotational fluid motion, and that the boundary conditions are those satisfied in nature.

What is really proved in my former paper is that the cases there mentioned are the only ones in which the velocity potential can be expanded in a Fourier series of the type

$$\Sigma R_s \cos\frac{s\pi\theta}{\alpha}\cos(mx-nt),$$

where r, θ, x are cylindrical coordinates of origin in the lowest line of the canal, θ the angle between the vector r and a side of the canal. What was assumed in the final statement was that the solution of the equation

$$\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}-m^2 f=0,$$

subject to $$\frac{\partial f}{\partial \theta} = 0, \quad \theta = 0 \text{ or } \alpha,$$

can always be expressed in the form

$$\Sigma R_s \cos \frac{s\pi\theta}{\alpha}.$$

This assumption is usually made, but, as an assumption, it is not legitimate; the possibility of such an expansion depends on the character of the other boundary conditions. The only case where the expansion can easily be shown to be possible is when the region within which the function is required can be extended, so that the boundary conditions, where $\theta = 0$ or α, are of the same character for all values of r. In this case it is evidently true, and the conditions here stated are satisfied in all the problems solved by such expansions which I have examined. It will be shown below that these conditions are satisfied for symmetric waves both along and across canals of uniform isosceles triangular cross section, vertical angles $\pi/2$ and $2\pi/3$, and for asymmetric waves when the vertical angle is $\pi/2$.

2. Taking the case when the vertical angle is $\pi/2$, if a velocity potential exists in the region AOB satisfying the required conditions,

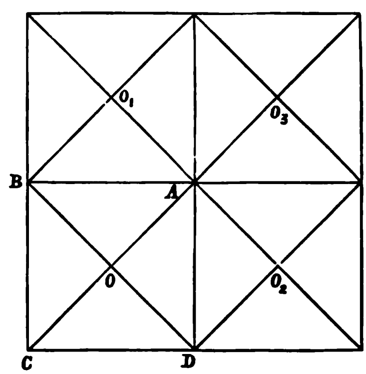

Fig. 1.

then functions exist in the spaces DOA, COD, BOC, and the function can be continued throughout the space $ABCD$, $\frac{\partial f}{\partial \theta}$ being zero along AC and BD. The function can then be continued throughout all space as the squares can be repeated so as to fill all space, and further

$\frac{\partial f}{\partial \theta} = 0$ all along the line $COAO_3$ produced to infinity, and also along DOB produced to infinity. Hence the velocity potential, both for symmetric and asymmetric waves along or across this canal, can be expressed in the form

$$\Sigma R_s \cos \frac{s\pi\theta}{a}.$$

When the vertical angle is $2\pi/3$, if a velocity potential exists in the space AOB (Fig. 2) satisfying the required conditions, then functions exist in the spaces AOC, BOC, and the function can be continued throughout the space ABC, $\frac{\partial f}{\partial \theta}$ being zero along OA, OB, and OC. The function can then be continued throughout all space, as the equilateral triangles can be repeated so as to fill all spaces. When the wave motion is symmetric, $\frac{\partial f}{\partial \theta}$ is zero along OD, OE, and OG; therefore $\frac{\partial f}{\partial \theta}$ vanishes all along the lines OA and OB produced indefinitely. Hence the velocity potential of a symmetric wave motion, either along

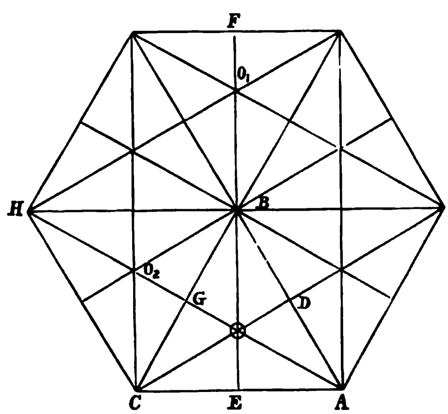

FIG. 2.

