# THEORY OF THE ALGEBRAIC FUNCTIONS OF A COMPLEX VARIABLE

BY

JOHN CHARLES FIELDS, PH. D.
ASSOCIATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF TORONTO

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DEDICATED TO THE MEMORY OF MY PARENTS

# PREFACE.

The theory of the algebraic functions developed in the following pages is algebraic in its methods and perfectly general. It holds for any algebraic equation reducible or irreducible, however complicated its singularities may be and whatever its character at infinity. The development of the theory may be said to culminate in the complementary theorem, in Chapter XII, from which theorem a number of well-known theorems in the theory of the algebraic functions immediately follow as corollaries.

The principles here presented have been in the possession of the writer for some eight years past and had in fact, in a somewhat different form, already been written up with a view to publication in the summer of 1898. Other matters however intervened and the work was laid aside for a long period. Since then further interruptions have occurred, — the theory however has been twice rewritten in the interval and has probably lost nothing by the delay in publication. The writer has not felt it to be necessary to go into the theory of the Abelian integrals, his object having been attained on presenting the purely algebraic side of his subject.

The principle embodied in the statement of the limitation on the orders of coincidence which a reduced form can have simultaneously with the $n$ branches of the fundamental algebraic equation, is evidently of wider scope than the theory of the algebraic functions. Also the method of the deformation of a product, or its equivalent, should find its application elsewhere, for example in the theory of the algebraic numbers and in the theory of the algebraic functions of several variables. In the latter connection the writer might say that he possesses a simple representation of the branches of an algebraic function of any number of variables in the neighborhood of a singular manifold and hopes to be able to utilize this representation in combination with the methods employed in the present volume.

In conclusion the writer desires to express his thanks to Professor Mittag-Leffler and Professor Phragmén for the interest they have taken in his work. Acknowledgement is also due to the publishers, Messrs Mayer and Müller, for the obliging readiness with which they have always met the wishes of the author.

Toronto, May, 1906.

J. C. FIELDS.

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# CHAPTER I.

## Introductory.

Reduction of an algebraic equation to an equation of integral algebraic form. The integral algebraic equation $F(z, v) = 0$ of degree $N$ in the two variables and of degree $n$ in the dependent variable $v$ alone. The *reduced form* of a rational function of $(z, v)$. The *orders of coincidence* of a rational function of $(z, v)$ with the $n$ branches of our equation corresponding to a given value of the independent variable $z$. The orders of coincidence $\mu_1, \ldots \mu_n$ of the $n$ branches each with the product of the remaining $n-1$ branches.

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Being given any algebraic equation

$$(1) \qquad E(z, u) = e_\nu u^\nu + e_{\nu-1} u^{\nu-1} + \cdots + e_0 = 0$$

in which the coefficients are integral rational functions of a variable $z$ it may happen that a multiple factor is present. Such factor would be detected on applying to $E(z, u)$ and $E'_u(z, u)$ the process for finding the greatest common divisor. Ridding ourselves of the repeated factors so discovered we obtain an equation which is free of multiple factors and which we shall find it convenient to write in the form

$$(2) \qquad f(z, u) = u^n + f_{n-1} u^{n-1} + \cdots + f_0 = 0$$

where the coefficients $f$ are rational functions of $z$ — integral or fractional. The least common denominator of these coefficients we shall represent by the letter $g$.

Multiplying the left-hand side of equation (2) by $g^n$ and writing $v = gu$  we obtain an algebraic equation

(3) $$F(z, v) = v^n + F_{n-1} v^{n-1} + \cdots + F_0 = 0$$

in which the coefficients $F$ are polynomials in $z$. The degree of this equation in $z$ and $v$ we shall indicate by the letter $N$. We shall then have $N \gtreqqless n$.

It is with equation (3) that we principally have to do in the present volume, and as we shall have occasion later on to refer to its constant coefficients it will be convenient to write it also in the form

(4) $$F(z, v) = \Sigma a_{s,t} z^s v^t = 0.$$

This equation then may be any integral algebraic equation reducible or irreducible subject only to the condition that it contain no multiple factor. In the neighborhood of any point in the $z$-plane it will split up into $n$ branches whose equations may be written in the form

(5) $$v - P_1 = 0, \ v - P_2 = 0, \ \ldots \ v - P_n = 0$$

where for a finite point $z = a$ the $P$'s represent series in $z - a$ involving only positive exponents, integral or fractional, while for the point at $\infty$ they represent series in $\frac{1}{z}$ involving integral or fractional exponents which may be either positive or negative, the number of the latter however being necessarily finite.

The $n$ branches will be made up of a number of simple branches involving only integral exponents and various groups of branches which constitute complete cycles. In the following there will be no distinction made between the two kinds of branches, a simple branch being regarded as constituting by itself a cycle of order 1.

By virtue of equation (3) any rational function of $(z, v)$ may be reduced, and that in one way only, to the form

(6) $$H(z, v) = h_{n-1} v^{n-1} + h_{n-2} v^{n-2} + \cdots + h_0$$

where the coefficients $h$ are rational functions of $z$. This of course is on the supposition that the reduced form continues to represent the same algebraic function of $z$ as the given form for each of the irreducible factors of the equation (3) simultaneously, in the case where this equation is reducible.



When we refer to the *reduced form* of a rational function of $(z, v)$ it will always be the form (6) which is meant, and it may be taken for granted that the functions with which we shall here have to do, have already been reduced to this form where nothing in the text or formulae implies the contrary.

We shall speak of the order of a function of $(z, v)$ relative to a branch $v-P=0$ or of the *order of coincidence* of the function with the branch, meaning thereby the lowest exponent which presents itself on developing the function in powers of $z-a$ or $\frac{1}{z}$, as the case may be, after substituting for $v$ from the equation to the branch. The order of coincidence of a function with a branch can then be either positive or negative, integral or fractional. Instead of saying the order of coincidence of a function with a branch it may sometimes be convenient to speak of the order of coincidence of the branch with the function and we shall employ the two forms of expression indifferently to denote identically the same thing.

The order of coincidence of a given function with the curve $F(z, v)=0$ for a given value of $z$ we shall define as the smallest order of coincidence of the function with one of the $n$ corresponding branches of the curve.

The order of coincidence of the function $v-P_k$ with the branch $v-P_l=0$ and that of the function $v-P_l$ with the branch $v-P_k=0$ are the same being both equal to the smallest exponent of $z-a$ or $\frac{1}{z}$, as the case may be, in the difference $P_k-P_l$. For the sake of brevity then and without any risk of ambiguity one may speak of the order of coincidence of the two functions $v-P_k$ and $v-P_l$ or of the order of coincidence of the two branches $v-P_k=0$ and $v-P_l=0$ with one another, meaning thereby the order of coincidence of the function $v-P_k$ with the branch $v-P_l=0$ or that of the function $v-P_l$ with the branch $v-P_k=0$.

The order of coincidence of the branch $v-P_k=0$ with $v-P_l$ we shall indicate by $\mu_{k,l}=\mu_{l,k}$. Its order of coincidence with

(7) $$(v-P_1)\ \dots\ (v-P_{k-1})(v-P_{k+1})\ \dots\ (v-P_n)$$



the product namely of the $n-1$ factors of $F(z,v)$ conjugate to $v-P_k$, will evidently be equal to the sum of its orders of coincidence with the separate factors. On indicating this order of coincidence by $\bar{\mu}_k$ we shall have

(8) $$\bar{\mu}_k = \mu_{k,1} + \cdots + \mu_{k,k-1} + \mu_{k,k+1} + \cdots + \mu_{k,n}.$$

The quantities $\mu_{k,l}$ which here appear are all finite as otherwise we should have a factor $v-P_l$, $(l \neq k)$, which is identical with $v-P_k$, in which case $F(z,v)$ would have a repeated factor contrary to hypothesis.

The orders of coincidence

(9) $$\bar{\mu}_1,\ \bar{\mu}_2,\ \dots\ \bar{\mu}_n$$

of the several branches of the equation (3) each with the product of the remaining $n-1$ branches are then all finite, and of the order of coincidence $\bar{\mu}_k$ corresponding to the branch $v-P_k=0$ it may further be said that it will be an integral multiple of $\frac{1}{\nu_k}$ where $\nu_k$ is the order of the cycle to which the branch belongs. To prove this it is only necessary to remark that the order of coincidence of the branch $v-P_k=0$ with the product (7) is made up of the sum of its orders of coincidence with the remaining $\nu_k-1$ branches of the cycle to which it belongs together with its order of coincidence with the product of the other $n-\nu_k$ branches of the curve. The $n-\nu_k$ branches in question constitute a number of complete cycles and their product therefore involves no fractional exponents. The order of coincidence of the branch $v-P_k=0$ with this product must then be an integral multiple of $\frac{1}{\nu_k}$, its order of coincidence with each of the branches of its own cycle is such a multiple of $\frac{1}{\nu_k}$ and its aggregate order of coincidence with the $n-1$ factors of the product (7) must therefore also be an integral multiple of $\frac{1}{\nu_k}$.

# CHAPTER II.

## Classification of values of $z$.

The cycles into which the $n$ branches corresponding to a given value of the variable $z$ group themselves. Three categories of values of the variable $z$, depending on the character of the corresponding set of values $\bar{\mu}_1, \bar{\mu}_2, \ldots \bar{\mu}_n$.

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We shall now examine a little more closely the nature of the numbers $\bar{\mu}_k$ which have been defined in the chapter preceding, confining ourselves for the present however to those numbers which correspond to finite values of the variable $z$.

Suppose that the $n$ branches of the curve corresponding to the value $z=a$ group themselves into $r$ cycles having the orders $\nu_1, \nu_2, \ldots \nu_r$ respectively. To this value of $z$ will correspond $r$ values $b_1, b_2, \ldots b_r$ of $v$ and these values may or may not happen to be all unequal.

The equations (5) in Chapter I are supposed to represent the $n$ branches of the curve and of these the equations representing a cycle of the order $\nu$ may be written more fully in the form

$$(1)\qquad \begin{cases} 0=v-b-\alpha\varepsilon_1(z-a)^{\frac{1}{\nu}}-\ldots\ldots \\ 0=v-b-\alpha\varepsilon_2(z-a)^{\frac{1}{\nu}}-\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0=v-b-\alpha\varepsilon_\nu(z-a)^{\frac{1}{\nu}}-\ldots\ldots \end{cases}$$

where $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_\nu$ are the $\nu$th roots of unity. This cycle we shall call an *isolated cycle* if its $\nu$ branches are the only ones of the $n$ branches of the curve which pass through the point $(a, b)$.

Of the set of $n$ values $\bar{\mu}$ those corresponding to the several branches of a cycle are evidently all equal to one another.



First occupying ourselves with the branches of an isolated cycle we can immediately dispose of the case in which $\nu = 1$ that is of the case of an isolated simple branch. Namely, the order of coincidence of an isolated simple branch with each of the other $n-1$ branches and therefore with their product is equal to 0. Assuming then in regard to our isolated cycle that we have $\nu \geqq 2$ we shall distinguish two cases according as we have $\alpha \neq 0$ or $\alpha = 0$ in the equations (1) above. In both these cases we may say that the order of coincidence of a branch of the cycle with the product of the other $n-1$ branches of the curve is equal to the sum of its orders of coincidence with the other $\nu - 1$ branches of the cycle, for its order of coincidence with a branch of another cycle will be equal to 0.

In the former of the two cases just mentioned the order of coincidence of two branches of the cycle with one another will evidently be equal to $\frac{1}{\nu}$ and the sum of the orders of coincidence of one of these branches with the remaining $\nu - 1$ branches of the cycle, and therefore with the remaining $n-1$ branches of the curve, will be equal to $\frac{\nu-1}{\nu}$, a value which is $<1$. In the latter case the order of coincidence of two branches of the cycle with one another will be $\geqq \frac{2}{\nu}$ and the sum of the orders of coincidence of one of these branches with the remaining $\nu - 1$ branches of the cycle, and therefore with the remaining $n-1$ branches of the curve, will be $\geqq \frac{2(\nu-1)}{\nu} = 2 - \frac{2}{\nu}$ a value which is $\geqq 1$.

In the former of the two cases under consideration the product of the $\nu$ branches (1) will contain a term in $z-a$ to the first power, as will therefore also the product of the $n$ branches of the curve when arranged according to powers of $z-a$ and $v-b$. The point $(a, b)$ in this case then will not be a multiple point on the curve $F(z, v) = 0$. In the latter of the two cases however the product of the $\nu$ branches will contain no such term of the first order in $z-a$ alone and the terms of lowest order in

$z-a$ and $v-b$ will never have an order which is $<2$. In this case then the point $(a, b)$ will be a multiple point on the curve.

This disposes of the isolated cycle in so far as is necessary for our purpose and we shall now consider the case in which the branches of a second cycle pass through the point $(a, b)$. The order of this second cycle we shall indicate by $\nu'$.

In the first place assuming $\nu \leqq \nu'$ the order of coincidence of a branch of the first cycle with a branch of the second will be $\geqq \frac{1}{\nu'}$; its order of coincidence therefore with the product of the $\nu'$ branches of the second cycle will be $\geqq 1$. Its order of coincidence with the product of the remaining $\nu - 1$ branches of the first cycle will be $\geqq \frac{\nu-1}{\nu}$ and its order of coincidence with the product of the other $n-1$ branches of the curve will therefore be $\geqq 1 + \frac{\nu-1}{\nu}$, a value which is $\geqq 1$. Secondly assuming $\nu > \nu'$ the order of coincidence of a branch of the first cycle with a branch of the second will be $\geqq \frac{1}{\nu}$, its order of coincidence therefore with the product of the $\nu'$ branches of the second cycle $\geqq \frac{\nu'}{\nu}$. Its order of coincidence then with the product of the branches of the second cycle and the remaining $\nu - 1$ branches of the first cycle, and therefore its order of coincidence with the product of the other $n-1$ branches of the curve, will be $\geqq \frac{\nu-1}{\nu} + \frac{\nu'}{\nu} = 1 + \frac{\nu'-1}{\nu}$ a value which is $\geqq 1$.

The order of coincidence of a branch of a cycle which is not isolated with the product of the remaining $n-1$ branches of the curve is then always $\geqq 1$. Also the product of the branches of either cycle separately contains no terms of order $<1$ in $z-a$ and $v-b$ so that the combined products of the branches of the two cycles, and therefore the product of the $n$ branches of the curve, arranged according to powers of $z-a$ and $v-b$, will contain no terms of order $<2$. The point $(a, b)$ will therefore in this case be a multiple point on the curve.

In accord with the results just obtained, any finite value of $z$ may be assigned to one of three categories for which the corresponding sets of orders of  coincidence $\bar{\mu}$ are characterized as follows:

A. To $z = a$ correspond $n$ different values of $v$ and therefore $n$ isolated simple branches. The orders of coincidence $\bar{\mu}_1, \ldots \bar{\mu}_n$ in this case are all equal to 0.

B. To $z = a$ correspond less than $n$ different values of $v$ but no multiple point. The orders of coincidence $\bar{\mu}_1, \ldots \bar{\mu}_n$ will then be all $< 1$ but not all of them will be equal to 0.

C. To $z = a$ corresponds, among other points, at least one multiple point. The orders of coincidence $\bar{\mu}_1, \ldots \bar{\mu}_n$ in this case will include ones which are $\geqq 1$.

The points of the curve which correspond to a value $z = a$ which belongs to the category (A) are neither multiple nor branch points. In the case of the category (B) they include among them a non-multiple branch point, and in the case of the category (C) a multiple point which may or may not happen to be at the same time a branch point. The values of $z$ belonging to the categories (B) and (C) are finite in number being, as we know, the roots of the discriminant of the equation $F(z, v) = 0$.

# CHAPTER III.

## Method of deformation of a product.

Proof that it is always possible to construct an integral rational function of $(z, v)$, in whose reduced form the coefficient of $v^{n-1}$ is not divisible by $z-a$, and whose orders of coincidence with the $n$ branches corresponding to the value $z=a$ are precisely the same as those of an arbitrarily assigned integral rational function whose reduced form is not divisible by $z-a$. Representation of such function in the form of a product. *Deformation of a product.* The orders of coincidence of an integral rational function which is not divisible by $z-a$, with the $n$ branches corresponding to the value $z=a$, cannot be simultaneously greater than the numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$.

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We shall now consider the integral rational functions of $(z, v)$, with a view to ultimately determining more precisely the nature of the sets of orders of coincidence which such functions may possess for the $n$ branches of the curve corresponding to a given value $z=a$.

Suppose

(1) $$G(z, v) = g_{n-1} v^{n-1} + g_{n-2} v^{n-2} + \cdots + g_0$$

to be an integral rational function of $(z, v)$ in its reduced form, the coefficients $g$ being therefore polynomials in $z$. Of the orders of coincidence of this function with the $n$ branches of the curve corresponding to a finite value of the independent variable, evidently none can be negative. For the value $z=\infty$ however one at least of the orders of coincidence of the corresponding branches with the function must be negative unless the function is everywhere finite, in which case, as is well-known, it must represent a constant — it may be a different constant — for each of the irreducible factors of the fundamental equation.

The effect of multiplication or division by a factor $z-a$ on the orders of coincidence of a function is evident. Namely on multiplying the function by this factor its orders of coincidence with the branches corresponding to the value $z=a$ would be each increased by 1, the orders of coincidence corresponding to the other finite values of $z$ would remain unchanged and its orders of coincidence with the branches at $\infty$ would be each decreased by 1. The reverse effect would result on dividing by the factor in question. For the study of the possible sets of orders of coincidence corresponding to a finite value $z=a$, which may happen to be offered by an integral rational function of $(z, v)$, it will then suffice to confine our considerations to functions $G(z, v)$ in which the coefficients $g$ do not have the common factor $z-a$.



Assuming now that we have to do with a given function $G(z, v)$ in which the coefficients $g$ are not all divisible by $z-a$, it may or may not happen that the coefficient $g_{n-1}$ is divisible by this factor. In the former case we shall prove that there always exists an integral rational function of $(z, v)$, in whose reduced form the coefficient of $v^{n-1}$ is not divisible by $z-a$, and whose orders of coincidence with the $n$ branches corresponding to the value $z=a$ are precisely the same as the orders of coincidence of the given function with these branches.

Suppose namely that $g_{n-\lambda}$ is the first coefficient in $G(z, v)$ which is not divisible by $z-a$. Multiplying this function by $v^{\lambda-1}$ we obtain

$$g_{n-1} v^{n+\lambda-2} + \cdots + g_{n-\lambda+1} v^{n} + g_{n-\lambda} v^{n-1} + \cdots + g_0 v^{\lambda-1}.$$

Reduced by aid of the equation $F(z, v)=0$, this expression will take the form

$$\Gamma(z, v) = \gamma_{n-1} v^{n-1} + \gamma_{n-2} v^{n-2} + \cdots + \gamma_0$$

in which the coefficient $\gamma_{n-1}$ evidently has the form $(z-a)\gamma + g_{n-\lambda}$, since the coefficients $g_{n-1}, \ldots g_{n-\lambda+1}$ are by hypothesis all divisible by $z-a$.