or across a canal vertical angle $2\pi/3$, can be expressed in the form

$$\Sigma R_s \cos \frac{s\pi\theta}{a}.$$

When the wave motion is asymmetric, the condition $\frac{\partial f}{\partial \theta}$ zero is no longer satisfied all along OA and OB, and it follows from the analysis of the previous paper that the velocity potential is not expressible in this form. When the vertical angle is $\pi/3$, all space can be partitioned by means of hexagons, but parts of the fixed boundaries when produced are free surfaces in other partitions, the condition $\frac{\partial f}{\partial \theta}$ zero all along them is satisfied neither for symmetric nor asymmetric waves, and in this case also the velocity potential cannot be expressed in the form

$$\Sigma R_s \cos \frac{s\pi\theta}{a}.$$

3. When, in the results obtained for the wave motion in these canals, the depth of the canal is made indefinitely great, velocity potentials for waves standing against a sloping bank or along it are obtained, when the bank makes angles $\pi/4$ or $\pi/6$ with the horizon. The velocity potential in these cases takes the form of a sum of exponentials, and it suggests the inquiry for what angles of slope is there a solution of this form. In the case of waves standing against a sloping bank, a solution of this form has been given by Kirchhoff, *Gesam. Abhand.*, Bd. II., p. 428, when the angle with the horizon is of the form $(2m+1)\pi/2n$, m and n being integers. It will be shown below that the solution in the form of a sum of exponentials makes the velocity finite at a great distance from the edge only when the angle of slope is of the form $\pi/2n$. Further, it is shown below that there is a solution of this form for waves parallel to the bank when the angle of slope is of the form $\pi/(4n+2)$, and that in this case the velocity is finite at a great distance from the edge. In all these cases where the condition of finite velocity at a great distance from the edge is satisfied, the velocity of propagation of the waves is that of deep sea waves.

4. Taking the axis of x along the intersection of the free surface and the bank, that of y horizontally and perpendicular to it, and that of z vertically upwards, the velocity potential satisfies the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{1}$$

throughout the fluid, subject to

$$\frac{\partial \phi}{\partial z} \cos a + \frac{\partial \phi}{\partial y} \sin a = 0, \tag{2}$$

when $z\cos\alpha+y\sin\alpha=0$, α being the angle which the bank makes with the horizon, and

$$l\frac{\partial\phi}{\partial z}-\phi=0, \tag{3}$$

when $z=0$, where $l=g/n^2$, $2\pi/n$ being the period of the motion. Further, the velocity at a great distance from the edge must be finite.

Considering first the case of waves standing against the bank, ϕ takes the form $f(y,z)\,e^{\iota nt}$, where f may be written

$$f=\left\{\cos\left(z\frac{\partial}{\partial y}\right)+\frac{\sin\left(z\dfrac{\partial}{\partial y}\right)}{l\dfrac{\partial}{\partial y}}\right\}f_0, \tag{4}$$

for the above expression for f satisfies (1) and (3), f_0 being a function of y only. It is proposed to investigate in what cases f_0 is of the form $\Sigma A_\lambda e^{\lambda y}$. When f_0 is of this form it may be written

$$f_0=\int e^{-\iota myt}\chi(t)\,dt,$$

where the path of integration is any closed curve enclosing all the poles of $\chi(t)$ which are simple.*

Then $$f=\tfrac{1}{2}\int\left[\left(1+\frac{1}{mlt}\right)e^{-\iota m(y+\iota z)t}+\left(1-\frac{1}{mlt}\right)e^{-\iota m(y-\iota z)t}\right]\chi(t)\,dt.$$

To satisfy (2) change to polars; then $y+\iota z=re^{\iota\theta}$, and (2) becomes

$$\int\left[\left(1+\frac{1}{mlt}\right)mte^{-\iota\alpha}e^{-\iota mre^{-\iota\alpha}t}\left(1-\frac{1}{mlt}\right)mte^{\iota\alpha}e^{-\iota mre^{\iota\alpha}t}\right]\chi(t)\,dt=0.$$

This is equivalent to

$$e^{-2\iota\alpha}\Sigma(t_s+\lambda)e^{-\iota mre^{-\iota\alpha}t_s}\left[\chi(t)(t-t_s)\right]_{t=t_s}$$
$$=\Sigma(t_s-\lambda)e^{-\iota mre^{\iota\alpha}t_s}\left[\chi(t)(t-t_s)\right]_{t=t_s}.$$

for all values of r, where $$\lambda=\frac{1}{ml}.$$

Therefore, if t_0 is a pole of $\chi(t)$, $t_0e^{-2\iota\alpha}$ is one and also $t_0e^{-4\iota\alpha}$, &c.

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* It may be shown that this form of f_0, when the poles of $\chi(t)$ are not simple, does not lead to a solution of the problem in any of the cases where the above form fails.

Hence $t_0 e^{-2qi\alpha} = t_0$, which requires $\alpha = p\pi/q$, where p and q are integers. Writing $\gamma = e^{-2i\alpha}$,

$$\chi(t) = \frac{\chi_1(t)}{(t-t_0)(t-t_0\gamma)\dots(t-t_0\gamma^{q-1})},$$

then

$$\frac{\gamma(t_0+\lambda)\chi_1(t_0)}{t_0^{q-1}(1-\gamma)(1-\gamma^2)\dots(1-\gamma^{q-1})} = \frac{(t_0\gamma-\lambda)\chi_1(t_0\gamma)}{t_0^{q-1}(\gamma-1)(\gamma-\gamma^2)\dots(\gamma-\gamma^{q-1})}, \text{ \&c.};$$

therefore $$\chi_1(t_0)(t_0+\lambda) = \chi_1(t_0\gamma)(t_0\gamma-\lambda),$$

$$\chi_1(t_0\gamma)(t_0\gamma+\lambda = \chi_1(t_0\gamma^2)(t_0\gamma^2-\lambda),$$

$$\text{\&c.} = \text{\&c.},$$

$$\chi_1(t_0\gamma^{q-1})(t_0\gamma^{q-1}+\lambda) = \chi_1(t_0)(t_0-\lambda);$$

that these equations may be consistent, q must be an even integer. Let $q = 2k$; then

$$\chi_1(t_0\gamma^s) = \frac{(t_0\gamma^{s-1}+\lambda)(t_0\gamma^{s-2}+\lambda)\dots(t_0+\lambda)}{(t_0\gamma^s-\lambda)(t_0\gamma^{s-1}-\lambda)\dots(t_0\gamma-\lambda)}\chi_1(t_0),$$

that is, $$\chi_1(t_0\gamma^s) = \frac{(t_0\gamma^{s+1}-\lambda)(t_0\gamma^{s+2}-\lambda)\dots(t_0\gamma^{k+s-1}-\lambda)\gamma^{ks}}{(t_0\gamma-\lambda)(t_0\gamma^2-\lambda)\dots(t_0\gamma^{k-1}-\lambda)}\chi_1(t_0);$$

therefore $$\chi_1(t) = A(t\gamma-\lambda)(t\gamma^2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k,$$

and $$f_0(y) = A\int e^{\iota m y t}\frac{(t\gamma-\lambda)(t\gamma_2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k dt}{t^{2k}-1}.$$

That the velocity may be finite everywhere, it will be sufficient to make $f_0(y)$ finite when y is infinite; that this may be so, $e^{-\iota m y \gamma^s}$ must be finite when y is infinite for all the values of s which can occur, and these are therefore the integers given by

$$\cos\left(\frac{\pi}{2}+2s\alpha\right) \leqslant 0,$$

which requires

$$k \geqslant (2p+1)s \geqslant 0, \quad 3k \geqslant (2p+1)s \leqslant 2k, \quad \text{\&c.},$$

where $$\alpha = \frac{2p+1}{2k}\pi.$$

When $p = 0$, $\alpha = \pi/2k$, s has values $0, 1$ to k; and

must have $$\frac{(t\gamma-\lambda)(t\gamma^2-\lambda)\dots(t\gamma^{k-1}-\lambda)t^k}{(t-\gamma^{k+1})(t-\gamma^{k+2})\dots(t-\gamma^{2k-1})}$$