The multiplication by the factor $v^{\lambda-1}$ does not affect the orders of coincidence with the branches corresponding to the value $z=a$, unless $v=o$ is a value of $v$ corresponding to this value of $z$. In the latter case we may replace the multiplier $v^{\lambda-1}$ by $(v-c)^{\lambda-1}$ where $v=c$ is not one of the values of $v$ corresponding to $z=a$. This multiplier will not affect the orders of coincidence corresponding to the value $z=a$ and, like the factor $v^{\lambda-1}$, it will also evidently give us for coefficient of $v^{n-1}$ in the reduced form an expression of the form $(z-a)\gamma+g_{n-\lambda}$. The function $\Gamma(z,v)$ so obtained is then an integral rational function of $(z,v)$ in its reduced form, in which the coefficient of $v^{n-1}$ is not divisible by $z-a$ and whose orders of coincidence with the branches corresponding to the value $z=a$ are precisely the same as those of the given function.



We have proved, that being given any integral rational function of $(z,v)$ which in its reduced form is not divisible by $z-a$, we can always find an integral rational reduced form in which the coefficient of $v^{n-1}$ is not divisible by $z-a$, and whose orders of coincidence with the several branches of $F(z,v)=0$ corresponding to the value $z=a$ are precisely the same as those of the given function. The study of the possible sets of orders of coincidence, which an integral rational function of $(z,v)$ may have with the $n$ branches of the fundamental curve corresponding to a value $z=a$, reduces itself then to the consideration of the orders of coincidence of functions of the type $G(z,v)$ in which the coefficient $g_{n-1}$ of $v^{n-1}$ is not divisible by $z-a$.

A function $G(z,v)$ may be factored in the form

(2) $$G(z,v)=g_{n-1}(v-Q_1)(v-Q_2)\ldots(v-Q_{n-1})$$

where the $Q$'s are series in powers of $z-a$. The exponents of $z-a$ which appear may be integral or fractional, and the factors of the product group themselves in cycles. Unless $g_{n-1}$ be divisible by $z-a$ however none of the exponents will be negative.

Supposing $g_{n-1}$ not to be divisible by $z-a$, the order of coincidence of $G(z,v)$ with any branch of the curve $F(z,v)=0$ corresponding to the value $z=a$, will be equal to the sum of the orders of coincidence of the $n-1$ factors $v-Q$ with the said branch. To determine the possible sets of orders of coincidence corresponding to a value $z=a$ which may be presented by a reduced form $G(z,v)$, which is not divisible by $z-a$, it will be sufficient then to consider those that may be offered by a product of the form

(3) $$(v-Q_1)(v-Q_2)\ldots(v-Q_{n-1}).$$

With reference to the product (2) we have said that the factors $v-Q$ group themselves in cycles. It will be convenient for us however in the consideration of the product (3), not to limit ourselves to the case in which the factors $v-Q$ constitute a set of complete cycles. In this product then we shall simply assume that the $Q$'s represent series in powers of $z-a$, integral or fractional — here too we do not exclude the case in which a finite number of negative exponents may present themselves. Whether or not the series $Q$ happen to represent algebraic functions is immaterial for the moment.

The point at $\infty$ it is to be understood is not excluded in the reasoning which follows, and in a product of the form (3) having reference to this point $z-a$ is of course replaced by $\frac{1}{z}$.

We shall now prove that the orders of coincidence of a product of the form (3) with the $n$ branches of the curve corresponding to a given value of $z$ cannot simultaneously be greater than the numbers $\bar{\mu}_1, \dots \bar{\mu}_n$ respectively as defined in Chapter I, for the value of the variable in question.

It will be convenient to employ the expression *deformation of a product* to describe the process of replacing in a given product one or more of its factors by as many new ones. Being given any product of the form (3) we shall show by a succession of deformations, none of which diminishes any of the orders of coincidence of the product with the several branches of the curve corresponding to the value of the variable $z$ in question, that we may derive another product whose orders of coincidence with the several branches are not simultaneously greater than the corresponding values in the set of numbers $\bar{\mu}_1, \dots \bar{\mu}_n$. It will then follow that the same holds good in regard to the orders of coincidence of our original product with the several branches.

To prove our proposition we shall compare the factors of the given product with factors of the product

(4) $$F(z, v) = (v - P_1)(v - P_2) \dots (v - P_n).$$

It may be that a factor $v-Q$ of the product (3) has a greater order of coincidence with a certain factor $v-P$ of the product (4) than with any other of the factors of this product. In that case we shall substitute the factor $v-P$ for the said factor in our product. This substitution does not diminish the order of coincidence of the product with any of the $n$ branches of the curve and gives an infinite order of coincidence with the branch $v-P=0$. — In case however the factor $v-Q$ in question has the same greatest order of coincidence with several of the factors in the product (4), we may substitute any arbitrary one of these factors for the factor $v-Q$ without diminishing any of the orders of coincidence of the product with the several branches. Suppose namely that the order of coincidence of the factor $v-Q$ with the branch $v-P_k=0$ is at least as great as its order of coincidence with any of the other $n-1$ branches. If then $v-P_l=0$ be one of these $n-1$ branches, it follows that the lowest exponent in the difference $P_l-Q$ is not greater than the lowest exponent in the difference $P_k-Q$, and therefore not greater than the lowest exponent in the difference $(P_l-Q)-(P_k-Q)=P_l-P_k$. The order of coincidence of the factor $v-Q$ with the branch $v-P_l=0$ is then not greater than the order of coincidence of $v-P_k$ with this branch, and as a consequence the order of coincidence of our product with the branch $v-P_l=0$ loses nothing on replacing $v-Q$ by $v-P_k$ in the product.

Substituting then for each of the $n-1$ factors $v-Q$ a factor $v-P$ in the manner just indicated, the original product (3) will be replaced by a product of $n-1$ factors selected from among the $n$ factors $v-P$, including it may be repetitions, and having orders of coincidence with the several branches which are in no case less than the orders of coincidence of the original product with these branches.

Of the $n$ factors of the product (4) one at least must be lacking in the product of $n-1$ factors which we have just constructed, and still more will be lacking if the same factor presents itself more than once among the $n-1$ factors of our product. In the latter case we shall rid our product of repeated factors by a further series of deformations. If, namely, a certain factor $v-P_k$ appears more than once in the product, we shall replace one of its repetitions by a factor $v-P$ which does not as yet appear in the product and namely by that one, or if there be several such, by one of those, with which it has the greatest order of coincidence. This deformation rids us of one of the repetitions in the product and evidently does not subtract from the orders of coincidence of the product with any of the branches, unless it be in the case of ones with which it still retains infinity as order of coincidence. Not accounting this a diminution, we may say then that the deformation in question has not diminished any of the orders of coincidence of the product with the several branches.

By a succession of such deformations we may rid ourselves of all repeated factors, without diminution in any of the orders of coincidence, obtaining as the factors of our final product $n-1$ different factors of the product (4). Supposing $v-P_s$ to be the lacking factor in the product this will have the form

(5) $$(v-P_1)\ldots(v-P_{s-1})(v-P_{s+1})\ldots(v-P_n).$$

The orders of coincidence of this product with all the branches of the curve excepting $v-P_s=0$ are infinite, and with this branch its order of coincidence is $\bar{\mu}_s$. Since none of the orders of coincidence with the several branches has been diminished by the successive deformations in passing from the original product to the product just considered, it follows that the order of coincidence of the given product with the branch $v-P_s=0$ must be $\leqq \bar{\mu}_s$.

Of any product of the form (3) then we may say, that one at least among its orders of coincidence with the several branches cannot exceed the corresponding number in the system $\bar{\mu}_1, \bar{\mu}_2, \ldots \bar{\mu}_n$. In other words the orders of coincidence of a product of the form (3) with the branches $v-P_1=0, \ldots v-P_n=0$, cannot simultaneously be greater than the numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$ respectively.

In particular any rational function of $(z, v)$ of the form

(6) $$v^{n-1}+h'_{n-2}v^{n-2}+\ldots+h'_0$$

where the coefficients $h'$ are rational functions of $z$, can be represented by a product of the type (3), and its orders of coincidence therefore with the several branches of the curve corresponding to a given value of the variable $z$, cannot simultaneously be greater than the respective members of the corresponding system of numbers $\bar{\mu}_1, \dots \bar{\mu}_n$. This holds whether the value of the variable $z$ in question be finite or infinite.



Since any integral rational function of $(z, v)$, involving $v^{n-1}$, may be obtained from a function of the form (6) by multiplication with an integral rational function $g_{n-1}$ of $z$, we conclude that any product of the form (2), and therefore any function of the form (1) in which $g_{n-1}$ is not divisible by $z-a$, has orders of coincidence with the branches of the curve corresponding to the finite value $z=a$, which are not simultaneously greater than the numbers of the corresponding system $\bar{\mu}_1, \dots \bar{\mu}_n$. It follows then that any function of the form (1), — that is, that any integral rational function of $(z, v)$ — which is not divisible by $z-a$, cannot have orders of coincidence with the branches of the curve corresponding to the value $z=a$, which are simultaneously greater than the numbers of the system $\bar{\mu}_1, \dots \bar{\mu}_n$ corresponding to this value of the variable.

# CHAPTER IV.

## Sets of orders of coincidence for a value $z=a$.

The $n$ branches corresponding to a given value $z=a$ regarded as grouped in $r$ cycles of orders $\nu_1, \ldots \nu_r$ respectively. The orders of coincidence of an integral rational function which is not divisible by $z-a$, with the branches of the several cycles, cannot simultaneously be greater than the numbers $\mu_1, \ldots \mu_r$ respectively. Construction of an integral rational function which is not divisible by $z-a$, which has $\mu_s - \frac{\sigma}{\nu_s}$ as its order of coincidence with the branches of the cycle of order $\nu_s$ — where $\sigma$ may have any one of the values $0, 1, \ldots \nu_s - 1$ — and whose orders of coincidence with the $n-\nu_s$ branches of the remaining cycles may be as large as we please. The greatest value of the exponent $i$ which can present itself in a rational function of the form $(z-a)^{-i} G(z,v)$ in which $G(z,v)$ is a polynomial which is not divisible by $z-a$, and where the function is finite for all the branches corresponding to the value $z=a$, is the greatest of the integers $[\mu_1], \ldots [\mu_r]$.

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We have shewn in the chapter preceding, that a product of the form (3) has orders of coincidence with the branches $v-P_1=0, \ldots v-P_n=0$ which cannot simultaneously be greater than the corresponding numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$ respectively. We have further constructed a product of the said form, namely

$$(v-P_1) \ldots (v-P_{s-1})(v-P_{s+1}) \ldots (v-P_n)$$

whose orders of coincidence with the several branches are infinite, excepting in the case of the branch $v-P_s=0$ with which its order of coincidence is equal to the corresponding number $\bar{\mu}_s$. The excepted branch may of course be any one of the $n$ branches.

In what precedes we have been dealing with the individual branches of the curve independently of their ordering in cycles. We shall now suppose that the $n$ branches corresponding to a given value of the variable $z$ group themselves into $r$ cycles of orders $\nu_1, \ldots \nu_r$ respectively, and the orders of coincidence of the individual branches of these cycles, each with the product of the other $n-1$ branches of the curve, we shall indicate by $\mu_1, \ldots \mu_r$ respectively. — That the orders of coincidence of the branches of the same cycle, each with the product of the remaining $n-1$ branches of the curve, are equal, is evident. — The numbers $\mu_1, \ldots \mu_r$ repeated $\nu_1, \ldots \nu_r$ times respectively will be identical with some arrangement of the numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$.

The orders of coincidence of a rational function of $(z, v)$ with the several branches of a cycle are the same. From the chapter preceding it follows that the orders of coincidence of a rational function of the form (6), with the branches of the several cycles, cannot simultaneously be greater than the numbers $\mu_1, \ldots \mu_r$ respectively corresponding to the value of the variable $z$ in question. Also for a finite value $z=a$ we derive that the orders of coincidence of a function of the form (1) in the chapter preceding, — that is of an integral rational function of $(z, v)$ which is not divisible by $z-a$ — with the branches of the several cycles cannot simultaneously be greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively.

We have seen that the orders of coincidence of all $n$ branches of the curve with a product of the form (3) in Chapter III, cannot be indefinitely great, and in fact that they cannot simultaneously be greater than the numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$ respectively corresponding to the value of the variable in question. Further, in formula (5) of that chapter we have constructed a product of the said form whose orders of coincidence with $n-1$ out of the $n$ branches are infinite, while its order of coincidence with the remaining branch $v-P_s=0$ is equal to $\bar{\mu}_s$. We shall now shew that it is possible to construct a product of the form in question, which at the same time represents a *rational* function of $(z, v)$, and whose orders of coincidence with the branches of an arbitrarily chosen set of $r-1$ out of the $r$ cycles are finite but as large as we may please, while its order of coincidence with the branches of the other cycle is equal to $\mu_s$, in case this cycle be the one of order $\nu_s$.

Our lettering of the $n$ branches of the curve has heretofore been arbitrary, and we may therefore for the moment assume it to have been so ordered that the $\nu_s$ branches $v-P_1=0, \ldots v-P_{\nu_s}=0$ constitute the cycle of order $\nu_s$. Consider the product of $n-1$ factors

(1) $$(v-P_2)\ (v-P_3) \ldots (v-P_{\nu_s})\ (v-P_{\nu_s+1}) \ldots (v-P_n).$$

The orders of coincidence of this product with all the branches excepting $v-P_1=0$ are infinite and with this branch its order of coincidence is $\mu_s$.

By deformation of the product (1) we shall derive another product

(2) $$(v-Q_2)\ (v-Q_3) \ldots (v-Q_{\nu_s})\ (v-Q_{\nu_s+1}) \ldots (v-Q_n)$$

where the $Q$'s are obtained by discarding terms of higher order in the series $P_2, P_3, \ldots P_n$. The $Q$'s in the last $n-\nu_s$ factors namely are obtained on dropping terms beyond certain orders in the $n-\nu_s$ series $P_{\nu_s+1}, \ldots P_n$, such orders being taken the same in the case of all the series belonging to the same cycle. The partial product

(3) $$R(z,v)=(v-Q_{\nu_s+1}) \ldots (v-Q_n)$$

thus arrived at will represent a rational function of $(z,v)$, for the $n-\nu_s$ series $P_{\nu_s+1}, \ldots P_n$ constitute a set of complete cycles. The function will further be an *integral* rational function of $(z,v)$ in the case where we have to do with a finite value of the variable $z=a$. Also the orders of coincidence of the function $R\ (z,v)$ with the $n-\nu_s$ branches $v-P_{\nu_s+1}=0, \ldots v-P_n=0$ may be made as large as we like, by retaining in the $Q$'s terms of sufficiently high order in $z-a$ or $\frac{1}{z}$ as the case may be.

The $Q$'s in the $\nu_s-1$ factors of the partial product

(4) $$(v-Q_2) \ldots (v-Q_{\nu_s})$$

we shall suppose to be obtained by retaining terms of sufficiently high order in the $\nu_s-1$ series $P_2, \ldots P_{\nu_s}$. In the combined system of $n-1$ series $Q_2, Q_3, \ldots Q_n$ we shall assume — what is evidently permissible — that terms of sufficiently high order from the series $P_2, P_3, \ldots P_n$ have been retained in order that the order of coincidence of the product (2) with the branch $v-P_1=0$ may be equal to $\mu_s$, and in particular with regard to the

$\nu_s - 1$ series $Q_2, \ldots Q_{\nu_s}$ we shall assume that terms of sufficiently high order from the series $P_2, \ldots P_{\nu_s}$ have been retained in order that the orders of coincidence of the product (2) with the $\nu_s - 1$ branches $v - P_2 = 0, \ldots v - P_{\nu_s} = 0$ may be all greater than $\mu_s$.

In the product (2) then as we have constructed it, the series $Q$ involve but a finite number of terms. Its order of coincidence with the branch $v - P_1 = 0$ is equal to $\mu_s$; with the branches $v - P_2 = 0, \ldots v - P_{\nu_s} = 0$ its orders of coincidence are greater than $\mu_s$ and may be, for that matter, as much greater as we please, while its orders of coincidence with the remaining $n - \nu_s$ branches of the curve may also be regarded as indefinitely large.

If the branches of which we have been speaking have reference to a finite value of the variable $z = a$, the function represented by the partial product (4) may evidently be written in the form

$$B_0(z,v) + B_1(z,v)(z-a)^{\frac{1}{\nu_s}} + \cdots + B_{\nu_s - 1}(z,v)(z-a)^{\frac{\nu_s - 1}{\nu_s}}$$

where $B_0(z,v)$, $B_1(z,v)$, $\ldots B_{\nu_s-1}(z,v)$ are integral rational functions of $(z,v)$. If we have to do with the point at $\infty$, it will only be necessary here and in what immediately follows to replace $z - a$ by $\frac{1}{z}$. In this case also the functions $B_0(z,v), \ldots B_{\nu_s-1}(z,v)$ would be rational but not in general integral.

Substituting $R(z,v)$ and the expression just written for the partial products (3) and (4) respectively in (2), we obtain as total product an expression of the form

$$(5) \quad B(z,v) = \{B_0(z,v) + B_1(z,v)(z-a)^{\frac{1}{\nu_s}} + \cdots + B_{\nu_s-1}(z,v)(z-a)^{\frac{\nu_s-1}{\nu_s}}\}.\ R(z,v).$$

It is always possible then to construct a function of the form (5) whose order of coincidence with the branch $v - P_1 = 0$ is equal to $\mu_s$, while its orders of coincidence with the remaining $n-1$ branches are as large as we may please; though so far as regards the $\nu_s - 1$ branches $v - P_2 = 0, \ldots v - P_{\nu_s} = 0$ it will be sufficient for our purpose that its orders of coincidence with these branches be greater than $\mu_s$.

Representing the $\nu_s$th roots of unity by $\varepsilon_1 = 1, \varepsilon_2, \ldots \varepsilon_{\nu_s}$ construct the $\nu_s$ functions

$$(6)\quad \begin{cases} B'(z,v) = \{B_0(z,v) + B_1(z,v)\,\varepsilon_1(z-a)^{\frac{1}{\nu_s}} + \cdots + B_{\nu_s-1}(z,v)\,\varepsilon_1^{\nu_s-1}(z-a)^{\frac{\nu_s-1}{\nu_s}}\}\,.\,R(z,v) \\ B''(z,v) = \{B_0(z,v) + B_1(z,v)\,\varepsilon_2(z-a)^{\frac{1}{\nu_s}} + \cdots + B_{\nu_s-1}(z,v)\,\varepsilon_2^{\nu_s-1}(z-a)^{\frac{\nu_s-1}{\nu_s}}\}\,.\,R(z,v) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B^{(\nu_s)}(z,v) = \{B_0(z,v) + B_1(z,v)\,\varepsilon_{\nu_s}(z-a)^{\frac{1}{\nu_s}} + \cdots + B_{\nu_s-1}(z,v)\,\varepsilon_{\nu_s}^{\nu_s-1}(z-a)^{\frac{\nu_s-1}{\nu_s}}\}\,.\,R(z,v). \end{cases}$$

The first of these functions is identical with the function $B(z,v)$ in (5); the other $\nu_s-1$ functions are derived from $B(z,v)$ by substituting $\varepsilon_2(z-a)^{\frac{1}{\nu_s}}, \ldots \varepsilon_{\nu_s}(z-a)^{\frac{1}{\nu_s}}$ respectively for $(z-a)^{\frac{1}{\nu_s}}$. The effect of such a substitution on the orders of coincidence of the branches of the cycle of order $\nu_s$ is simply to interchange these orders of coincidence with one another. Supposing then that the function $B'(z,v)$ has $\mu_s$ as its order of coincidence with the branch $v-P_1=0$ while its orders of coincidence with the branches $v-P_2=0, \ldots v-P_{\nu_s}=0$ are greater than $\mu_s$, the function $B^{(\varkappa)}(z,v)$ will have the order of coincidence $\mu_s$ with that branch which is derived from $v-P_1=0$ by substituting $\varepsilon_\varkappa(z-a)^{\frac{1}{\nu_s}}$ for $(z-a)^{\frac{1}{\nu_s}}$, while its orders of coincidence with the remaining $\nu_s-1$ branches of the cycle will be greater than $\mu_s$.

The only one of the $\nu_s$ functions $B'(z,v), B''(z,v), \ldots B^{(\nu_s)}(z,v)$ whose order of coincidence with the branch $v-P_1=0$ is equal to $\mu_s$ is the first, and the orders of coincidence of the remaining $\nu_s-1$ functions with this branch are all greater than $\mu_s$. Any linear combination with constant multipliers of these $\nu_s$ functions will then have $\mu_s$ as its order of coincidence with the branch $v-P_1=0$ in case the multiplier of $B'(z,v)$ is other than 0. Multiplying the functions by $\varepsilon_1^{\nu_s-\sigma}, \varepsilon_2^{\nu_s-\sigma}, \ldots \varepsilon_{\nu_s}^{\nu_s-\sigma}$ respectively and adding, we obtain such a linear combination

$$\varepsilon_1^{\nu_s-\sigma}B'(z,v) + \varepsilon_2^{\nu_s-\sigma}B''(z,v) + \cdots + \varepsilon_{\nu_s}^{\nu_s-\sigma}B^{(\nu_s)}(z,v).$$

Giving $\sigma$ any one of the values $0, 1, \ldots \nu_s-1$ and substituting for $B'(z,v), B''(z,v), \ldots B^{(\nu_s)}(z,v)$ the expressions by which they are defined in (6), the terms will all disappear with the exception of those involving the factor $(z-a)^{\frac{\sigma}{\nu_s}}$, for we have

$$\sum_{\varkappa=1}^{\nu_s} \varepsilon_{\varkappa}^{\nu_s-\sigma+\tau} = \nu_s \text{ or } 0$$

according as $\tau \equiv \sigma$ or $\not\equiv \sigma$ (mod. $\nu_s$).

It follows that

$$\nu_s B_\sigma(z, v)(z-a)^{\frac{\sigma}{\nu_s}}. R(z, v) = \varepsilon_1^{\nu_s-\sigma} B'(z, v) + \cdots + \varepsilon_{\nu_s}^{\nu_s-\sigma} B^{(\nu_s)}(z, v)$$

is a function whose order of coincidence with the branch $v-P_1=0$ is equal to $\mu_s$. The order of coincidence of the rational function $B_\sigma(z, v) . R(z, v)$ with this branch, and therefore also with the remaining $\nu_s-1$ branches of the cycle, will be equal to $\mu_s - \frac{\sigma}{\nu_s}$.

The orders of coincidence of the rational functions

(7) $$B_0(z, v) . R(z, v),\ B_1(z, v) . R(z, v),\ \ldots\ B_{\nu_s-1}(z, v) . R(z, v)$$

with the branches of the cycle of order $\nu_s$ will then have the values

$$\mu_s,\ \mu_s - \frac{1}{\nu_s},\ \ldots\ \mu_s - \frac{\nu_s-1}{\nu_s}$$

respectively. Furthermore with the remaining $n-\nu_s$ branches of the curve belonging to the other $r-1$ cycles, the orders of coincidence of the functions (7) may be supposed to be as large as we please in virtue of the factor $R(z, v)$.

On comparing the product (2) with its expression in the form (5), we see that the term $v^{n-1}$ will be contained in the function $F_0(z, v) . R(z, v)$. The first of the $\nu_s$ functions (7) may then be written in the form

$$B_0(z, v) . R(z, v) = v^{n-1} + h'_{n-2} v^{n-2} + \cdots + h'_0$$

and is therefore expressible as a product of the form (III, 3)*. The coefficients $h'$ are here rational functions of $z$, and in fact integral rational functions of $z$ in case we have to do with a finite value of the variable

* We shall find it convenient to refer in this manner to formulae other than those of the current chapter. Namely we enclose in parentheses the number of the chapter, followed by the number of the formula.

$z=a$. We see then that we can construct a product of the form (III, 3) representing a rational function of $(z,v)$ and having $\mu_s$ as its order of coincidence with the branches of the cycle of order $\nu_s$, while its orders of coincidence with the branches of the remaining $r-1$ cycles are as large as we may please.

We have just seen that the first of the $\nu_s$ functions (7) involves the term $v^{n-1}$, and we see further that this is the only one of the $\nu_s$ functions which involves $v$ to as high a power. On adding this function then to any one of the remaining $\nu_s-1$ functions $B_\sigma(z,v).R(z,v)$ in (7), we obtain a function

$$\{B_0(z,v)+B_\sigma(z,v)\}.R(z,v)$$

involving the term $v^{n-1}$ and having $\mu_s-\frac{\sigma}{\nu_s}$ as its order of coincidence with the branches of the cycle of order $\nu_s$. This function will then have the form (III, 6) and will therefore be expressible as a product of the form (III, 3). Also its orders of coincidence with the $n-\nu_s$ branches $v-P_{\nu_s+1}=0, \ldots v-P_n=0$ may be supposed to be indefinitely large by virtue of the factor $R(z,v)$.

The $\nu_s$ functions

(8) $B_0(z,v).R(z,v),\ \{B_0(z,v)+B_1(z,v)\}.R(z,v), \ldots \{B_0(z,v)+B_{\nu_s-1}(z,v)\}.R(z,v)$

then constitute a set of $\nu_s$ rational functions of $(z,v)$ each of which is expressible as a product of the form (III, 3) — or, what is the same thing, each of which may be written in the form (III, 6) — and of which the orders of coincidence with the branches of the cycle of order $\nu_s$ have the values

$$\mu_s,\ \mu_s-\frac{1}{\nu_s}, \ldots \mu_s-\frac{\nu_s-1}{\nu_s}$$

respectively, while their orders of coincidence with the branches of the other $r-1$ cycles may be made as large as we like by a proper choice of the factor $R(z,v)$. If our formulae here have reference to a finite value $z=a$ of the variable the functions (7) and (8) will be integral functions of $(z,v)$, for in that case the factors of the product (1) and therefore those of the product (2) involve no negative powers of $z-a$, and the same will then be true of the product (5) and therefore also of the functions (7) and (8).

Since the term $v^{n-1}$ appears in each of the functions (8) none of these functions is divisible by $z-a$. We see then that it is always possible to construct an integral rational function of $(z, v)$ which is not divisible by $z-a$ and whose order of coincidence with the branches of the cycle of order $\nu_s$ is equal to $\mu_s - \frac{\sigma}{\nu_s}$ — where $\sigma$ may have any one of the values $0, 1, \ldots \nu_s - 1$ — while its orders of coincidence with the branches of the other $r-1$ cycles are as large as we may please.

The results already obtained in this chapter, in so far as they regard a finite value $z=a$, may be combined in the one statement. — The orders of coincidence of an integral rational function of $(z, v)$ which is not divisible by $z-a$, with the branches of the several cycles, cannot simultaneously be greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively, but it is always possible to construct such a function whose orders of coincidence with the branches of an arbitrarily chosen set of $r-1$ out of the $r$ cycles are as large as we may please, while its order of coincidence with the branches of the other cycle is equal to $\mu_s - \frac{\sigma}{\nu_s}$, in case this cycle be the one of order $\nu_s$ — where $\sigma$ may have any one of the values $0, 1, \ldots \nu_s - 1$.

The former of the two theorems here combined might also be stated in a somewhat modified form as follows: — If the orders of coincidence of an integral rational function of $(z, v)$ with the branches of the several cycles are simultaneously equal to or greater than the numbers

$$\mu_1 + \frac{1}{\nu_1}, \ldots \mu_r + \frac{1}{\nu_r}$$

respectively, the function must be divisible by $z-a$. From this it will follow, that if the orders of coincidence of an integral rational function of $(z, v)$ with the branches of the several cycles be simultaneously equal to or greater than the numbers

$$\mu_1 + i - 1 + \frac{1}{\nu_1}, \ldots \mu_r + i - 1 + \frac{1}{\nu_r}$$

respectively, where $i$ is any positive integer, the function must be divisible by $(z-a)^i$.



It may also be noted as a consequence of the two theorems stated above that the highest possible order of coincidence corresponding to the value $z=a$, which an integral rational function $G(z, v)$ which is not divisible by $z-a$ can have with the curve $F(z, v)=0$, is the greatest of the numbers $\mu_1, \ldots \mu_r$. From this it follows further that the greatest value of the exponent $i$ which can occur in a rational fractional function of the form $(z-a)^{-i} G(z, v)$, which is infinite for none of the branches corresponding to the value $z=a$ and in which the numerator $G(z, v)$ is not divisible by $z-a$, is the greatest of the integers $[\mu_1], \ldots [\mu_r]$.

As to the point at $\infty$, we have shewn in the present chapter that the orders of coincidence of a rational function of the form (III, 6) with the branches of the several cycles at $\infty$, cannot simultaneously be greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively, but that it is always possible to construct such a function whose orders of coincidence with the branches of an arbitrarily chosen set of $r-1$ out of the $r$ cycles are as large as we please, while its order of coincidence with the branches of the other cycle is equal to $\mu_s - \frac{\sigma}{\nu_s}$ — in case this cycle be the one of order $\nu_s$ — where $\sigma$ may have any one of the values $0, 1, \ldots \nu_s - 1$. Functions of $(z, v)$ of this description will not in general be integral and the functions (8) which we have actually constructed, if expressed in the form (III, 6), would have coefficients $h'$ consisting of a finite number of terms in powers of $\frac{1}{z}$ involving as a rule both positive and negative exponents, since exponents of both characters appear in the factors of the product (1) from which the functions (8) were derived.

In the reduced form of any rational function of $(z, v)$

$$h_{n-1} v^{n-1} + h_{n-2} v^{n-2} + \cdots + h_0$$

we may suppose the coefficients $h$ to be developed in powers of $\frac{1}{z}$. Both positive and negative exponents may present themselves, the number of the latter however being in any case finite.

If we have $h_{n-1} \neq 0$ the function can also be written in the form



(9) $$\left(\frac{1}{z}\right)^k g\left(\frac{1}{z}\right)\left(v^{n-1}+h'_{n-2}v^{n-2}+\cdots+h'_0\right)$$

where $k$ may happen to be o or an integer positive or negative, and where $g\left(\frac{1}{z}\right)$ is a series which involves only positive powers of $\frac{1}{z}$ and in which the constant term is different from 0. Since the factor $v^{n-1}+h'_{n-2}v^{n-2}+\cdots+h'_0$ cannot have orders of coincidence with the branches of the several cycles at $\infty$ which are simultaneously greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively, it follows that the orders of coincidence of a function of the form (9) with the branches of the several cycles, cannot simultaneously be greater than the numbers $\mu_1+k, \ldots \mu_r+k$ respectively.

The orders of coincidence of any rational function of $(z, v)$ then with the branches of the several cycles at $\infty$, in the case where the function in its reduced form involves a term $h_{n-1}v^{n-1}$, cannot simultaneously be greater than the numbers $\mu_1+k, \ldots \mu_r+k$ respectively, where $k$ is the smallest exponent which appears in the development of $h_{n-1}$ in powers of $\frac{1}{z}$. Both in this case and in the case where $h_{n-1}=0$, further results will be obtained in Chapter VI in regard to the connection between the form of a function and its orders of coincidence with the branches at $\infty$.

It may be remarked that it is also possible by direct deformation of the factors of the product (1) to obtain a set of $\nu_s$ products of the form (III, 3), which represent rational functions of $(z, v)$ and whose orders of coincidence with the branches of the cycle of order $\nu_s$ are, as in the case of the functions (7) and (8), equal to

$$\mu_s,\ \mu_s-\frac{1}{\nu_s},\ \ldots\ \mu_s-\frac{\nu_s-1}{\nu_s}$$

respectively, while their orders of coincidence with the branches of the other cycles are as large as we may please.

In the case of a finite value $z=a$, where for the order of one of the cycles we have $\nu_s=1$, the cycle in question reduces to a single simple branch, say $v-P_1=0$. By what precedes it is then possible to construct an integral rational function of $(z, v)$, — and in fact an integral rational function which is not divisible by $z-a$ — whose order of coincidence with the branch $v-P_1=0$ is just equal to $\mu_s$, while its orders of coincidence with the remaining $n-1$ branches are as large as we may please. We may also prove that if the orders of coincidence of an integral rational function with these $n-1$ branches are sufficiently large, its order of coincidence with the branch $v-P_1=0$ must be $\geqq \mu_s$.

We shall first consider an integral rational function

$$G(z, v)=g_{n-1}v^{n-1}+\cdots+g_0$$

in which the coefficient $g_{n-1}$ is not divisible by $z-a$. This function we may represent in the form of a product

$$G(z, v)=g_{n-1}(v-Q_2)\ \ldots\ (v-Q_{n-1}).$$

The orders of coincidence of the function $G(z, v)$ with the several branches of the curve corresponding to the value $z=a$ will be equal to the orders of coincidence of the product

$$(v-Q_2)\ \ldots\ (v-Q_{n-1})$$

with these branches since $g_{n-1}$ is not divisible by $z-a$.

Supposing now that the orders of coincidence of this product with the $n-1$ branches $v-P_2=0, \ldots\ v-P_n=0$ are sufficiently large, it must be that each one of the $n-1$ factors $v-Q$ corresponds to a different one of these $n-1$ branches, in this sense, that it has a higher order of coincidence with this branch than with any other of the $n$ branches of the curve. If namely the order of coincidence $\mu_{k,l}$ of the branch $v-P_k=0$ with the branch $v-P_l=0$ be the largest order of coincidence of the branch $v-P_k=0$ with any of the other $n-1$ branches, it will follow that this branch cannot have with the product in question an order of coincidence which is $>(n-1)\mu_{k,l}$, unless it has with some one factor $v-Q$ at least, an order of coincidence which is $>\mu_{k,l}$ and therefore greater than its order of coincidence with any of the other $n-1$ branches. The order of coincidence of the factor $v-Q$ in question with the branch $v-P_k=0$ would then be greater than its order of coincidence with any of the other $n-1$ branches.

Indicating by $m_k$ the number $(n-1)\mu_{k,l}$ just constructed relatively to the branch $v-P_k=0$, we can construct a set of numbers $m_2, m_3, \ldots m_n$ corresponding to the branches $v-P_2=0$, $v-P_3=0, \ldots v-P_n=0$ respectively, such that if the orders of coincidence of the function $G(z,v)$ with these $n-1$ branches be simultaneously greater than the numbers $m_2, m_3, \ldots m_n$ respectively, to each of the branches will correspond a factor $v-Q$ having with this branch an order of coincidence greater than its order of coincidence with any of the other $n-1$ branches. The order of coincidence of the branch $v-P_1=0$ with any one of the factors $v-Q$ will then be equal to its order of coincidence with the branch $v-P=0$ which corresponds to this factor, and its order of coincidence with the function $G(z,v)$ will therefore be equal to the sum of its orders of coincidence with the branches $v-P_2=0, \ldots v-P_n=0$. The order of coincidence of the branch $v-P_1=0$ with the function $G(z,v)$ must then be $\mu_s$, for the cycle of order $\nu_s=1$ we have supposed to be constituted by this single branch.

In the case where the function $G(z,v)$ has orders of coincidence with the branches $v-P_2=0, \ldots v-P_n=0$ which are simultaneously greater than the numbers $m_2, \ldots m_n$ respectively, and where at the same time the coefficient $g_{n-1}$ is not divisible by $z-a$, it must, as we have just seen, have $\mu_s$ as its order of coincidence with the branch $v-P_1=0$. This will hold good also in the case where $g_{n-1}$ is divisible by $z-a$ so long as the function itself is not divisible by $z-a$, for in this case adding to the function $G(z,v)$ an integral rational function of $(z,v)$, whose orders of coincidence with the branches $v-P_2=0, \ldots v-P_n=0$ are simultaneously greater than the numbers $m_2, \ldots m_n$ respectively and in which the coefficient of $v^{n-1}$ is not divisible by $z-a$, both the sum and the function added will be functions of the form already considered, in which the coefficient of $v^{n-1}$ is not divisible by $z-a$ and whose orders of coincidence with the $n-1$ branches $v-P_2=0, \ldots v-P_n=0$ are simultaneously greater than the numbers $m_2, \ldots m_n$ respectively. The orders of coincidence of the sum and of the function added, with the branch $v-P_1=0$, must then be both equal to $\mu_s$, and therefore the order of coincidence of the sum less the function added, that is the order of coincidence of the original function $G(z,v)$ with the branch in question, must be $\geqq \mu_s$. In all cases then the order of coincidence of an integral rational function $G(z, v)$ with the branch $v - P_1 = 0$ is $\geqq \mu_s$ when its orders of coincidence with the branches $v - P_2 = 0, \ldots v - P_n = 0$ are greater than the numbers $m_2, \ldots m_n$ respectively, and in case the function is not divisible by $z - a$ its order of coincidence with the branch $v - P_1 = 0$ will be $\mu_s$, since its orders of coincidence with the branches of the several cycles cannot in such case be simultaneously greater than the numbers $\mu_1, \ldots \mu_n$, which by the definition of the numbers $m$ they all are with the exception of the order of coincidence with the branch $v - P_1 = 0$.



Still supposing the branch $v - P_1 = 0$ to be a simple branch, the like reasoning would evidently also suffice to prove that its order of coincidence with an integral rational function $G(z, v)$ must be $\geqq \mu_s$, if this function have sufficiently large orders of coincidence with all the other branches which pass through the same point of the fundamental curve as the branch $v - P_1 = 0$. If namely the equations $v - P_1 = 0$, $v - P_2 = 0, \ldots v = P_k = 0$ represent all the branches of the curve passing through a point $(a, b)$, and if the orders of coincidence of the function $G(z, v)$ with the branches $v - P_2 = 0, \ldots v - P_k = 0$ be greater than the respective numbers $m_2, \ldots m_k$ constructed relatively to these branches, its order of coincidence with the branch $v - P_1 = 0$ will certainly be $\geqq \mu_s$. Also in case the function $G(z, v)$ be not divisible by the factor $z - a$ its order of coincidence with the branch $v - P_1 = 0$ will be just $= \mu_s$, for this is evidently the case if the coefficient of $v^{n-1}$ in $G(z, v)$ be not divisible by $z - a$, and if this coefficient be divisible by $z - a$ it is nevertheless possible, as we have seen in Chapter III, to construct an integral rational function which possesses precisely the same orders of coincidence with the $n$ branches corresponding to the value $z = a$ and in which the coefficient of $v^{n-1}$ is not divisible by $z - a$.