[redacted]

for a factor; this requires that $\lambda = 1$, that is, $ml = 1$, and then

$$f_0(y) = A\int e^{-\iota myt}\frac{t^k\,dt}{(t-1)(t-\gamma)\dots(t-\gamma^k)}.$$

When p has any other value than zero, the condition of finite velocity everywhere cannot be satisfied, as is easily seen from the form of $f_0(y)$, it being necessary that λ should be real. In the case where $\alpha = \pi/2k$, f is given by

$$f = A\int\left(\cosh mzt + \frac{\sinh mzt}{mlt}\right)e^{-\iota myt}\frac{t^k\,dt}{(t-1)(t-\gamma)\dots(t-\gamma^k)}.$$

The velocity potential can also be expressed in the form of a series as follows:—

$$f = A'\left\{\cos\left(z\frac{\partial}{\partial y}\right) + \frac{\sin\left(z\dfrac{\partial}{\partial y}\right)}{l\dfrac{\partial}{\partial y}}\right\}$$

$$\sum_{s=0}^{s=k} e^{-\frac{1}{2}(\pi\iota)(k-1)-s(\alpha-\frac{1}{2}\pi)\iota-\iota my\gamma^s}\cos(s-1)\alpha\dots\cos(s-k+1)\alpha.$$

There are two cases, according as k is an odd or an even integer.

(1) k odd $= 2k'+1$; then

$$f = C\sum_{s=0}^{s=k'}\cos(s-1)\alpha\cos(s-2)\alpha\dots\cos(s-2k')\alpha$$

$$\times\left[e^{-m(y\sin 2s\alpha - z\cos 2s\alpha)}\cos s\alpha\cos\left\{m(z\sin 2s\alpha + y\cos 2s\alpha) + (k'-s)\frac{\pi}{2}\right\}\right.$$

$$\left.+e^{-m(y\sin 2s\alpha + z\cos 2s\alpha)}\sin s\alpha\sin\left\{m(z\sin 2s\alpha - y\cos 2s\alpha) - (k'-s)\frac{\pi}{2}\right\}\right].$$

(2) k even $= 2k'$; then

$$f = C\sum_{0}^{k'-1}\cos(s-1)\alpha\dots\cos(s-2k'+1)\alpha$$

$$\times\left[e^{-m(y\sin 2s\alpha - z\cos 2s\alpha)}\cos s\alpha\cos\left\{m(z\sin 2s\alpha + y\cos 2s\alpha) + k'-s-\tfrac{1}{2})\frac{\pi}{2}\right\}\right.$$

$$\left.+e^{-m(y\sin 2s\alpha + z\cos 2s\alpha)}\sin s\alpha\sin\left\{m(z\sin 2s\alpha - y\cos 2s\alpha) - (k'-s-\tfrac{1}{2})\frac{\pi}{2}\right\}\right]$$

$$+\frac{C}{2}e^{-my}(\cos mz + \sin mz).$$

Particular cases—

$$\alpha = \pi/4,\quad f = \tfrac{1}{2}C\left\{e^{mz}(\cos my - \sin my) + e^{-my}(\cos mz + \sin mz)\right\};$$

$$\alpha = \pi/6,\quad f = C\left\{e^{-m\frac{1}{2}(y\sqrt{3}+z)}\sin m\frac{z\sqrt{3}-y}{2}\right.$$

$$\left.+\sqrt{3}\,e^{-m\frac{1}{2}(y\sqrt{3}-z)}\cos m\frac{z\sqrt{3}+y}{2} - e^{mz}\sin my\right\}.$$

In all these cases the wave motion is that due to the reflection of deep sea waves at the bank, as appears from the relation $ml = 1$.