# CHAPTER V.

## Adjoint orders of coincidence.

*Adjoint orders of coincidence.* Construction of a rational function possessing any assigned set of adjoint orders of coincidence corresponding to a given value of the variable $z$. Definition of an *adjoint function*. $F'_v(z, v)$ an adjoint function. A rational function which is adjoint for all finite values of $z$ is necessarily integral. Every coincidence of an integral rational function with the branches of a cycle corresponding to a finite value of $z$, over and above the coincidences requisite to adjointness, imposes an extra condition on the coefficients of the function. Every extra coincidence with the branches of a cycle corresponding to the value $z=\infty$ of a rational function of degree $N-1$, assumed to be adjoint relatively to the value $z=\infty$, imposes an extra condition on the coefficients of the function.

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The set of functions (7) in the last chapter has been constructed with reference to some particular one of the cycles corresponding to a given value of $z$. Such a set of functions might be constructed for each of the cycles corresponding to the value of the variable in question.

Suppose a system of $r$ such sets of functions, each corresponding to a different cycle, to be constituted by the following $\sum_{s=1}^{r} \nu_s = n$ functions: —

$$(1)\quad \begin{cases} B_{1,0}(z,v)\,.\,R_1(z,v),\; B_{1,1}(z,v)\,.\,R_1(z,v),\;\ldots\; B_{1,\nu_1-1}(z,v)\,.\,R_1(z,v) \\ B_{2,0}(z,v)\,.\,R_2(z,v),\; B_{2,1}(z,v)\,.\,R_2(z,v),\;\ldots\; B_{2,\nu_2-1}(z,v)\,.\,R_2(z,v) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_{s,0}(z,v)\,.\,R_s(z,v),\; B_{s,1}(z,v)\,.\,R_s(z,v),\;\ldots\; B_{s,\nu_s-1}(z,v)\,.\,R_s(z,v) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B_{r,0}(z,v)\,.\,R_r(z,v),\; B_{r,1}(z,v)\,.\,R_r(z,v),\;\ldots\; B_{r,\nu_r-1}(z,v)\,.\,R_r(z,v) \end{cases}$$

Here the $\nu_s$ functions in the $s$th row are supposed to have reference to the cycle of order $\nu_s$, and their orders of coincidence with any branch of this cycle are assumed to have the values


$$\mu_s, \ \mu_s - \frac{1}{\nu_s}, \dots \ \mu_s - \frac{\nu_s - 1}{\nu_s}$$

respectively, while their orders of coincidence with the branches of the other cycles may be assumed to have values which are greater than a set of arbitrarily assigned values, by virtue of our choice of the factor $R_s(z, v)$. Any function $B_{s,\sigma}(z, v) \,.\, R_s(z, v)$ in the above system then is supposed to have $\mu_s - \frac{\sigma}{\nu_s}$ as order of coincidence with the branches of the cycle of order $\nu_s$, while its orders of coincidence with the branches of the other cycles are indefinitely large.

By the aid of the system of functions (1) we shall now be able to construct a rational function of $(z, v)$ — and, in case the system have reference to a finite value of the variable $z$, an integral rational function of $(z, v)$ — whose orders of coincidence with the branches of the several cycles have the values

$$\frac{n_1}{\nu_1}, \ \frac{n_2}{\nu_2}, \dots \ \frac{n_r}{\nu_r} \tag{2}$$

respectively, where $n_1, \ n_2, \dots n_r$ may be any given integers, subject only to the condition that the values (2) be not less than the respective numbers

$$\mu_1 - 1 + \frac{1}{\nu_1}, \ \mu_2 - 1 + \frac{1}{\nu_2}, \dots \ \mu_r - 1 + \frac{1}{\nu_r}. \tag{3}$$

If the numbers (3) have reference to a finite value of the variable $z$ none of them will be negative. The same will then be true of the numbers (2) also. If however we have to do with the point at $\infty$ negative values may happen to present themselves, both among the numbers in (3) and among those in (2). In both these cases too the value 0 may occur, and in fact for finite values of the variable $z$ other than those belonging to the category (C), the numbers (3) will all be equal to 0.

The numbers $\mu_1, \dots \mu_r$, as we have already seen in Chapter I, are integral multiples of $\frac{1}{\nu_1}, \dots \frac{1}{\nu_r}$ respectively, and the orders of coincidence (2) may therefore be expressed in the form

$$m_1+\mu_1-\frac{\sigma_1}{\nu_1},\ m_2+\mu_2-\frac{\sigma_2}{\nu_2},\ \ldots\ m_r+\mu_r-\frac{\sigma_r}{\nu_r}$$



where the numbers $m$ are positive integers or zero and where the numbers $\sigma_s$ are to be found among the $\nu_s$ integers $0, 1, \ldots \nu_s-1$.

To construct a function possessing the orders of coincidence in question, select the $r$ functions

(4) $$B_{1,\sigma_1}(z,v).R_1(z,v), B_{2,\sigma_2}(z,v).R_2(z,v), \ldots B_{r,\sigma_r}(z,v).R_r(z,v)$$

out of the $r$ rows in (1), and multiply these functions by the respective factors

$$\lambda_1(z-a)^{m_1}, \lambda_2(z-a)^{m_2}, \ldots \lambda_r(z-a)^{m_r}$$

where $\lambda_1, \ldots \lambda_r$ are constants which are different from 0. Adding the $r$ products we obtain as sum

(5) $$\lambda_1(z-a)^{m_1}B_{1,\sigma_1}(z,v).R_1(z,v)+\cdots+\lambda_r(z-a)^{m_r}B_{r,\sigma_r}(z,v).R_r(z,v).$$

The order of coincidence of the $s$th element in this sum with the branches of the cycle of order $\nu_s$ *is* $m_s+\mu_s-\frac{\sigma_s}{\nu_s}$. Its orders of coincidence with the branches of the other $r-1$ cycles may at the same time be assumed to be greater than a given set of arbitrary numbers, this only requiring a proper choice of the factor $R_s(z,v)$ as appears from Chapter IV. Suppose then, that for each of the values $s=1, 2, \ldots r$ the factor $R_s(z,v)$ has been so constructed, that the function

$$(z-a)^{m_s}B_{s,\sigma_s}(z,v).R_s(z,v)$$

has, as its orders of coincidence with the branches of the several cycles, the following

(6) $$m_1+\mu_1-\frac{\sigma_1}{\nu_1}+,\ m_2+\mu_2-\frac{\sigma_2}{\nu_2}+,\ \ldots\ m_s+\mu_s-\frac{\sigma_s}{\nu_s},\ \ldots\ m_r+\mu_r-\frac{\sigma_r}{\nu_r}+$$

where the symbol + attached to any number denotes that the corresponding order of coincidence is greater than the indicated number. Here the $s$th number in (6) is the only one to which the symbol does not find itself attached.

The several elements of the sum in (5) then have orders of coincidence with the branches of the cycle of order $\nu_s$ which are greater than the number $m_s + \mu_s - \frac{\sigma_s}{\nu_s}$, with the exception of the $s$th element whose order of coin-

cidence with the branches of the cycle in question is precisely this number. The number $m_s + \mu_s - \frac{\sigma_s}{\nu_s}$ is therefore the order of coincidence of the sum (5) with the branches of the cycle of order $\nu_s$. Since this is true for the $r$ values $s = 1, 2, \ldots r$, it follows that the sum (5) represents a rational function of $(z, v)$ whose orders of coincidence with the branches of the cycles of orders $\nu_1, \ldots \nu_r$ respectively are given by the numbers

$$m_1 + \mu_1 - \frac{\sigma_1}{\nu_1},\ m_2 + \mu_2 - \frac{\sigma_2}{\nu_2},\ \ldots\ m_r + \mu_r - \frac{\sigma_r}{\nu_r}.$$

This function then has as its orders of coincidence with the branches of the $r$ cycles the required set of values (2).

A function whose orders of coincidence with the branches of the several cycles corresponding to a given value of the variable $z$ are equal to or greater than the corresponding numbers in the system (3) belonging to this value of the variable, we shall say is *adjoint* to the fundamental curve $F(z, v) = 0$ for the value of the variable in question, and such a set of orders of coincidence we shall call a set of *adjoint orders of coincidence.* Also of an individual branch we shall say that its order of coincidence with a function is *adjoint*, if such order of coincidence happens to be equal to or greater than the corresponding number $\mu - 1 + \frac{1}{\nu}$. This is intended to define our use of the word adjoint not only for any finite value of the variable $z$, but also for the value $z = \infty$.

We shall find it convenient at times to make use of the word *extra-adjoint*, to designate an order of coincidence which is equal to or greater than the number $\mu$ corresponding to the branch in question. A function would then be said to possess a set of *extraadjoint* orders of coincidence for a given value of the variable $z$, if its orders of coincidence with the branches of the several cycles were simultaneously equal to or greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively. An extraadjoint order of coincidence is of course always also an adjoint order of coincidence. A distinction between adjoint and extraadjoint orders of coincidence exists only in the case of a branch which belongs to a cycle of order $\nu > 1$, for in this case only do we have $\bar{\mu} \neq \mu - 1 + \frac{1}{\nu}$. For a simple branch then the terms adjoint and extraadjoint are synonymous.



It may happen that the same function possesses the property of adjointness for every value of the variable $z$. For example $F'_v(z, v)$ is such a function, as can readily be shewn. — Namely, on taking the partial derivative with regard to $v$ of $F(z, v)$ as represented in (III, 4), we obtain

$$(7) \qquad F'_v(z, v) = (v - P_1) \ldots (v - P_n) \left\{ \frac{1}{v - P_1} + \cdots + \frac{1}{v - P_n} \right\}$$

and on substituting in this expression $v = P_s$, we get the same result as on making this substitution in the product

$$(v - P_1) \ldots (v - P_{s-1})(v - P_{s+1}) \ldots (v - P_n).$$

The order of coincidence of this product with the branch $v - P_s = 0$ is equal to $\bar{\mu}_s$ as we have seen in Chapter I, and this will therefore also be the order of coincidence of the function $F'_v(z, v)$ with the branch in question. The orders of coincidence of the function $F'_v(z, v)$ with the branches $v - P_1 = 0, \ldots v - P_n = 0$ respectively will then be equal to $\bar{\mu}_1, \ldots \bar{\mu}_n$, and its orders of coincidence with the branches of the several cycles into which these $n$ branches group themselves will consequently have the values $\mu_1, \ldots \mu_r$. The orders of coincidence of the function $F'_v(z, v)$ are therefore extraadjoint for the value of $z$ to which the product (III, 4) corresponds. To every value of $z$ however corresponds a representation of $F(z, v)$ in the form of such a product, and the function $F'_v(z, v)$ is therefore extraadjoint to the curve $F(z, v) = 0$ for every value of $z$ — the value $z = \infty$ included. It is then also adjoint to the curve for all values of the variable $z$.

If we define an extraadjoint function as one which possesses a set of extraadjoint orders of coincidence for every value of the variable $z$ the function $F'_v(z, v)$ will be an extraadjoint function, and to a constant factor the only extraadjoint function, for the quotient of any extraadjoint function by $F'_v(z, v)$ would be nowhere infinite and would therefore be a constant*.

* The constant might of course have a different value for each of the irreducible equations included in the fundamental algebraic equation in the case where this equation is reducible.

A function which is adjoint to the curve for all values of the variable $z$ we shall call an adjoint function. The function $F'_v(z, v)$ is then also an adjoint function. This function, it may be remarked, is integral in $(z, v)$, and the same will be true of any rational function of $(z, v)$ which is an adjoint function. In fact, in its reduced form, any rational function of $(z, v)$ which is adjoint for all finite values of $z$ must be integral.

To prove the statement just made consider any function in its reduced form. It may be written as a fraction $\frac{G(z, v)}{g(z)}$ where the numerator is an integral function of $(z, v)$ and the denominator an integral function of $z$, and where further $G(z, v)$ and $g(z)$ have no factor in common. Supposing $g(z)$ to be other than a constant it will contain some factor $z-a$. The function $G(z, v)$ is not divisible by this factor, and its orders of coincidence with the branches of the several cycles corresponding to the value $z=a$ therefore cannot simultaneously be greater than the corresponding numbers $\mu_1, \ldots \mu_r$ respectively. The orders of coincidence of the function $\frac{G(z, v)}{g(z)}$ with the branches of the several cycles then cannot simultaneously be greater than the numbers $\mu_1-1, \ldots \mu_r-1$ respectively. Its order of coincidence with the branches of at least one of the cycles must therefore be less than the number of the system (3) corresponding to this cycle. The function under consideration is consequently not adjoint for the value $z=a$.

If then the denominator $g(z)$ of the function $\frac{G(z, v)}{g(z)}$ be other than a constant, there will be some finite value of $z$ for which the function is not adjoint. It follows that a rational function of $(z, v)$ which is adjoint for all finite values of $z$, must at the same time be an integral function of these variables.

The converse of the last statement is in so far true that we may say, that any integral function of $(z, v)$ is adjoint for all values of $z$ belonging to the categories (A) and (B) of Chapter II, for the numbers (3) corresponding to any such value of $z$ are all equal to 0 and the orders of coincidence of an integral function with branches corresponding to a finite value of $z$ are never less than 0. For a value $z=a$ belonging to the category (A) namely, we have $r=n$, $\mu_1=\mu_2=\ldots=\mu_r=0$, $\nu_1=\nu_2=\ldots=\nu_r=1$ and the numbers in (3) are all of the type $0-1+1=0$. For a value of $z$ belonging to the category (B) we have $r<n$ and the numbers in (3) which are not of the type just mentioned are nevertheless of the type $\frac{\nu-1}{\nu}-1+\frac{1}{\nu}=0$, for as we have seen in Chapter II the order of coincidence of a branch of a cycle of order $\nu$ with the product of the other $n-1$ branches is equal to $\frac{\nu-1}{\nu}$ in case the cycle does not correspond to a multiple point.

To say then that a rational function of $(z,v)$ is adjoint for all finite values of $z$ is equivalent to saying that it is integral and adjoint for all values of $z$ belonging to the category (C), that is for all values of $z$ to which correspond multiple points.

For a value of $z$ belonging to the category (C) the corresponding numbers $\mu_1, \ldots \mu_r$ are not all less than 1, and among the numbers (3) therefore there will be one at least which is greater than 0.

The number of values of $z$ belonging to the category (C) is as we know, finite, and the number of conditions to which we must subject the undetermined coefficients of an integral rational function of $(z,v)$ in order that it may be adjoint for one of these values is also finite. The number of conditions necessary to adjointness for all the values of $z$ belonging to the category (C) then is finite, and the number of linearly independent rational functions of $(z,v)$ which are adjoint for all finite values of the variable $z$ will therefore be infinite. The number of linearly independent rational adjoint functions is however finite, and more generally the number of linearly independent integral rational functions of $(z,v)$ which are adjoint for the value $z=\infty$ is finite as we shall see in chapter VI, for we shall there shew that the degree of such a function is $\leqq N-1$.

Returning to the consideration of the function (5) — supposed to be constructed with reference to a finite value of the variable $z=a$ — we see that it is an integral function of $(z,v)$, for the exponents $m_1, \ldots m_r$ are zero or positive and the factors $B_{s,\sigma_s}(z,v) \,.\, R_s(z,v)$ appearing in the individual elements of the sum are integral, as we have seen in Chapter IV. We have shewn that it is always possible for a given value of the variable to construct a function of the form (5), having as its orders of coincidence with the branches of the corresponding cycles an arbitrary set of adjoint orders of coincidence.



Let us now build an integral rational function of $(z, v)$

$$g_{n-1}v^{n-1} + g_{n-2}v^{n-2} + \cdots + g_0$$

in which the polynomials $g$ are of sufficiently high degree in $z$, their constant coefficients also being arbitrary. First subject these arbitrary constant coefficients to the conditions necessary to the adjointness of the function for a given finite value of $z$. The number of such necessary conditions for adjointness we shall indicate by the letter $A$. Now subject the coefficients to the still further conditions implied in the function having as its orders of coincidence for the value of the variable in question, the set of adjoint orders of coincidence

$$\frac{n_1}{\nu_1}, \frac{n_2}{\nu_2}, \cdots \frac{n_r}{\nu_r}.$$

From the set of orders of coincidence necessary to adjointness, to the set of adjoint orders of coincidence in question, we may pass by a series of steps each individual one of which involves an addition to the order of coincidence of the function with the branches of one and of only one of the cycles, the addition to the order of coincidence being $\frac{1}{\nu_s}$ in case the cycle in question be the one of order $\nu_s$. That this is possible follows from the fact that we can construct an integral rational function of $(z, v)$, having as its orders of coincidence with the branches of the cycles corresponding to a given value of the variable $z$ an arbitrary set of adjoint orders of coincidence.

Every step in the process just described implies a further condition on the coefficients of the function, and only one further condition as is evident, for the order of coincidence of a rational function of $(z, v)$ with the branches of a cycle of order $\nu_s$ is always measured by an integral multiple of $\frac{1}{\nu_s}$. The number of independent conditions to which the coefficients of an arbitrary integral rational function of $(z, v)$ must be subjected, in order that it may have as its orders of coincidence with the branches of the several cycles corresponding to the finite value of the variable $z=a$ the set of adjoint orders of coincidence here in question, is therefore equal to

$$A+\left(\frac{n_1}{\nu_1}-\mu_1+1-\frac{1}{\nu_1}\right)\nu_1+\cdots+\left(\frac{n_r}{\nu_r}-\mu_r+1-\frac{1}{\nu_r}\right)\nu_r$$

or we may say that the number of independent conditions which must be satisfied by the coefficients of the function, in order that it may have as its orders of coincidence with the branches of the several cycles a certain set of adjoint orders of coincidence $\mu'_1, \ldots \mu'_r$, is equal to

$$(8) \qquad A+\left(\mu'_1-\mu_1+1-\frac{1}{\nu_1}\right)\nu_1+\cdots+\left(\mu'_r-\mu_r+1-\frac{1}{\nu_r}\right)\nu_r .$$

What the number $A$ is will be determined later on.

If instead of a finite value of the variable it were the value $z=\infty$ with which we had to do, we should have instead of (5) an expression of the form

$$(9) \qquad \lambda_1\left(\frac{1}{z}\right)^{m_1} B_{1,\sigma_1}(z,v) \,.\, R_1(z,v)+\cdots+\lambda_r\left(\frac{1}{z}\right)^{m_r} B_{r,\sigma_r}(z,v) \,.\, R_r(z,v)$$

having as its orders of coincidence with the branches of the several cycles at $\infty$ an arbitrary set of adjoint orders of coincidence (2), the numbers and functions here in question being supposed of course to be defined or constructed, as the case may be, with reference to the value $z=\infty$.

The function (9) will not in general be integral in $(z, v)$, for the exponents $m$ are positive or zero and the functions $B_{s,\sigma_s}(z,v) \,.\, R_s(z,v)$ constructed with reference to the value $z=\infty$ will as a rule involve both positive and negative powers of $\frac{1}{z}$. Also the number of powers of $\frac{1}{z}$ of both descriptions which occur in the expression of these particular functions, as we have constructed them, is finite.

It will be convenient for us to here anticipate the results of the chapter following, in order that we may include in the conclusions of the present chapter the value $z=\infty$ as well as the finite values of the variable. In Chapter VI namely, it will be proved that the degree of a rational function of $(z, v)$ which is adjoint for the value $z = \infty$ cannot be greater than $N-1$. The degree of a function, it is to be here understood, has reference to the function in its reduced form and is supposed to be defined not only for integral rational functions of $(z, v)$ but also for non-integral functions of the variables. If, namely, a function of the form

$$h_{n-1}v^{n-1} + h_{n-2}v^{n-2} + \cdots + h_0$$

be not integral we assume that the coefficients $h$ have been developed according to powers of $\frac{1}{z}$ and the degree of the function is then defined, as in the case of an integral function, by the sum of the exponents of $z$ and $v$ in the term or terms of highest order in these variables.

A rational function of $(z, v)$ of degree $N-1$ may be written in the form

$$(10) \qquad z^{N-1}G\left(\frac{1}{z}, v\right) = h_{n-1}v^{n-1} + h_{n-2}\;{}^{n-2} + \ldots + h_0$$

where in the function $G\left(\frac{1}{z}, v\right)$ the coefficients of the several powers of $v$, supposed to be developed according to powers of $\frac{1}{z}$, involve no negative exponents, and in fact no exponents which are less than the exponents of the powers of $v$ which they multiply. Under this form then will be included, among others, all rational functions of $(z, v)$ which are adjoint for the value $z = \infty$.

The statement just made is of course to be accepted only provisionally, on the understanding that we are to supply in the following chapter, a proof of the proposition that the degree of a rational function of $(z, v)$ which is adjoint for the value $z = \infty$ cannot be greater than $N-1$. The function (9) we have supposed to be adjoint for this value of $z$ and it will therefore be included in the form (10).

Consider the general function of the form (10), that is the general rational function of $(z, v)$ of degree $N-1$. We shall suppose the coefficients $h$ to be developed according to powers of $\frac{1}{z}$. The constant coefficients of these powers of $\frac{1}{z}$ we may assume to be arbitrary up to terms of as high an order as we please. Assuming to begin with that the coefficients are arbitrary for all terms up to ones of sufficiently high order in $\frac{1}{z}$ for the

purpose which we have in view, — namely for the purpose of determining in the most general case the number of conditions which must be satisfied by the constant coefficients in a rational function of degree $N-1$, in order that it may have a certain set of adjoint numbers as its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ — we shall first subject these coefficients to the conditions just necessary for the adjointness of the function relative to the value $z=\infty$. The as yet undetermined number of such conditions of adjointness we shall indicate by the letter $A$.

We shall now subject the coefficients to the still further conditions implied in the function having as its orders of coincidence for the value $z=\infty$, a certain set of adjoint numbers

$$\frac{n_1}{\nu_1}, \frac{n_2}{\nu_2}, \dots \frac{n_r}{\nu_r}.$$

From the set of orders of coincidence necessary to adjointness to the set of adjoint orders of coincidence in question, we may pass by a series of steps each individual one of which involves an addition to the order of coincidence of the function with the branches of one and of only one of the cycles, the addition to the order of coincidence being $\frac{1}{\nu_s}$ in case the cycle in question be the one of order $\nu_s$. That this is possible follows from the fact that we can construct a function of the form (9), and therefore one of the form (10), having as its orders of coincidence with the branches of the cycles corresponding to the value $z=\infty$ an arbitrary set of adjoint orders of coincidence.

Each step in the process just described will imply one and only one extra condition on the coefficients of the function. The number of extra conditions implied in passing from the set of orders of coincidence just requisite to adjointness to the set of adjoint orders of coincidence here in question, will therefore be given by the sum

$$\left(\frac{n_1}{\nu_1}-\mu_1+1-\frac{1}{\nu_1}\right)\nu_1+\left(\frac{n_2}{\nu_2}-\mu_2+1-\frac{1}{\nu_2}\right)\nu_2+\cdots+\left(\frac{n_r}{\nu_r}-\mu_r+1-\frac{1}{\nu_r}\right)\nu_r.$$

The total number of independent conditions to which the constant coefficients  in an arbitrary function of the form (10) must be subjected, in order that it may have as its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ the set of adjoint orders of coincidence (2), will then be equal to

$$A+\left(\frac{n_1}{\nu_1}-\mu_1+1-\frac{1}{\nu_1}\right)\nu_1+\cdots+\left(\frac{n_r}{\nu_r}-\mu_r+1-\frac{1}{\nu_r}\right)\nu_r$$

where we intend the letter $A$ to have the meaning which has just been attached to it, and where the other symbols also are supposed of course to be defined with reference to the value $z=\infty$.

Making use of the notation $\mu'_1, \mu'_2, \ldots \mu'_r$, instead of that employed above, to indicate an arbitrary set of adjoint orders of coincidence corresponding to the value $z=\infty$, the expression just obtained assumes the form (8). The form (8) then may be regarded as having reference to any given value of the variable $z$, the value $z=\infty$ included, the symbols involved in the form being supposed in each case to be defined with reference to the particular value of the variable under consideration.

If we have to do with a given finite value of the variable $z$ the symbol $A$, as we have seen, indicates the number of conditions just sufficient to the adjointness of an integral rational function of $(z, v)$ for the particular value of $z$ in question, while the whole expression (8) gives the number of independent conditions which must be satisfied by its coefficients, in order that it may have as its orders of coincidence with the branches of the corresponding cycles the set of adjoint orders of coincidence $\mu'_1, \mu'_2, \ldots \mu'_r$, supposed to be defined for the value of $z$ under consideration. If $z=\infty$ is the value of the variable to which the expression (8) is supposed to have reference, the letter $A$ indicates the number of independent conditions just sufficient to the adjointness of a function of the form (10), that is of a function of degree $N-1$, for the value of $z$ in question; while the whole expression (8) gives the number of independent conditions which must be satisfied by its coefficients, in order that it may have as its orders of coincidence with the branches of the corresponding cycles the set of adjoint orders of coincidence $\mu'_1, \mu'_2, \ldots \mu'_r$, supposed to be defined for the value $z=\infty$.

We have seen that every coincidence of an integral rational function with the branches of a cycle corresponding to a finite value $z=a$, over and above the coincidences requisite to adjointness, imposes an extra condition on the coefficients of the function. Indicate, as before, by $A$, the number of conditions requisite to the adjointness of a sufficiently general integral rational function. Over and above the coincidences requisite to adjointness impose $B$ further coincidences on the function. In all we thus subject the coefficients of the function to $A+B$ independent conditions. Now suppose the branch $v-P_1=0$ to be a simple branch. Suppose also that the $B$ additional adjoint coincidences here in question have reference to the $n-1$ branches $v-P_2=0, \dots v-P_n=0$ and indicate by $A'$ the number of conditions requisite to the adjointness of an integral rational function relative to these $n-1$ branches alone. If now we impose on an integral rational function the coincidences requisite to adjointness relative to the $n-1$ branches $v-P_2=0, \dots v-P_n=0$, and over and above these coincidences the $B$ additional adjoint coincidences here in question, we subject its coefficients in all to $A'+B$ independent conditions. If however $B$ is a sufficiently large number and if we attribute to each of the cycles, whose branches are included among the $n-1$ branches $v-P_2=0, \dots v-P_n=0$, a sufficient number of the $B$ coincidences, we know from the preceding chapter that the order of coincidence of the integral rational function with the branch $v-P_1=0$ must also be adjoint. It follows that the $A'+B$ conditions requisite to adjointness relative to the $n-1$ branches $v-P_2=0, \dots v-P_n=0$ and to the possession of the $B$ additional adjoint coincidences here in question, suffice also to insure adjointness relative to the $n$ branches $v-P_1=0, \dots v-P_n=0$ together with the possession of the $B$ additional adjoint coincidences.

It follows that we must have $A'+B=A+B$ and therefore also $A'=A$. The number of conditions requisite to the adjointness of an integral rational function of $(z,v)$ relative to the $n-1$ branches $v-P_2=0, \dots v-P_n=0$, is then the same as the number of conditions requisite to adjointness relative to all $n$ branches $v-P_1=0, \dots v-P_n=0$. From this it follows that if an integral rational function of $(z,v)$ have adjoint orders of coincidence with all the branches corresponding to a given finite value $z=a$ save with a single simple branch, its order of coincidence with this simple branch must also be adjoint. In the statement of this proposition it is evidently not necessary to retain the word simple, since if the order of coincidence of a rational function with one branch of a cycle be adjoint its order of coincidence with any other branch of the cycle must also be adjoint, for the orders of coincidence of a rational function with the branches of a cycle must all be the same. We may then say that if an integral rational function of $(z, v)$ have adjoint orders of coincidence with all the branches corresponding to a given finite value $z = a$ excepting with a single branch, its order of coincidence with this branch must also be adjoint.

We might add that an integral rational function of $(z, v)$ which possesses extraadjoint orders of coincidence with $n-1$ out of the $n$ branches corresponding to a finite value $z = a$, must also possess an extraadjoint order of coincidence with the remaining branch. For in the case of a simple branch there is no distinction between adjoint and extraadjoint orders of coincidence, and in any other case the orders of coincidence of a rational function with the several branches of a cycle are the same.

If $G((z-a)^{\frac{1}{\nu}}, v)$ be an integral rational function of $(z-a)^{\frac{1}{\nu}}$ and $v$ whose orders of coincidence with all but one of the $n$ branches corresponding to the value $z = a$ are extraadjoint, its order of coincidence with the remaining branch will also be extraadjoint. To see this it is only necessary to write $(z-a)^{\frac{1}{\nu}} = \varsigma$ when our function $G((z-a)^{\frac{1}{\nu}}, v)$ becomes an integral rational function of $(\varsigma, v)$, whose orders of coincidence with all but one of the branches of the transformed fundamental equation corresponding to the value $\varsigma = 0$ are extraadjoint, and whose order of coincidence with the remaining branch must therefore also be extraadjoint. The truth of our theorem is then evident on retransforming to terms of $(z-a)^{\frac{1}{\nu}}$.

# CHAPTER VI.

## Degree of function as related to orders of coincidence at ∞.

Connection between the degree of a rational function and its orders of coincidence with the branches at ∞. In a rational function which is adjoint relatively to the value $z=\infty$ the degree of the element involving $v^{n-1}$ is $\overline{\overline{<}}\, n-1$ and the degree of the function itself is $\overline{\overline{<}}\, N-1$.

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We have defined the degree of a rational function of $(z, v)$ as the sum of the exponents of $z$ and $v$ in the term or terms of highest order in these variables which appear in the expression of the function in its reduced form, the coefficients of the powers of $v$ being supposed to be developed according to powers of $\frac{1}{z}$. The same definition we may assume to apply also in the case of functions of $(z, v)$ possessing a reduced form in which the coefficients of the powers of $v$ involve irrationalities, and where in their development according to powers of $\frac{1}{z}$ fractional exponents may happen to present themselves. The definition does not exclude the presence of fractional powers of $v$ though such powers will not appear in the functions with which we are going to occupy ourselves.

We shall now seek to determine a connection between the degree of a function and its orders of coincidence with the branches at ∞. The cycles at ∞ we shall suppose to be $r$ in number, of orders $\nu_1, \ldots \nu_r$ respectively, and the symbols $\mu_1, \ldots \mu_r$ will be defined as heretofore.

In one respect it will be convenient to modify our previous notation for the purposes of the present chapter. We shall have namely to do with products of the type (III, 3) constructed with reference to the value $z=\infty$, and products of this description we shall here represent in the form

$$(1)\qquad \left[v-z^{\beta_1}Q_1\left(\frac{1}{z}\right)\right]\left[v-z^{\beta_2}Q_2\left(\frac{1}{z}\right)\right]\cdots\left[v-z^{\beta_{n-1}}Q_{n-1}\left(\frac{1}{z}\right)\right]$$

where the series $Q$ — in case they do not happen to be identically equal to 0 — commence with a constant term which is different from 0 and where the exponents $\beta$ may happen to be positive or negative, integral or fractional.

The function $F(z,v)$ too we shall write in the form

$$(2)\qquad F(z,v)=\left[v-z^{\alpha_1}P_1\left(\frac{1}{z}\right)\right]\left[v-z^{\alpha_2}P_2\left(\frac{1}{z}\right)\right]\cdots\left[v-z^{\alpha_n}P_n\left(\frac{1}{z}\right)\right]$$

instead of employing the representation given in (III, 4). Here the series $P$, like the series $Q$, are supposed to commence with a constant term which is different from 0.

We shall assume that the elements of the two sets of exponents $\beta_1, \beta_2, \ldots \beta_{n-1}$ and $\alpha_1, \alpha_2, \ldots \alpha_n$ as they stand, are arranged in order of ascending magnitude, not excluding of course the possibility that several successive elements of either series may happen to be equal to one another.

The numbers of elements of the $\beta$-set and of the $\alpha$-set which are $\overline{\gtrless} 1$ we shall indicate by $t$ and $s$ respectively. The $\beta$'s which are $>1$ will then be $\beta_{t+1}, \ldots \beta_{n-1}$ and the degree of the function represented by the product (1) will evidently be equal to

$$(3)\qquad t+\beta_{t+1}+\beta_{t+2}+\cdots+\beta_{n-1}\,.$$

The $\alpha$'s which are $>1$ will be $\alpha_{s+1}, \ldots \alpha_n$ and the degree $N$ of the function $F(z,v)$ will therefore be equal to

$$s+\alpha_{s+1}+\alpha_{s+2}+\cdots+\alpha_n\,.$$

As in Chapter I we shall indicate the order of coincidence of the two factors $v-z^{\alpha_k}P_k\left(\frac{1}{z}\right)$ and $v-z^{\alpha_l}P_l\left(\frac{1}{z}\right)$ with each other by the symbol $\mu_{k,l}$ and the orders of coincidence of the factors

$$v-z^{\alpha_1}P_1\left(\frac{1}{z}\right),\ v-z^{\alpha_2}P_2\left(\frac{1}{z}\right),\ \ldots\ v-z^{\alpha_n}P_n\left(\frac{1}{z}\right)$$



each with the product of the other $n-1$ factors, we shall indicate by $\bar{\mu}_1, \bar{\mu}_2, \ldots \bar{\mu}_n$.

We shall now suppose that we have to do with a product of the form (1) whose orders of coincidence with the several branches of the curve are simultaneously greater than $\bar{\mu}_1-\lambda, \bar{\mu}_2-\lambda, \ldots \bar{\mu}_n-\lambda$ respectively, where $\lambda$ is any given positive number, and we shall determine a limit for the degree of a function which can be represented by such a product. That the number $\lambda$ cannot be zero or negative, follows from the fact that a product of the form (1) cannot have orders of coincidence with the several branches of the curve which are simultaneously greater than the numbers $\bar{\mu}_1, \ldots \bar{\mu}_n$ respectively.

Comparing the factors of the product (1) with those of the product (2), it may be that a number of the exponents $\beta$ are equal to a number of the exponents $\alpha$, and it may further happen that $Q$'s and $P$'s corresponding to such equal exponents coincide in a number of their coefficients. In any case there will be at least one value which occurs more frequently among the $\alpha$'s than among the $\beta$'s, for the latter exponents are in the aggregate one less in number than the former.

We shall distinguish two cases: —

I. The smallest value which occurs more frequently among the $\alpha$'s than among the $\beta$'s is $\overline{\gtrless} 1$.

II. Of the values which occur more frequently among the $\alpha$'s than among the $\beta$'s one at least is $<1$.

For the present confining our attention to the former of the two cases, we shall suppose that the said smallest value occurs $q$ times among the exponents $\alpha$ and $r$ $(<q)$ times among the exponents $\beta$. The latest element in the set of exponents $\alpha_1, \ldots \alpha_n$ which has this value we shall indicate by $\alpha_\sigma$, and the number of elements of the set $\beta_1, \ldots \beta_{n-1}$ which are not greater than this value we shall indicate by $\tau$. The elements $\alpha_{\sigma-q+1}, \alpha_{\sigma-q+2}, \ldots \alpha_\sigma$ and $\beta_{\tau-r+1}, \beta_{\tau-r+2}, \ldots \beta_\tau$ will then be the aggregate of those which have this value in the two sets of exponents.

In the product (1) replace each one of the $r$ factors

$$\left[v - z^{\beta_{\tau-r+1}} Q_{\tau-r+1}\left(\frac{1}{z}\right)\right], \ldots \left[v - z^{\beta_\tau} Q_\tau\left(\frac{1}{z}\right)\right]$$



by that one of the $q$ factors

$$\left[v - z^{\alpha_{\sigma-q+1}} P_{\sigma-q+1}\left(\frac{1}{z}\right)\right], \ldots \left[v - z^{\alpha_\sigma} P_\sigma\left(\frac{1}{z}\right)\right]$$

with which its order of coincidence is greatest. This deformation of the product (1) makes no alteration in the values of the set of exponents $\beta$ and evidently also does not diminish the order of coincidence of the product with any one of the $n$ branches. It may be that certain of the $q$ factors $v - z^\alpha P\left(\frac{1}{z}\right)$ here in question have been repeated in the deformation. In that case we replace a repeated factor by one of the $q$ factors which has not as yet been substituted and namely by that one — or by one of those — with which it has the greatest order of coincidence. If repeated factors still remain we replace them in like manner successively by others of the $q$ factors which do not as yet appear, until finally for the $r$ factors $v - z^\beta Q\left(\frac{1}{z}\right)$ of the original product we have substituted $r$ different ones from among the $q$ factors $v - z^\alpha P\left(\frac{1}{z}\right)$ here considered. The set of exponents $\beta$ has not been altered by the deformations just effected, and the product has lost nothing from its orders of coincidence with any of the branches, save perhaps in the case of ones with which its orders of coincidence are still infinite.

The deformed product may be written in the form

$$(4)\quad \left[v - z^{\beta_1} Q_1\left(\frac{1}{z}\right)\right] \ldots \left[v - z^{\beta_{\tau-r}} Q_{\tau-r}\left(\frac{1}{z}\right)\right]\left[v - z^{\alpha_{\sigma-q+1}} P_{\sigma-q+1}\left(\frac{1}{z}\right)\right] \ldots$$
$$\ldots \left[v - z^{\alpha_{\sigma-q+r}} P_{\sigma-q+r}\left(\frac{1}{z}\right)\right]\left[v - z^{\beta_{\tau+1}} Q_{\tau+1}\left(\frac{1}{z}\right)\right] \ldots \left[v - z^{\beta_{n-1}} Q_{n-1}\left(\frac{1}{z}\right)\right]$$

on supposing the $q$ factors $v - z^\alpha P\left(\frac{1}{z}\right)$ considered above to have been properly lettered for this purpose. From among these $q$ factors then one at least — the factor $v - z^{\alpha_\sigma} P_\sigma\left(\frac{1}{z}\right)$ — is lacking in the product (4).