5. For waves parallel to the bank the velocity potential ϕ may be written

$$f(y, z)\cos(mx - nt);$$

then, as before, putting

$$f = \left\{ \cos z \sqrt{\frac{\partial^2}{\partial y^2} - m^2} + \frac{\sin z \sqrt{\frac{\partial^2}{\partial y^2} - m^2}}{l \sqrt{\frac{\partial^2}{\partial y^2} - m^2}} \right\} f_0(y),$$

the equations (1) and (3) are satisfied, and it is proposed to investigate when the remaining conditions can be satisfied by the assumption

$$f_0(y) = \Sigma A_s e^{myt_s}.$$

This assumption being made, f is given by

$$f = \Sigma A_s \left\{ \cosh(mz \sin\psi_s) + \frac{\sinh(mz\sin\psi_s)}{ml\sin\psi_s} \right\} e^{my\cos\psi_s},$$

where the constants are determined by the conditions

$$\frac{\partial f}{\partial z}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = 0,$$

when $z\cos\alpha + y\sin\alpha = 0$ and $f_0(y)$ is finite when y is infinite. Hence, from the first of these,

$$\Sigma A_s \left[\left\{ m\sin\psi_s \sinh(mz\sin\psi_s) + \frac{1}{l}\cosh(mz\sin\psi_s) \right\} \cos\alpha \right.$$
$$\left. + \left\{ \cosh(mz\sin\psi_s) + \frac{\sinh(mz\sin\psi_s)}{ml\sin\psi_s} \right\} m\cos\psi_s \sin\alpha \right] e^{my\cos\psi_s} = 0,$$

when

$$z\cos\alpha + y\sin\alpha = 0;$$

that is, transforming to polars,

$$\Sigma A_s \left[\left(m + \frac{1}{l\sin\psi_s}\right) \sin(\psi_s + \alpha)\, e^{mr\cos(\psi_s+\alpha)} \right.$$
$$\left. - \left(m - \frac{1}{l\sin\psi_s}\right) \sin(\psi_s - \alpha)\, e^{mr\cos(\psi_s-\alpha)} \right] = 0,$$

for all positive values of r. Therefore

$$\psi_{s+1} = \psi_s + 2\alpha \quad \text{and} \quad \alpha = p\pi/q,$$

[redacted]

where p and q are integers, and

$$A_0\left(m+\frac{1}{l}\operatorname{cosec}\psi_0\right) = A_1\left(m-\frac{1}{l}\operatorname{cosec}\psi_1\right),$$

$$A_1\left(m+\frac{1}{l}\operatorname{cosec}\psi_1\right) = A_2\left(m-\frac{1}{l}\operatorname{cosec}\psi_2\right),$$

$$\&c. = \&c.,$$

$$A_{q-1}\left(m+\frac{1}{l}\operatorname{cosec}\psi_{q-1}\right) = A_0\left(m-\frac{1}{l}\operatorname{cosec}\psi_0\right);$$

that these equations may be consistent q must be an even integer. Let $q = 2k$; then

$$A_s\operatorname{cosec}\psi_s = \frac{ml\sin\psi_{s-1}+1}{ml\sin\psi_s - 1}A_{s-1}\operatorname{cosec}\psi_{s-1},$$

that is,

$$A_s\operatorname{cosec}\psi_s = \frac{(ml\sin\psi_{s-1}+1)(ml\sin\psi_{s-2}+1)\dots(ml\sin\psi_0+1)}{(ml\sin\psi_s-1)(ml\sin\psi_{s-1}-1)\dots(ml\sin\psi_1-1)}A_0\operatorname{cosec}\psi_0,$$

whence, $2k\alpha$ being an odd multiple of π,

$$A_s\operatorname{cosec}\psi_s = \frac{(-)^s(ml\sin\psi_{s+1}-1)(ml\sin\psi_{s+2}-1)\dots(ml\sin\psi_{k+s-1}-1)}{(ml\sin\psi_1-1)(ml\sin\psi_2-1)\dots(ml\sin\psi_{k-1}-1)}$$
$$\times A_0\operatorname{cosec}\psi_0.$$

Now that $f_0(y)$ may be finite for all positive values of y, $\cos\psi_s \not< 0$ for all values of s which can occur; taking the case

$$\alpha = \pi/2k, \quad 3\pi/2 \geqslant \psi_0 + 2s\alpha \geqslant \pi/2,$$

let
$$\psi_0 = \pi/2+\beta,$$

where β lies between 0 and 2α (this includes all possibilities); then

$$2k \geqslant \frac{2\beta k}{\pi} + 2s \geqslant 0,$$

and s can have values 0 to $k-1$ if β is not zero, 0 to k if β is zero.