Since the exponents $\beta_1, \ldots \beta_{\tau-r}$ are all less and the exponents $\beta_{\tau+1}, \ldots \beta_{n-1}$ all greater, than the exponent $\alpha_\sigma$, it follows that the order of coincidence of the branch $v - z^{\alpha_\sigma} P_\sigma\left(\frac{1}{z}\right) = 0$ with the product (4) is equal to

$$-(\tau - r)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-q+r} - \beta_{\tau+1} - \ldots - \beta_{n-1}.$$

Now, by hypothesis, the order of coincidence of the branch here in question with the original product (1) was $> \bar{\mu}_\sigma - \lambda$, and since it has not been diminished by the deformation its order of coincidence with the product (4) will also be $> \bar{\mu}_\sigma - \lambda$. We therefore have

$$-(\tau - r)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-q+r} - \beta_{\tau+1} - \cdots - \beta_{n-1} > \bar{\mu}_\sigma - \lambda$$

and since

$$\bar{\mu}_\sigma = \mu_{\sigma,\,1} + \cdots + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-1} + \mu_{\sigma,\,\sigma+1} + \cdots + \mu_{\sigma,\,n}$$
$$= -(\sigma - q)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-1} - \alpha_{\sigma+1} - \cdots - \alpha_n$$

it follows that

$$-(\tau - r)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-q+r} - \beta_{\tau+1} - \cdots - \beta_{n-1} >$$
$$-(\sigma - q)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+1} + \cdots + \mu_{\sigma,\,\sigma-1} - \alpha_{\sigma+1} - \cdots - \alpha_n - \lambda$$

whence

$$-(\tau - r)\alpha_\sigma - \beta_{\tau+1} - \cdots - \beta_{n-1} > -(\sigma - q)\alpha_\sigma + \mu_{\sigma,\,\sigma-q+r+1} + \cdots$$
$$\cdots + \mu_{\sigma,\,\sigma-1} - \alpha_{\sigma+1} - \cdots - \alpha_n - \lambda.$$

The numbers $\mu_{\sigma,\,\sigma-q+r+1} \cdots \mu_{\sigma,\,\sigma-1}$ are however each $\geqq -\alpha_\sigma$, so that

$$-(\tau - r)\alpha_\sigma - \beta_{\tau+1} - \cdots - \beta_{n-1} > -(\sigma - q)\alpha_\sigma - (q - r - 1)\alpha_\sigma - \alpha_{\sigma+1} - \cdots - \alpha_n - \lambda$$

and consequently

(5) $$\beta_{\tau+1} + \cdots + \beta_{n-1} < (\sigma - \tau - 1)\alpha_\sigma + \alpha_{\sigma+1} + \cdots + \alpha_n + \lambda.$$

By hypothesis we have $\beta_{\tau+1} > \alpha_\sigma \geqq 1$ and $\alpha_{\sigma+1} > \alpha_\sigma$. Also the first of the set of exponents $\alpha_1, \ldots \alpha_n$ and the first of the set $\beta_1, \ldots \beta_{n-1}$ which are $> 1$, we have indicated by $\alpha_{s+1}$ and $\beta_{t+1}$ respectively. It follows therefore that we must have $\alpha_{\sigma+1} \geqq \alpha_{s+1}$ and $\beta_{\tau+1} \geqq \beta_{t+1}$. Adding $\beta_{t+1} + \cdots + \beta_\tau$ then to both sides of the inequality (5) we obtain

$$\beta_{t+1} + \cdots + \beta_{\tau+1} + \cdots + \beta_{n-1} < (\sigma - \tau - 1)\alpha_\sigma + \beta_{t+1} + \cdots + \beta_\tau + \alpha_{\sigma+1} + \cdots + \alpha_n + \lambda.$$

From our hypothesis that $\alpha_{\sigma-q+1} = \cdots = \alpha_\sigma$ is the smallest value which occurs more frequently among the $\alpha$'s than among the $\beta$'s, it follows that among the $\tau - r$ exponents $\beta_1, \ldots \beta_{\tau-r}$ are to be found the $\sigma - q$ elements $\alpha_1, \ldots \alpha_{\sigma-q}$ and the value $\alpha_\sigma$ repeated $r$ times, since $\beta_{\tau-r+1} = \cdots = \beta_\tau = \alpha_{\sigma-q+1} = \cdots = \alpha_\sigma$. Among the exponents $\beta_{t+1}, \ldots \beta_\tau$ will then be found the $\sigma - s$ elements $\alpha_{s+1}, \ldots \alpha_\sigma$ less the elements $\alpha_{\sigma-q+r+1}, \ldots \alpha_\sigma$, whence we derive

$$\beta_{t+1} + \cdots + \beta_\tau \leqq \alpha_{s+1} + \cdots + \alpha_\sigma - (q-r)\,\alpha_\sigma + \{(\tau - t) - (\sigma - s - q + r)\}\alpha_\sigma$$

since none of the exponents $\beta$ here in question, and therefore in particular none of the $\tau - t - (\sigma - s - q + r)$ exponents among these, for whose values we have not more precisely accounted, can have a value which is $> \alpha_\sigma$.

On substituting the expression on the right of the inequality last obtained, for $\beta_{t+1} + \cdots + \beta_\tau$ on the right of the inequality next preceding, we obtain

$$\beta_{t+1} + \cdots + \beta_{n-1} < (s - t - 1)\,\alpha_\sigma + \alpha_{s+1} + \cdots + \alpha_\sigma + \alpha_{\sigma+1} + \cdots + \alpha_n + \lambda.$$

Adding $t$ to both sides of this inequality, the resulting inequality may be written in the form

$$t + \beta_{t+1} + \cdots + \beta_{n-1} < (s - t - 1)\,(\alpha_\sigma - 1) + s + \alpha_{s+1} + \cdots + \alpha_n + \lambda - 1.$$

Here we have $\alpha_\sigma \geqq 1$, and in the case where we have $\alpha_\sigma > 1$ we shall have at the same time $t > s - 1$, for the exponents $\alpha_1, \ldots \alpha_s$, being $\leqq 1$ and therefore $< \alpha_\sigma$, will all be found among the exponents $\beta$ and therefore in particular among the exponents $\beta_1, \ldots \beta_t$ which are $\leqq 1$. This follows from our hypothesis that $\alpha_\sigma$ is the least value which does not occur as frequently among the $\beta$'s as among the $\alpha$'s. We shall therefore either have $\alpha_\sigma = 1$, or simultaneously $\alpha_\sigma > 1$ and $t > s - 1$, whence in either case we have $(s - t - 1)\,(\alpha_\sigma - 1) \leqq 0$. From our inequality above we then derive the inequality

$$(6) \qquad t + \beta_{t+1} + \cdots + \beta_{n-1} < s + \alpha_{s+1} + \cdots + \alpha_n + \lambda - 1.$$

Expressed in words, this inequality states that the number of the exponents $\beta$ which are $\leqq 1$ plus the sum of those $\beta$'s which are $> 1$, is less than the number of the $\alpha$'s which are $\leqq 1$ plus the sum of those which are $> 1$ increased by the number $\lambda - 1$.

This disposes for the moment of Case I and we shall now occupy ourselves with Case II. In this case there are one or more values which are $<1$ and which occur more frequently among the $\alpha$'s than among the $\beta$'s. It will be convenient here to make our notation run parallel to that already employed in the treatment of Case I, and with that end in view we shall employ $\alpha_\sigma$ to indicate the latest element in the series $\alpha_1, \ldots \alpha_n$ which is not $>1$, and whose value at the same time appears more frequently among the $\alpha$'s than among the $\beta$'s. The number of elements of the set of exponents $\beta_1, \ldots \beta_{n-1}$ which are not greater than this value we shall indicate by $\tau$. The number of the $\alpha$'s and the number of the $\beta$'s which have the value $\alpha_\sigma$ we shall indicate by $q$ and $r$ $(<q)$ respectively, so that the elements $\alpha_{\sigma-q+1}, \ldots \alpha_\sigma$ and $\beta_{\tau-r+1}, \ldots \beta_\tau$ will be the aggregate of those which have this value in the two sets of exponents.

The reasoning in Case II would now begin in precisely the same way as in Case I. We should derive (4) by deformation of the product (1) and the text would remain unaltered up to and inclusive of the inequality (5). Starting out then with the inequality (5), the $\beta$'s there appearing will include all those which are $>1$ since we have $\beta_\tau \leqq \alpha_\sigma \leqq 1$. They will therefore include $\beta_{t+1}$, the first of the $\beta$'s which is $>1$. Further among the $t-\tau$ exponents $\beta_{\tau+1}, \ldots \beta_t$ will be found all the elements $\alpha_{\sigma+1}, \ldots \alpha_s$, since by hypothesis $\alpha_\sigma$ was the latest of the set of exponents $\alpha_1, \ldots \alpha_n$ which was not $>1$ and whose value at the same time occurred more frequently among the $\alpha$'s than among the $\beta$'s. The remaining $(t-\tau)-(s-\sigma)$ of the $\beta$'s here in question will be each $>\alpha_\sigma$ since $\beta_\tau$ was the latest $\beta$ which was not $>\alpha_\sigma$, and we shall therefore have

$$\beta_{\tau+1} + \cdots + \beta_t \geqq (t-\tau-s+\sigma)\,\alpha_\sigma + \alpha_{\sigma+1} + \cdots + \alpha_s\,.$$

Subtracting the left-hand side and the right-hand side of this inequality from the left-hand side and the right-hand side respectively of the inequality (5), we derive

$$\beta_{t+1} + \cdots + \beta_{n-1} < (s-t-1)\,\alpha_\sigma + \alpha_{s+1} + \cdots + \alpha_n + \lambda$$

whence

$$t + \beta_{t+1} + \cdots + \beta_{n-1} < (s-t-1)\,(\alpha_\sigma - 1) + s + \alpha_{s+1} + \cdots + \alpha_n + \lambda - 1\,.$$

Here we have $\alpha_\sigma \leqq 1$. In case we have $\alpha_\sigma = 0$, or if we have simultaneously $\alpha_\sigma < 1$ and $t \leqq s-1$, we shall at the same time have $(s-t-1)(\alpha_\sigma - 1) \leqq 0$, and therefore

$$t + \beta_{t+1} + \cdots + \beta_{n-1} < s + \alpha_{s+1} + \cdots + \alpha_n + \lambda - 1$$

the same inequality (5) already found to hold true in Case I. If however we have $\alpha_\sigma < 1$ and $t > s-1$ this inequality does not immediately follow. In this case, going back to the product (1) we shall substitute the factors

$$v - z^{\alpha_1} P_1\left(\frac{1}{z}\right), \ \ldots \ v - z^{\alpha_s} P_s\left(\frac{1}{z}\right)$$

respectively for the first $s$ factors of the product. The deformed product will have the form

$$(7) \quad \left[v - z^{\alpha_1} P_1\left(\frac{1}{z}\right)\right] \cdots \left[v - z^{\alpha_s} P_s\left(\frac{1}{z}\right)\right]\left[\left[v - z^{\beta_{s+1}} Q_{s+1}\left(\frac{1}{z}\right)\right] \cdots \left[v - z^{\beta_{n-1}} Q_{n-1}\left(\frac{1}{z}\right)\right]\right].$$

Since we have $t > s-1$ and $\beta_t \leqq 1$ the exponents $\beta_1, \ldots \beta_s$ are none of them $> 1$. The same is also true of the exponents $\alpha_1, \ldots \alpha_s$ since $\alpha_{s+1}$ is the first of the $\alpha$'s which is $> 1$. It follows therefore that the orders of coincidence of the branches

$$v - z^{\alpha_{s+1}} P_{s+1}\left(\frac{1}{z}\right) = 0, \ \ldots \ v - z^{\alpha_n} P_n\left(\frac{1}{z}\right) = 0$$

with the product have not been affected by the deformation. Furthermore the orders of coincidence of the $s$ branches

$$v - z^{\alpha_1} P_1\left(\frac{1}{z}\right) = 0, \ \ldots \ v - z^{\alpha_s} P_s\left(\frac{1}{z}\right) = 0$$

with the deformed product are infinite. It consequently follows that the orders of coincidence of this product with the several branches are simultaneously greater than the numbers $\bar{\mu}_1 - \lambda, \ldots \bar{\mu}_n - \lambda$ respectively, for such by hypothesis was the case for the orders of coincidence with these branches of the original product (1).

The smallest value which occurs more frequently among the exponents $\alpha_1, \ldots \alpha_n$ than among the exponents of the product (7) will evidently be $\geqq \alpha_{s+1}$ and therefore $> 1$. The product (7) then comes under the head of those products which have been handled in Case I, and the inequality (6) will therefore apply to its exponents. The only exponents however which appear in this inequality are the ones which are $>1$, and these exponents have not been affected by the deformation of (1) into (7). In (7) then, as in (1), the exponents which are $>1$ are $\beta_{t+1}, \dots \beta_{n-1}$ and these exponents therefore satisfy the inequality

$$t + \beta_{t+1} + \cdots + \beta_{n-1} < s + \alpha_{s+1} + \cdots + \alpha_n + \lambda - 1.$$

We have established the existence of the inequality (6) then in the Cases I and II and it therefore holds in regard to any product of the form (1), whose orders of coincidence with the several branches are simultaneously greater than the numbers $\bar{\mu}_1 - \lambda, \dots \bar{\mu}_n - \lambda$ respectively. As we have already seen, the expression on the left of this inequality is equal to the degree of the function represented by the product (1). We have also seen that the expression $s + \alpha_{s+1} + \cdots + \alpha_n$ is equal to the degree $N$ of $F(z, v)$. It follows then from the inequality (6), that the degree of a function represented by a product of the form (1) must be $< N + \lambda - 1$, in case its orders of coincidence with the several branches of the curve be simultaneously greater than the numbers $\bar{\mu}_1 - \lambda, \dots \bar{\mu}_n - \lambda$ respectively. In particular if a rational function of $(z, v)$ of the form

$$v^{n-1} + h'_{n-2} v^{n-2} + \cdots + h'_0$$

have orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$, which are simultaneously greater than the corresponding numbers $\mu_1 - \lambda, \dots \mu_r - \lambda$ respectively, its degree will be $< N + \lambda - 1$.

The theorem just stated for the rational form above will also hold good for any rational function of $(z, v)$ in its reduced form, and that too for any value of the number $\lambda$ positive, negative or zero. For suppose that the orders of coincidence of the rational function

$$h_{n-1} v^{n-1} + h_{n-2} v^{n-2} + \cdots + h_0$$

with the branches of the several cycles are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively, and assume to begin with that we have $h_{n-1} \neq 0$. We may write the function in the form (IV, 9)

$$\left(\frac{1}{z}\right)^k g\left(\frac{1}{z}\right) \left(v^{n-1} + h'_{n-2}v^{n-2} + \cdots + h'_0\right).$$

The orders of coincidence of the factor

$$v^{n-1} + h'_{n-2}v^{n-2} + \cdots + h'_0$$

with the branches of the several cycles will then be simultaneously greater than the numbers $\mu_1-\lambda-k, \ldots \mu_r-\lambda-k$ respectively, and its degree — in accord with what has been already proved for a function of its form — must therefore be $<N+\lambda+k-1$. Multiplying by the factor $\left(\frac{1}{z}\right)^k g\left(\frac{1}{z}\right)$ the degree of the product will be $<N+\lambda-1$. We have proved then, with regard to any rational function in whose reduced form the coefficient of $v^{n-1}$ is different from 0, that its degree is $<N+\lambda-1$, in case its orders of coincidence with the branches of the several cycles are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively.

The existence of a function of the form here in question, having orders of coincidence with the branches of the several cycles which are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively, implies further a limitation on the value of the exponent $k$. Namely we saw that the orders of coincidence of the factor

$$v^{n-1} + h'_{n-2}v^{n-2} + \cdots + h'_0$$

with the branches of the several cycles, must be simultaneously greater than the numbers $\mu_1-\lambda-k, \ldots \mu_r-\lambda-k$ respectively, and we know from Chapter IV, that for a function of this form the orders of coincidence with the branches of the several cycles cannot simultaneously be greater than the numbers $\mu_1, \ldots \mu_r$ respectively. It follows that we must have $\lambda+k>0$ and consequently $-k<\lambda$. The degree of the element involving $v^{n-1}$ in the general rational function (IV, 9) is however $n-k-1$, and this degree in the case of the functions here in question must therefore be $<n+\lambda-1$.

We may say then of a rational function in whose reduced form we have $h_{n-1} \neq 0$, and whose orders of coincidence with the branches of the several cycles are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively, not only that its degree is $<N+\lambda-1$, but also that the degree of the element $h_{n-1}v^{n-1}$ must be $<n+\lambda-1$. For example $F'_v(z, v)$ is a function which involves the element $nv^{n-1}$, and whose orders of coincidence with the branches of the several cycles, as we have seen in the chapter preceding, are equal to $\mu_1, \ldots \mu_r$ respectively. These orders of coincidence however are simultaneously greater than the numbers $\mu_1-1, \ldots \mu_r-1$. The degree of the function then in accordance with the theory developed above would have to be $<N$, and it is as a matter of fact equal to $N-1$ except in the case where the degree of $F(z, v)$ depends on a term which does not involve $v$, in which case the degree of $F'_v(z, v)$ is $<N-1$. The degree of the function $z^{-k}F'_v(z, v)$, it may be remarked, is $\overline{\gtrless} N-k-1$ and its orders of coincidence with the branches of the several cycles are equal to the numbers $\mu_1+k, \ldots \mu_r+k$ respectively.

Turning now to the case of a rational function in which we have $h_{n-1}=0$, we shall prove that its degree is subject to the same limitation which has been shown to exist in the case where we have $h_{n-1}\neq 0$. That is we shall prove that the degree of a function

$$h_{n-2}v^{n-2}+\cdots+h_0 \tag{8}$$

must be $<N+\lambda-1$, if its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively. — To the function (8) we add any rational function in which the coefficient of $v^{n-1}$ is different from 0, and whose orders of coincidence with the branches of the several cycles are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively. Such a function, for example, is $z^{[\lambda]-1}F'_v(z, v)$. The sum

$$z^{[\lambda]-1}F'_v(z, v)+h_{n-2}v^{n-2}+\cdots+h_0$$

involves the power $v^{n-1}$ with a coefficient which is different from 0. Also its orders of coincidence with the branches of the several cycles are simultaneously greater than the numbers $\mu_1-\lambda, \ldots \mu_r-\lambda$ respectively. Its degree must therefore be $<N+\lambda-1$. The degree of $z^{[\lambda]-1}F'_v(z, v)$ however is $\overline{\gtrless} N+[\lambda]-2<N+\lambda-1$. Therefore the degree of the difference obtained on

subtracting this function from the sum must be $< N + \lambda - 1$ — that is the degree of the function (8) is $< N + \lambda - 1$.

With regard to any rational function of $(z, v)$ then, whose orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ are simultaneously greater than the numbers $\mu_1 - \lambda, \ldots \mu_r - \lambda$ respectively, we have proved that the degree of its reduced form must be $< N + \lambda - 1$, and further that the degree of the element $h_{n-1}v^{n-1}$, in case it presents itself, must be $< n + \lambda - 1$.

When $\lambda$ is an integer the theorem just stated may evidently also be worded as follows: — Any rational function of $(z, v)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ are equal to or greater than the numbers

$$(9) \qquad \mu_1 - \lambda + \frac{1}{\nu_1}, \ldots \mu_r - \lambda + \frac{1}{\nu_r}$$

respectively, will, when expressed in its reduced form, have a degree which is $\leqq N + \lambda - 2$ and the degree of the element $h_{n-1}v^{n-1}$ in this form, in case it presents itself, will be $\leqq n + \lambda - 2$.