A_s by the above can only vanish for $k-1$ values of ψ_s; therefore $k+1$ values of s must occur, and hence β is zero and the values of s from 0 to k occur. It will be observed that each exponential now appears twice for

$$\sin 2s\alpha = \sin 2(k-s)\alpha,$$

and it is therefore necessary to rewrite the expression for $f_0(y)$; there are two cases, k odd and k even. When k is odd $= 2k'+1$,

$$f_0(y) = \sum_{k'+1}^{3k'+1} A_s e^{-my\sin 2s\alpha}.$$

and, when k is even $= 2k'$,

$$f_e(y) = \sum_{k'}^{3k'} A_s e^{-my\sin 2s\alpha}.$$

Then, (1) k odd,

$$f = \sum_{k'+1}^{3k'+1} A_s \left[\cosh(mz\cos 2s\alpha) + \frac{\sinh(mz\cos 2s\alpha)}{ml\cos 2s\alpha}\right] e^{-my\sin 2s\alpha},$$

where $$A_{s+1}\left[ml - \sec 2(s+1)\alpha\right] = A_s(ml + \sec 2s\alpha).$$

Writing f in the form

$$f = \tfrac{1}{2}\sum_{k'+1}^{3k'+1} A_s \left[e^{m(z\cos 2s\alpha - y\sin 2s\alpha)}\left(1 + \frac{1}{ml\cos 2s\alpha}\right)\right.$$
$$\left. + e^{-m(z\cos 2s\alpha + y\sin 2s\alpha)}\left(1 - \frac{1}{ml\cos 2s\alpha}\right)\right],$$

it appears that the velocity will be finite for all values of y and z which occur, if $ml = 1$, for then

$$A_{2k'+2} = 0,$$

and the other constants A_s up to $A_{3k'+1}$, and further the coefficient

$$1 + \frac{1}{ml\cos 2(2k'+1)\alpha}$$

of $$e^{m\{z\cos 2(2k'+1)\alpha - y\sin 2(2k'+1)\alpha\}}$$ vanishes.

The other constants are given by the relation

$$A_s\sec 2s\alpha = \frac{\left[\cos 2(s-1)\alpha + 1\right]\left[\cos 2(s-2)\alpha + 1\right]\ldots\left[\cos 2(k'-1)\alpha + 1\right]}{(\cos 2s\alpha - 1)\left[\cos 2(s-1)\alpha - 1\right]\ldots\left[\cos 2(k'+2)\alpha - 1\right]}$$
$$\times A_{k'+1}\sec 2(k'+1)\alpha,$$

that is,

$$A_s\sec 2s\alpha = \frac{\left[\cos 2(s+1)\alpha - 1\right]\left[\cos 2(s+2)\alpha - 1\right]\ldots\cos\left[2(s+2k')\alpha - 1\right]}{\left[\cos 2(k'+2)\alpha - 1\right]\left[\cos 2(k'+3)\alpha - 1\right]\ldots\left[\cos 2(3k'+1)\alpha - 1\right]}$$
$$\times(-)^{s-k'-1}A_{k+1}\sec 2(k'+1)\alpha;$$

hence

$$A_s = (-)^s B\cos 2s\alpha\sin^2(s+1)\alpha\sin^2(s+2)\alpha\ldots\sin^2(s+2k')\alpha,$$

for values of s from $k'+1$ to $2k'+1$, and is zero for other values.

It follows that

$$f = B \sum_{s=k'+1}^{s=2k'+1} (-)^s \sin^2 (s+1) \alpha \sin^2 (s+2) \alpha \ldots \sin^2 (s+2k') \alpha$$
$$\times \left\{ \cosh (mz \cos 2s\alpha) \cos 2s\alpha + \sinh (mz \cos 2s\alpha) \right\} e^{-my \sin 2s\alpha}.$$

The velocity of propagation of the waves is given by

$$V^2 = g/m,$$

that is,

$$V^2 = g\lambda/2\pi,$$

where λ is the wave-length, and the velocity potential by

$$\phi = B \sum_{s=k'+1}^{s=2k'+1} (-)^s \sin^2 (s+1) \alpha \sin^2 (s+2) \alpha \ldots \sin^2 (s+2k') \alpha$$
$$\times \left\{ \cos^2 s\alpha\, e^{-m (y \sin 2s\alpha - z \cos 2s\alpha)} - \sin^2 s\alpha\, e^{-m (y \sin 2s\alpha + z \cos 2s\alpha)} \right\} \cos (mx - nt),$$

where

$$\alpha = \pi/(4k'+2).$$

When k is even $= 2k'$,

$$f = A_{k'} e^{-my} \left(1 + \frac{z}{l}\right)$$
$$+ \sum_{k'-1}^{3k'-1} A_s \left\{ \cosh (mz \cos 2s\alpha) + \frac{\sinh (mz \cos 2s\alpha)}{ml \cos 2s\alpha} \right\} e^{-my \sin 2s\alpha}$$
$$+ A^{3k'} e^{my} \left(1 + \frac{z}{l}\right) + Bye^{mz};$$

but the constants cannot be determined to satisfy the conditions that the velocity is everywhere finite, and that the motion is the same at a great distance from the bank, as in the case of deep sea waves. It should be observed that Sir George Stokes' solution is included in the general form.

If α is of the form $\dfrac{2p+1}{2k}\pi$, where p is other than zero, the conditions cannot be satisfied, and it follows that the only cases where the velocity potential can be expressed as a sum of exponentials, the velocity being sensible at a distance from the edge, is when the angle of the bank is of the form $\pi/(4k+2)$, and that then the velocity of propagation of the waves is that of deep sea waves.

Particular cases :—

(1) $\alpha = \dfrac{\pi}{2}$, $\phi = Ce^{mz} \cos (mx - nt)$;

(2) $\alpha = \dfrac{\pi}{6}$, $\phi = C \left\{ e^{mz} + e^{-m \frac{1}{2} (y\sqrt{3}+z)} - 3e^{-m \frac{1}{2} (y\sqrt{3}-z)} \right\} \cos (mx - nt)$;

and so on.

The following presents to the Library were received during the Recess :—

"Proceedings of the Royal Society," Vol. lix., Nos. 357, 358; Vol. lx., Nos. 359, 360.

"Journal of the Institute of Actuaries," Vol. xxxii., Pt. 6, July, 1896.

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"Rendiconti del Circolo Matematico," Tomo x., Fasc. 4 ; Palermo, 1896.

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Frey, M. von.—"Untersuchungen über die Sinnesfunctionen der menschlichen Haut," 8vo; Leipzig, 1896.

Drude, P.—"Zur Theoriestehender elektrischer Drahtwellen," 8vo; Leipzig, 1896.

"Proceedings of the Royal Irish Academy," Vol. III., No. 5; Dublin, 1896.

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Murani, O.—"Sperimenti sul Raggi Röntgen," 4to; Milano, 1896.

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Cayley, A.—"Collected Mathematical Papers," Vol. X., 4to; Cambridge, 1896. (Two copies.)

"On 2 as a 16-ic Residue and Note," by Lt.-Col. Allan Cunningham, R.E. (Corrected offprint of pp. 85–122 of Vol. XXVII. of the Society's *Proceedings*.)

ERRATA AND ADDENDA.

VOL. XXVI.

Page 144. The results, in this paper, stated for surfaces of revolution should be deleted, as the differential equations there given are obeyed, not by H and E, as stated, but by dH/dp and dE/dp.

VOL. XXVII.

Page 103, table, last line but one, column of $\frac{1}{2}e$, *for* "32," *read* "16."