In particular when $\lambda = 1$ the numbers (9) represent the orders of coincidence just requisite to adjointness for the value $z = \infty$. We conclude that the reduced form of a rational function of $(z, v)$ which is adjoint for the value $z = \infty$ will have a degree which is $\leqq N - 1$ and that the degree of the element $h_{n-1}v^{n-1}$ in this form will be $\leqq n - 1$. It evidently follows that the coefficient $h_{n-1}$ in this case can contain no negative exponents in its development according to powers of $\frac{1}{z}$.

An important case is that in which $\lambda = -1$. For this value of $\lambda$ the theorem will read as follows: — A rational function of $(z, v)$ whose orders of coincidence with the branches of the several cycles are equal to or greater than the numbers

$$(10) \qquad \mu_1 + 1 + \frac{1}{\nu_1}, \ldots \mu_r + 1 + \frac{1}{\nu_r}$$

respectively, must have in its reduced form a degree which is $\leqq N - 3$, and the degree of the element $h_{n-1}v^{n-1}$ in this form, in case it presents itself, must be $\leqq n - 3$.

In the case of an integral rational function to which the theorem just stated applies an element $h_{n-1}v^{n-1}$ will not present itself, for the degree of such an element would in this case be $>n-3$. We may then say of an integral rational function of $(z, v)$ whose orders of coincidence with the branches of the several cycles are equal to or greater than the respective numbers of the set (10), that it must on reduction take the form (8) and have a degree which is $\leqq N-3$.

Integral rational functions of the degree $N-3$ and possessing for the value $z = \infty$ orders of coincidence such as those here in question will reappear in the theory later on.

# CHAPTER VII.

## Sets of complementary adjoint orders of coincidence.

Functions which are *complementary adjoint* to each other for a given value of the variable $z$. The product of integral rational functions which are complementary adjoint to the order $i$ for a value $z=a$ is divisible by $(z-a)^i$. The orders of coincidence of an integral rational function $\psi(z,v)$ for a given value $z=a$ will be complementary adjoint to the order $i$ to a certain set of adjoint orders of coincidence corresponding to the same value of the variable, if the product of $\psi(z,v)$ by the general integral rational function possessing the latter set of orders of coincidence be divisible by $(z-a)^i$.

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Returning now to the consideration of finite values of the variable $z$, we have seen in Chapter IV that the orders of coincidence of an integral rational function of $(z,v)$ which is not divisible by $z-a$, with the branches of the several cycles corresponding to the value $z=a$, cannot simultaneously be greater than the numbers $\mu_1, \ldots \mu_r$ respectively corresponding to the value of the variable in question. We have furthermore seen, that if an integral rational function of $(z,v)$ have orders of coincidence with the branches of the several cycles which are equal to or greater than the numbers

(1) $$\mu_1+i-1+\frac{1}{\nu_1}, \ldots \mu_r+i-1+\frac{1}{\nu_r}$$

respectively, where $i$ is a positive integer, then must the function be divisible by $(z-a)^i$.

Let us now consider the product of two integral rational functions of $(z,v)$. This product expressed in its reduced form may happen to be divisible by $z-a$ though this is not the case for either of the factors separately. We shall suppose $\varphi(z,v)$ and $\psi(z,v)$ to be two integral rational functions of $(z,v)$, which may or may not happen to be divisible by $z-a$. Their orders of coincidence with the branches of the several cycles we shall designate by $\mu_1', \ldots \mu_r'$ and $\mu_1'', \ldots \mu_r''$ respectively. The orders of coincidence of the product of the two functions with the branches of the cycles, will be equal to the sums of the pairs of the corresponding orders of coincidence of the functions with these branches. Its orders of coincidence with the branches of the several cycles will therefore be

$$\mu_1' + \mu_1'', \ldots \mu_r' + \mu_r''.$$

If we now have

$$(2) \qquad \mu_1 + \mu_1'' \geqq \mu_1 + i - 1 + \frac{1}{\nu_1}, \ldots \mu_r' + \mu_r'' \geqq \mu_r + i - 1 + \frac{1}{\nu_r}$$

or, what amounts to the same thing, if we have

$$\mu_1' + \mu_1'' > \mu_1 + i - 1, \ldots \mu_r' + \mu_r'' > \mu_r + i - 1$$

where $i$ is a positive integer, it follows that the product in its reduced form must be divisible by $(z-a)^i$, and we therefore have

$$(3) \qquad \varphi(z,v) \,.\, \psi(z,v) = (z-a)^i \,\Omega(z,v)$$

where $\Omega(z,v)$ is an integral rational function of $(z,v)$ supposed to be expressed in its reduced form. The functions $\varphi(z,v)$ and $\psi(z,v)$ also we suppose to be expressed in their reduced forms, and the relation (3) is then equivalent to an identity of the form

$$(4) \qquad \varphi(z,v) \,.\, \psi(z,v) = \vartheta(z,v) \,.\, F(z,v) + (z-a)^i \Omega(z,v)$$

where $\vartheta(z,v)$ is an integral rational function of $(z,v)$ not involving $v$ to a power higher than $v^{n-2}$, for $F(z,v)$ involves the term $v^n$ and powers of $v$ higher than $v^{2n-2}$ do not present themselves, since neither of the factors on the left of the identity involves $v$ to a power higher than $v^{n-1}$.

Suppose, for example, that we have to do with a value $z=a$ to which correspond $n$ different points of the curve. The $n$ branches in this case are all simple and we have $r=n$. We also have $\mu_1 = \mu_2 = \cdots = \mu_n = 0$. If furthermore the functions $\varphi(z,v)$ and $\psi(z,v)$ have as orders of coincidence with the several branches of the curve the numbers $\mu'_1 = i, \mu'_2 = i, \dots \mu'_l = i, \mu'_{l+1} = 0 \dots$ $\dots \mu'_n = 0$ and $\mu''_1 = 0, \mu''_2 = 0, \dots \mu''_l = 0, \mu''_{l+1} = i, \dots \mu''_n = i$ respectively, we have

$$\mu'_1 + \mu''_1 = i > \mu_1 + i - 1, \mu'_2 + \mu''_2 = i > \mu_2 + i - 1, \dots \mu'_n + \mu''_n = i > \mu_n + i - 1$$

and the product of the functions reduces to the form given on the right of (3).

Let us now consider an example in which we have to do with a value $z = a$ to which correspond less than $n$ different values of $v$. The orders of coincidence of the functions $\varphi(z,v)$ and $\psi(z,v)$ with the branches of the several cycles we shall suppose to be numbers $\mu'_1, \dots \mu'_r$ and $\mu''_1, \dots \mu''_r$ respectively, which satisfy the conditions

$$\mu'_1 \geqq i - 1 + \frac{1}{\nu_1}, \dots \mu'_r \geqq i - 1 + \frac{1}{\nu_r}; \mu''_1 \geqq \mu_1, \dots \mu''_r \geqq \mu_r$$

where, as before, $i$ is supposed to represent an integer. The inequalities (2) are then satisfied and the product of the two functions is reducible to the form given on the right of (3). The same would evidently also hold true in case the orders of coincidence of the functions satisfied the conditions

$$\mu'_1 \geqq i, \dots \mu'_r \geqq i; \mu''_1 \geqq \mu_1 - 1 + \frac{1}{\nu_1}, \dots \mu''_r \geqq \mu_r - 1 + \frac{1}{\nu_r}.$$

In the preceding the functions $\varphi(z,v)$ and $\psi(z,v)$ may or may not happen to be divisible by $z-a$. In the case of the last example however it is evident that the greatest value of the integer $i$ consistent with the function $\varphi(z,v)$ not being divisible by $z-a$ is the greatest of the integers $[\mu_1]$, $\dots [\mu_r]$, for we have seen that no integral rational function of $(z,v)$, which is not divisible by $z-a$, can have orders of coincidence with the branches of the several cycles which are simultaneously greater than the numbers $\mu_1, \dots \mu_r$ respectively.

The term adjoint has already been defined in Chapter V. If the product of two functions be adjoint for a certain value of the variable $z$, we shall say that the functions are *complementary adjoint* to each other for such value of the variable. Also if the orders of coincidence of the product with the branches of the several cycles be not less than the numbers

$$\mu_1 + i - 1 + \frac{1}{\nu_1}, \ldots \mu_r + i - 1 + \frac{1}{\nu_r}$$

respectively, we shall say that the functions are *complementary adjoint to the order i* for the value of the variable in question. The sets of orders of coincidence of the functions too, in such case, we shall say are complementary adjoint to the order $i$. For our purposes it here suffices to regard $i$ as an integer.

A function which is adjoint for a given finite value of $z$ is, for such value of the variable, complementary adjoint to any integral rational function of $(z, v)$, for the product of the two functions would evidently also be adjoint for the value of the variable in question.

There is no limit to the order to which two integral rational functions of $(z, v)$ may be complementary adjoint for a given value $z = a$, and that too — one case excepted — without either of the functions being divisible by $z - a$. In Chapter IV namely, we have seen that it is always possible to construct an integral rational function of $(z, v)$ which is not divisible by $z - a$, and whose orders of coincidence with the branches of an arbitrary set of $r - 1$ out of the $r$ cycles are as large as we may please while its order of coincidence with the branches of the remaining cycle, in case this be the one of order $\nu_s$, is $\mu_s$. Excepting in the case where $r = 1$ then, we might construct two functions $\varphi(z, v)$ and $\psi(z, v)$ neither of which is divisible by $z - a$ and which are such that the orders of coincidence of one of them with the branches of $r - k$ out of the $r$ cycles are indefinitely large, while the orders of coincidence of the other one with the branches of the remaining $k$ cycles are as large as we may please. The orders of coincidence of the product of the functions with the branches of all $r$ cycles could thus be made indefinitely large without either of the functions being divisible by $z - a$. Where we have $r = 1$ however this would be impossible, for in this case the $n$ branches constitute a single cycle of order $n$ and the set of numbers $\mu_1, \ldots \mu_r$ reduces to the single number $\mu_1$, so that it would not be possible to construct an integral rational function of $(z, v)$ which is not divisible by $z-a$ and whose order of coincidence with the branches of the cycle is greater than $\mu_1$. This is included namely under the general theorem that an integral rational function of $(z, v)$ which is not divisible by $z-a$ cannot have orders of coincidence with the branches of the several cycles which are simultaneously greater than $\mu_1, \ldots \mu_r$ respectively.



In (3) we have an expression for the product of two integral rational functions of $(z, v)$ which are complementary adjoint to each other to the order $i$ for the value $z=a$. More generally we can write the product of any two integral rational functions $\varphi(z, v)$ and $\psi(z, v)$ in the form

$$(5) \qquad \varphi(z, v) \,.\, \psi(z, v) = \chi^{(i)}(z, v) + (z-a)^i \,\Omega(z, v)$$

where the product in its reduced form is supposed to be separated into two parts $\chi^{(i)}(z, v)$ and $(z-a)^i\,\Omega(z, v)$, of which the latter is divisible by $(z-a)^i$ while the former involves no power of $z-a$ higher than $(z-a)^{i-1}$. Corresponding to a given integer $i$ there is evidently only one such representation of the product.

In case the functions $\varphi(z, v)$ and $\psi(z, v)$ happen to be complementary adjoint to the order $i$ for the value $z=a$, the function $\chi^{(i)}(z, v)$ must vanish identically and the relation (5) will assume the form (3). We may also write (5) in the form of an identity

$$(6) \qquad \varphi(z, v) \,.\, \psi(z, v) = \vartheta(z, v) \,.\, F(z, v) + \chi^{(i)}(z, v) + (z-a)^i\,\Omega(z, v)$$

which reduces to the identity (4) when $\varphi(z, v)$ and $\psi(z, v)$ are complementary adjoint to each other to the order $i$. If then for the value $z=a$ an integral rational function $\psi(z, v)$ possess a set of orders of coincidence which is complementary adjoint to the order $i$ to a set of numbers $\mu'_1, \ldots \mu'_r$, it follows that the function $\chi^{(i)}(z, v)$ must vanish identically in the product of $\psi(z, v)$ by any integral rational function $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are equal to or greater than the numbers $\mu'_1, \ldots \mu'_r$ respectively.

We shall limit ourselves to the case in which the numbers $\mu'_1, \ldots \mu'_r$ constitute an adjoint set relative to the value of the variable in question, and shall prove that in this case the converse of the proposition just stated also holds good. We shall prove namely, that the orders of coincidence $\mu_1'', \ldots \mu_r''$ of an integral rational function $\psi(z,v)$ with the branches of the several cycles, must be complementary adjoint to the order $i$ to a set of adjoint numbers $\mu_1', \ldots \mu_r'$, if the function $\chi^{(i)}(z,v)$ always vanishes identically in the product of $\psi(z,v)$ by any integral rational function $\varphi(z,v)$ whose orders of coincidence with the branches of the several cycles are equal to or greater than the numbers $\mu_1', \ldots \mu_r'$ respectively. Otherwise expressed, the orders of coincidence $\mu_1'', \ldots \mu_r''$ of $\psi(z,v)$ must satisfy the inequalities

$$\mu_1' + \mu_1'' > \mu_1 + i - 1, \ldots \mu_r' + \mu_r'' > \mu_r + i - 1$$

if the function $\chi^{(i)}(z,v)$ always vanishes identically in the product of $\psi(z,v)$ by any integral rational function $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are equal to or greater than the adjoint numbers $\mu_1', \ldots \mu_r'$ respectively.

To prove this we shall have to make use of the irrational function $B(z,v)$ whose form has been given in (IV, 5). We have seen that it is always possible to construct such a function having orders of coincidence as large as we please with $n-1$ out of the $n$ branches and having $\mu_s$ as its order of coincidence with the remaining branch, in case this branch be a member of the cycle of order $\nu_s$. Indicating the branch in question by the equation $v - P_1 = 0$ and multiplying the function $B(z,v)$ by the factor $(z-a)^{\mu_s' - \mu_s + 1 - \frac{1}{\nu_s}}$, we obtain a function

$$(7) \quad (z-a)^{\mu_s' - \mu_s + 1 - \frac{1}{\nu_s}} B(z,v) = (z-a)^{\mu_s' - \mu_s + 1 - \frac{1}{\nu_s}} \{B_0(z,v) + (z-a)^{\frac{1}{\nu_s}} B_1(z,v) + \cdots + (z-a)^{\frac{\nu_s - 1}{\nu_s}} B_{\nu_s - 1}(z,v)\} . R(z,v)$$

whose order of coincidence with the branch $v - P_1 = 0$ is $\mu_s' + 1 - \frac{1}{\nu_s}$, while its orders of coincidence with the other branches may be regarded as indefinitely large. Let us now consider one of the elements $(z-a)^{\frac{\sigma}{\nu_s}} B_\sigma(z,v) . R(z,v)$ in the sum which constitutes the function $B(z,v)$. The order of coincidence of such an element with the branch $v - P_1 = 0$ is equal to $\mu_s$, as we have seen in Chapter IV. This then will evidently also be the order of coincidence of the element with each of the branches belonging to the cycle of order $\nu_s$. In the expression on the right-hand side of (7) then, the order of coincidence of an element

$$(8)\qquad (z-a)^{\mu'_s-\mu_s+1+\frac{\sigma-1}{\nu_s}} B_\sigma(z,v) \,.\, R(z,v)$$

with the branches of the cycle of order $\nu_s$ will be equal to $(\mu'_s-\mu_s+1-\frac{1}{\nu_s})+\mu_s=\mu'_s+1-\frac{1}{\nu_s}$, while by virtue of the factor $R(z,v)$ its orders of coincidence with the branches of the other cycles may be made as large as we please.

Indicating by $i_\sigma=[\mu'_s-\mu_s+1+\frac{\sigma-1}{\nu_s}]$ the greatest integer contained in the exponent of $z-a$ in (8), we evidently have

$$i_\sigma \overline{\gtrless} (\mu'_s-\mu_s+1+\frac{\sigma-1}{\nu_s})-(1-\frac{1}{\nu_s})$$

and the order of coincidence of the rational function

$$(9)\qquad (z-a)^{i_\sigma} B_\sigma(z,v) \,.\, R(z,v)$$

with the branches of the cycle of order $\nu_s$ is therefore $\overline{\gtrless}\mu'_s$, while its orders of coincidence with the branches of the other cycles may by proper choice of the factor $R(z,v)$ be made greater than any assigned set of values. We shall assume then that it has been so chosen that the orders of coincidence of the function (9) with the branches of the cycles of orders $\nu_1, \ldots \nu_{s-1}, \nu_{s+1}, \ldots \nu_r$ are not less than the numbers $\mu'_1, \ldots \mu'_{s-1}, \mu'_{s+1}, \ldots \mu'_r$ respectively. The function (9) is therefore an integral rational function of $(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the numbers $\mu'_1, \ldots \mu'_{s-1}, \mu'_s, \mu'_{s+1}, \ldots \mu'_r$ respectively. It is then one of those functions $\varphi(z,v)$ in the product of which by the given function $\psi(z,v)$ the function $\chi^{(i)}(z,v)$ vanishes identically.

If therefore in the identity (6) we replace $\varphi(z,v)$ by the function (9) we arrive at an identity which may be written in the form

(10) $(z-a)^{i_\sigma} B_\sigma(z,v) \,.\, R(z,v) \,.\, \psi(z,v) = \bar{\vartheta}_\sigma(z,v) \,.\, F(z,v) + (z-a)^i \bar{\Omega}_\sigma(z,v).$



Multiplying both sides of this identity by the factor

$$(z-a)^{\mu_s' - \mu_s + 1 + \frac{\sigma-1}{\nu_s} - i_\sigma},$$

we obtain

(11) $(z-a)^{\mu_s' - \mu_s + 1 + \frac{\sigma-1}{\nu_s}} B_\sigma(z,v) \,.\, R(z,v) \,.\, \psi(z,v) = \vartheta_\sigma(z,v) \,.\, F(z,v) + (z-a)^i \Omega_\sigma(z,v)$

where however $\vartheta_\sigma(z,v)$ and $\Omega_\sigma(z,v)$ may contain a fractional power of $z-a$ as factor. Summing the expressions (11) for $\sigma = 0, 1, \ldots. (\nu_s - 1)$ we arrive at an identity

(12) $(z-a)^{\mu_s' - \mu_s + 1 - \frac{1}{\nu_s}} B(z,v) \,.\, \psi(z,v) = \vartheta(z,v) \,.\, F(z,v) + (z-a)^i \Omega(z,v)$

where $\vartheta(z,v)$ and $\Omega(z,v)$ are integral rational functions of $((z-a)^{\frac{1}{\nu_s}}, v)$, of which the former involves $v$ at most to the power $v^{n-2}$ whereas the latter involves $v$ to the power $v^{n-1}$ at most.