,, 104, result (117), *for* "n^4," *read* "$16n^4$."

,, 105, table, last line, *for* "$e =$," *read* "$2e =$."

,, 109, table, *for* "5,576,681," *read* "5,576,881."

,, 117, line 5 up, *for* "556,313," *read* "556,513."

,, 118, line of 155,377, column of $\frac{p-1}{\xi}$, *for* "8 . 9 . 23," *read* "8 . 9 . 13."

Page 119, column of p, *for* "1,554,713," *read* "1,554,913."

,, 122, table. Remove the whole line of $p = 23,057$ to left-hand side of table, with the correction "$b = 16$," *instead of* "256."

,, 166, line 2, equation (490), *read*

$$\text{“}\sigma = \frac{(1+a)^2(1+a+2a^2-2a^3)}{(1-a)^3(1-a-a^2)^3}.\text{”}$$

,, 168, equation (519), *read*

$$\text{“}Z_2 = z^2 - \frac{\sqrt{(51)}+2\sqrt{(15)}}{6}z + \tfrac{1}{3} + \frac{2\sqrt{(85)}}{15}.\text{”}$$

,, 178, equation (598), *read*

$$\text{“} = -q^3+3q^2+6q+13-3\sqrt{Q}.\text{”}$$

,, 427, line 4, *read* "$+\frac{y''}{\lambda'^2}$."

,, 429, equation (186), *read*

$$\text{“}c = b_3,\ c_4,\ c_5,\ b_4,\ 0,\ \ldots\text{.”}$$

,, 475, equation (480), *read* "$c_{22} = \frac{1}{c\,(\theta+\frac{2}{5}\omega)}$."

,, 487, 3rd line from bottom, *for* "(2,2)" *read* "(2,3)."

,, 488, line 19, *after* "the cells on (p,q)," *insert* "(p, q being prime to each other)."

,, 489, line 23, *for* "10th" *read* "5th."

,, 490, 2nd line, *for* "D" *read* "E."

,, 494, line 6, *after* "line (2,3)," *insert* "2, 3 being factors of 6, (2,3), (3,2) will be called *factor paths*."

,, 494, bottom row of Diagram K, *read* "λ_4" *for* "λ," in last cell but one.

,, 495, line 4, *for* "each appears," *read* "each pair appears."

,, 496, line 10, *after* "two rows of 12 cells," *insert* "the symbols in."

,, 505, last line, *for* "0,24," &c., *read* "0,30, the 0,0 cell of the upper square"; *and add*:—"When $n = 24$, the cells along 2,3 and 3,2 measured from 0,0 are different from those of 2,3 and 3,2 traced inward from the cell 0,24; but those of 4,3 and 3,4 from 0,0 and those traced inward from 0,24 are identical. When $n = 18$, the cells 3,2 and 2,3 from 0,0 are identical with those of 3,4 and 4,3 traced inward from 0,18; and the cells of 2,3 and 3,2 are identical with those of 4,3 and 3,4 traced inward from 0,18."

,, 590, equation (268), *transpose* "$\cos^{-1}$" *and* "$\sin^{-1}$."

,, 601, equation (308), *transpose* "$\cos^{-1}$" *and* "$\sin^{-1}$."

Explanations.

Page 496, line 16, *instead of* "II indicates," as far as "K, and the diagonals of L," *substitute* "In the line of sq.$(1,2)_\lambda$, $\Sigma\lambda$ denotes $\lambda, \lambda_1, \lambda_2, \ldots \lambda_6$; II under the paths 2,1, 5,1 indicates recurring groups of the same two different λ's; III under the paths 3,2, 1,0, 1,4 indicates recurring groups of the same three different λ's. These are seen in Diagram K; and similarly for the symbols on the line of sq.$(3,1)_b$.—*Vide* Diagram L."

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the paper either has been printed* in extenso, *or an abstract of it given, in the "Proceedings."*]

† A paper by Mr. Basset, with the above title, is printed in the *Math. Annalen*, Bd. XLVIII., pp. 89–96.

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