The function $B(z,v)$, as we have seen, may be supposed to have orders of coincidence as large as we please with the $n-1$ branches of the curve other than $v - P_1 = 0$, and the same will therefore hold true for the function $\Omega(z,v)$. If however the orders of coincidence of $\Omega(z,v)$ with the $n-1$ branches in question be sufficiently large, we know that its order of coincidence with the remaining branch $v - P_1 = 0$ must be $\gtreqless \mu_s$. If, for example, its orders of coincidence with $n-1$ of the $n$ branches be extraadjoint, we know from the proposition at the end of Chapter V that its order of coincidence with the remaining branch must also be extraadjoint. In constructing the function $B(z,v)$ then, we shall suppose in the first place that we have assigned extraadjoint orders of coincidence to all $n$ branches excepting only the branch $v - P_1 = 0$. Its order of coincidence with this branch also will then be extraadjoint, and will as we know be precisely equal to $\mu_s$. In the second place the extraadjoint orders of coincidence which we have assigned to the $n-1$ branches in question, we shall assume to have been chosen sufficiently large to assure orders of coincidence with the branches of the several cycles on the part of the product on the left-hand side of the identity (12.), which are simultaneously equal to or greater than the numbers $i+\mu_1, \dots i+\mu_r$ respectively — the branch $v-P_1=0$ being always excepted. The order of coincidence of the product with the branch $v-P_1=0$ will evidently be equal to the sum $(\mu'_s-\mu_s+1-\frac{1}{\nu_s})+\mu_s+\mu''_s=\mu'_s+\mu''_s+1-\frac{1}{\nu_s}$. Now the orders of coincidence of the several branches with the product are equal to their orders of coincidence with the element $(z-a)^i\Omega(z,v)$ on the right-hand side of the identity (12). The orders of coincidence of this element with the branches of the several cycles must then be equal to or greater than the numbers $i+\mu_1, \dots i+\mu_r$ respectively — the branch $v-P_1=0$ being excepted. The function $\Omega(z,v)$ must then have extraadjoint orders of coincidence with all the branches — excepting only the branch $v-P_1=0$. Its order of coincidence with this branch must then also be extraadjoint. The order of coincidence of the branch $v-P_1=0$ with the element $(z-a)^i\Omega(z,v)$ will therefore be $\geqq i+\mu_s$. Its order of coincidence with the element however is equal to its order of coincidence with the product on the left-hand side of the identity (12) and must therefore have the value $\mu'_s+\mu''_s+1-\frac{1}{\nu_s}$. It follows that we must have $\mu'_s+\mu''_s+1-\frac{1}{\nu_s}\geqq i+\mu_s$, whence

$$\mu'_s+\mu''_s\geqq\mu_s+i-1+\frac{1}{\nu_s}.$$

The inequality just arrived at has been derived with reference to an arbitrary one of the $n$ branches, represented for convenience by $v-P_1=0$ and supposed to belong to the cycle of order $\nu_s$. Such an inequality corresponding to any arbitrary one of the $r$ cycles will then exist, and we shall have simultaneously the $r$ inequalities

$$\mu'_1+\mu''_1\geqq\mu_1+i-1+\frac{1}{\nu_1}, \quad \dots \quad \mu'_r+\mu''_r\geqq\mu_r+i-1+\frac{1}{\nu_r}.$$

The function $\psi(z,v)$ must therefore be complementary adjoint to the order $i$ to the integral rational functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of the set of adjoint numbers $\mu'_1, \dots \mu'_r$.

To recapitulate. — If the function $\chi^{(i)}(z, v)$ vanish identically in the product of a given integral rational function $\psi(z, v)$ by any arbitrary integral rational function $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of a certain set of adjoint numbers $\mu'_1, \ldots \mu'_r$, the said function $\psi(z, v)$ must be complementary adjoint to the order $i$ to the system of functions $\varphi(z, v)$ in question. Conversely, we have seen that the function $\chi^{(i)}(z, v)$ in (6.) must vanish identically, in case the functions $\varphi(z, v)$ and $\psi(z, v)$ be complementary adjoint to the order $i$.

We may say then that the necessary and sufficient condition that an integral rational function $\psi(z, v)$ should be complementary adjoint to the order $i$ to all integral rational functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of a certain set of adjoint numbers $\mu'_1, \ldots \mu'_r$, is that the functions $\chi^{(i)}(z, v)$ in (6.) corresponding to the products of $\psi(z, v)$ by the functions $\varphi(z, v)$ in question, shall in all cases vanish identically.

# CHAPTER VIII.

## Conditions for certain sets of orders of coincidence for $z=a$.

The number of the linearly independent conditions which must be satisfied by the coefficients of an integral rational function $\psi(z.v)$ whose orders of coincidence corresponding to a finite value of the variable $z$ are complementary adjoint to the order $i$ to a given set of adjoint orders of coincidence corresponding to the value of the variable in question. The number of the linearly independent conditions which must be satisfied by the coefficients of an integral rational function of $(z,v)$ in order that it may be adjoint relatively to a given finite value of the variable $z$. Extension of results to the value $z=\infty$.

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In order that the function $\chi^{(i)}(z,v)$ in (VII. 6) may vanish identically the coefficients of the functions $\varphi(z,v)$ and $\psi(z,v)$ must satisfy a number of conditions, depending on the value of the exponent $i$. The coefficients of $\chi^{(i)}(z,v)$ are bilinear in the coefficients of $\varphi(z,v)$ and $\psi(z,v)$, and evidently also only involve such coefficients of these functions as belong to terms in which $z-a$ appears to a power lower than $(z-a)^i$. If one of the functions $\varphi(z,v)$ or $\psi(z,v)$ be given, the coefficients of the other function must satisfy a number of linear conditions in order that the corresponding function $\chi^{(i)}(z,v)$ may vanish identically. If the functions $\chi^{(i)}(z,v)$ in the products of the members of a given system of integral rational functions $\varphi_1(z,v)$, $\varphi_2(z,v)$, ... by the same integral rational function $\psi(z,v)$ are all to vanish identically, the coefficients of $\psi(z,v)$ must satisfy a number of linear conditions corresponding to each of the functions $\varphi(z,v)$ in question. These conditions, it may be, are not independent of one another. The number of the conditions which are independent of one another is in any case however finite, for whatever the functions $\varphi(z,v)$ may be, so long as they are integral, and however many they may be, it will suffice to equate to 0 the coefficients of $\psi(z,v)$ belonging to terms involving $z-a$ to a power lower than $(z-a)^i$, in order that the corresponding functions $\chi^{(i)}(z,v)$ may all vanish.

If an integral rational function $\psi(z,v)$ is to be complementary adjoint to the order $i$ to the system of integral rational functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of a given set of adjoint numbers $\mu'_1, \dots \mu'_r$, we have seen that the functions $\chi^{(i)}(z,v)$ in the system of products $\varphi(z,v).\psi(z,v)$ must all vanish identically. Among other conditions, this implies that the coefficient of the term in $(z-a)^{i-1}v^{n-1}$ in each of the functions $\chi^{(i)}(z,v)$ in question must be equal to 0. We shall now shew conversely, that if the coefficient of $(z-a)^{i-1}v^{n-1}$ is always 0 in the functions $\chi^{(i)}(z,v)$ corresponding to the products of a certain integral rational function $\psi(z,v)$ by the members of the system of integral rational functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of a certain set of adjoint numbers $\mu'_1, \dots \mu'_r$, then must the system of functions $\chi^{(i)}(z,v)$ corresponding to the set of products in question all vanish identically.

We shall suppose that the coefficient of $(z-a)^{i-1}v^{n-1}$ is equal to 0 in each of the system of functions $\chi^{(i)}(z,v)$ in question, and we shall assume at the same time, if possible, that one of the functions — $\overline{\chi}^{(i)}(z,v)$ — does not vanish identically. We shall then have a relation of the form

$$\overline{\varphi}(z,v).\psi(z,v)=\overline{\chi}^{(i)}(z,v)+(z-a)^i\,\overline{\Omega}(z,v)$$

in which $\overline{\chi}^{(i)}(z,v)$ does not vanish identically and where $\overline{\varphi}(z,v)$ is one of the system of functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1, \dots \mu'_r$ respectively. We shall further assume that $v^t$ is the highest power of $v$ which occurs in the expression of $\overline{\chi}^{(i)}(z,v)$ and that a term in $(z-a)^s v^t$ presents itself. — Here we have of course $s \leqq i-1, t \leqq n-1$. — On multiplying the equation above by the factor $(z-a)^{i-s-1}v^{n-t-1}$ we obtain

$$(z-a)^{i-s-1} v^{n-t-1} \bar{\varphi}(z, v) \, . \, \psi(z, v) = (z-a)^{i-s-1} v^{n-t-1} \bar{\chi}^{(i)}(z, v) +$$

$$(z-a)^{i} \, . \, (z-a)^{i-s-1} v^{n-t-1} \bar{\Omega}(z, v).$$

Since the function $\bar{\chi}^{(i)}(z, v)$ involves a term in $(z-a)^{s} v^{t}$, the function $(z-a)^{i-s-1} v^{n-t-1} \bar{\chi}^{(i)}(z, v)$ will involve a term in $(z-a)^{i-1} v^{n-1}$. Furthermore this function as it stands is in its reduced form, for it involves no power of $v$ higher than $v^{n-1}$. It will then consist of terms divisible by $(z-a)^{i}$, together with a function $\bar{\chi}^{(i)}(z, v)$ which involves a term in $(z-a)^{i-1} v^{n-1}$.

On representing the reduced form of $(z-a)^{i-s-1} v^{n-t-1} \bar{\varphi}(z, v)$ by $\varphi(z, v)$, the last equation will assume the form

$$\varphi(z, v) \, . \, \psi(z, v) = \chi^{(i)}(z, v) + (z-a)^{i} \Omega(z, v)$$

where $\chi^{(i)}(z, v)$ involves a term in $(z-a)^{i-1} v^{n-1}$, and where the orders of coincidence of the function $\varphi(z, v)$ with the branches of the several cycles are evidently not less than those of the function $\bar{\varphi}(z, v)$ with the same branches, and therefore not less than the adjoint numbers $\mu'_1, \dots \mu'_r$ respectively. This result however is at variance with our original hypothesis according to which the coefficient of $(z-a)^{i-1} v^{n-1}$ is 0 in every function $\chi^{(i)}(z, v)$ corresponding to one of the system of integral rational functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the numbers $\mu'_1, \dots \mu'_r$ respectively. If then the coefficient of $(z-a)^{i-1} v^{n-1}$ is 0 in each one of the system of functions $\chi^{(i)}(z, v)$ here in question, it must be that these functions themselves all vanish identically. In the preceding chapter however we have seen if these functions all vanish identically, that the function $\psi(z, v)$ must be complementary adjoint to the order $i$ to the system of functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the numbers $\mu'_1, \dots . \mu'_r$ respectively. It follows therefore, that if the coefficient of $(z-a)^{i-1} v^{n-1}$ is always 0 in the functions $\chi^{(i)}(z, v)$ corresponding to the products of the integral rational function $\psi(z, v)$ by the members of the system of integral rational functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1, \dots \mu'_r$ respectively, then must the function $\psi(z, v)$ be complementary adjoint to the order $i$ to the functions

$\varphi(z, v)$ in question. As the converse of this statement has already been seen to hold good we arrive at the following theorem. — The necessary and sufficient condition that an integral rational function $\psi(z, v)$ should be complementary adjoint to the order $i$ to the system of integral rational functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the respective members of a certain set of adjoint numbers $\mu'_1, \ldots \mu'_r$, is that the coefficient of $(z-a)^{i-1} v^{n-1}$ in the functions $\chi^{(i)}(z, v)$ corresponding to the products of $\psi(z, v)$ by the functions $\varphi(z, v)$ in question, shall in all cases be equal to 0.

Write the functions $\varphi(z, v)$ in the form

$$\varphi(z, v) = \varphi^{(i)}(z, v) + (z-a)^i ((z-a, v))$$

where we employ the notation $((z-a, v))$ to represent an expression which, arranged according to powers of $z-a$ and $v$, involves no negative exponents. The only terms in $\varphi(z, v)$ which affect the corresponding function $\chi^{(i)}(z, v)$ are those which appear in $\varphi^{(i)}(z, v)$. The same function $\chi^{(i)}(z, v)$ then appears in the expression of the product $\varphi^{(i)}(z, v) . \psi(z, v)$ in the form (VII, 6), as in that of the product $\varphi(z, v) . \psi(z, v)$ in such form. To each one of the functions $\varphi(z, v)$ corresponds a function $\varphi^{(i)}(z, v)$. The number of these $\varphi^{(i)}$-functions which are linearly independent of one another is finite. Their number we shall indicate by $l$ and shall represent by $\varphi_1^{(i)}(z, v), \varphi_2^{(i)}(z, v), \ldots \varphi_l^{(i)}(z, v)$ a complete system of linearly independent $\varphi^{(i)}$-functions corresponding to the functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1, \ldots \mu'_r$ respectively.

We shall have a system of $l$ identities

$$\text{(I)} \quad \varphi_s^{(i)}(z, v) . \psi(z, v) = \vartheta_s(z, v) . F(z, v) + \chi_s^{(i)}(z, v) + (z-a)^i \Omega_s(z, v), \quad (s = 1, \ldots l)$$

in which, if the coefficient of $(z-a)^{i-1} v^{n-1}$ in each of the $l$ functions $\chi_1^{(i)}(z, v), \ldots \chi_l^{(i)}(z, v)$ be equal to 0, these functions themselves will all vanish identically and the function $\psi(z, v)$ will be complementary adjoint to the order $i$ to the system of functions $\varphi(z, v)$. For, if $\varphi(z, v)$ be any one of the system of functions in question and $\varphi^{(i)}(z,v)$ the corresponding $\varphi^{(i)}$-function, we shall have an identity of the form



$$\varphi^{(i)}(z,v)\,.\,\psi(z,v)=\vartheta(z,v)\,.\,F(z,v)+\chi^{(i)}(z,v)+(z-a)^i\Omega(z,v)$$

where $\varphi^{(i)}(z,v)$ is a linear function of the $l$ functions $\varphi_1^{(i)}(z,v),\ldots\varphi_l^{(i)}(z,v)$, and where therefore evidently also the function $\chi^{(i)}(z,v)$ is the same linear function of the $l$ functions $\chi_1^{(i)}(z,v),\ldots\chi_l^{(i)}(z,v)$ in the identities (1). If then the coefficient of $(z-a)^{i-1}v^{n-1}$ in each of these $l$ functions be 0, its coefficient in the function $\chi^{(i)}(z,v)$ corresponding to any one of the functions $\varphi(z,v)$ will be 0. The functions $\chi^{(i)}(z,v)$ themselves will therefore vanish identically by what we have proved above, and the function $\psi(z,v)$ will be complementary adjoint to the order $i$ to the functions $\varphi(z,v)$ in question. Conversely, if $\psi(z,v)$ be complementary adjoint to the order $i$ to the system of functions $\varphi(z,v)$, we have seen that the coefficients of $(z-a)^{i-1}v^{n-1}$ in the corresponding set of functions $\chi^{(i)}(z,v)$, and therefore in particular in the $l$ functions $\chi_1^{(i)}(z,v),\ldots\chi_l^{(i)}(z,v)$, must all be equal to 0.

In order that the integral rational function $\psi(z,v)$ should be complementary adjoint to the order $i$ to the system of functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1,\ldots\mu'_r$ respectively, it is then necessary and sufficient that the coefficient of $(z-a)^{i-1}v^{n-1}$ should be equal to 0 in each of the $l$ functions $\chi_1^{(i)}(z,v),\ldots\chi_l^{(i)}(z,v)$ corresponding to the $l$ functions $\varphi_1^{(i)}(z,v),\ldots\varphi_l^{(i)}(z,v)$ in the identities (1). On regarding the coefficients of $\psi(z,v)$ as undetermined, and equating to 0 the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_1^{(i)}(z,v),\ldots\chi_l^{(i)}(z,v)$ appearing in the identities (1), we obtain $l$ linear equations between the coefficients of $\psi(z,v)$, expressing the necessary and sufficient conditions that $\psi(z,v)$ should be complementary adjoint to the order $i$ to the system of functions $\varphi(z,v)$.

That the $l$ equations just referred to are linearly independent of one another may readily be seen. For if the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_1^{(i)}(z,v),\ldots\chi_l^{(i)}(z,v)$ were linearly connected, the like linear expression in the functions $\varphi_1^{(i)}(z,v),\ldots\varphi_l^{(i)}(z,v)$ would give us a function $\varphi^{(i)}(z,v)$ such that the coefficient of $(z-a)^{i-1}v^{n-1}$ in the function $\chi^{(i)}(z,v)$ corresponding to the product $\varphi^{(i)}(z,v)\,.\,\psi(z,v)$ would be equal to 0, no matter what the coefficients of $\psi(z,v)$ might happen to be. The function $\varphi^{(i)}(z,v)$ cannot vanish identically, since by hypothesis the $l$ functions $\varphi_1^{(i)}(z\,v), \ldots \varphi_l^{(i)}(z,v)$ are linearly independent of one another. Suppose $v^s$ to be the highest power of $v$ which appears in the expression of the function $\varphi^{(i)}(z,v)$ and suppose furthermore that a term $\alpha(z-a)^r v^s$ actually presents itself. On choosing for $\psi(z,v)$ the function $\beta(z-a)^{i-1-r}v^{n-1-s}$, the function $\chi^{(i)}(z,v)$ corresponding to the product $\varphi^{(i)}(z,v).\psi(z,v)$ will evidently involve a term $\alpha\beta(z-a)^{i-1}v^{n-1}$ in which the coefficient is not equal to 0 unless we have $\beta=0$. We cannot therefore have a function $\varphi^{(i)}(z,v)$, such that the coefficient of $(z-a)^{i-1}v^{n-1}$ in the function $\chi^{(i)}(z,v)$ corresponding to the product $\varphi^{(i)}(z,v).\psi(z,v)$ is equal to 0 independently of the values of the coefficients of $\psi(z,v)$. It follows that the $l$ equations in the coefficients of the function $\psi(z,v)$, obtained on equating to 0 the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_1^{(i)}(z,v), \ldots \chi_l^{(i)}(z,v)$, are independent of one another.

While our more immediate object in what just precedes was to prove the linear independence of the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi^{(i)}(z,v)$ corresponding to the particular set of $l$ functions $\varphi^{(i)}(z,v)$ which appear in the identities (1), it is to be remarked that the proof itself has implied nothing in regard to these functions, other than that they are linearly independent of one another and that they do not involve $z-a$ to a power higher than $(z-a)^i$. We may therefore say that the coefficients of $(z-a)^{i-1}v^{n-1}$ in the functions $\chi^{(i)}(z,v)$ appearing in any number of identities of the form

$$\varphi^{(i)}(z,v).\psi(z,v)=\vartheta(z,v).F(z,v)+\chi^{(i)}(z,v)+(z-a)^i\Omega(z,v)$$

are linearly independent of one another so long as the corresponding functions $\varphi^{(i)}(z,v)$ are linearly independent of one another, the function $\psi(z,v)$ being supposed to have arbitrary coefficients.

In Chapter V we have employed the letter $A$ to indicate the number of conditions to which the coefficients of the general integral rational function of $(z,v)$ must be subjected, in order that it may be adjoint to the fundamental curve for a given value $z=a$. We are now in a position to determine the number $A$ in terms of known quantities corresponding to the value of the variable in question. Suppose $r$ to be the number of the cycles corresponding to this value of the variable, $\nu_1, \dots \nu_r$ respectively their orders and $\mu_1, \dots \mu_r$ the numbers which have heretofore been indicated by these symbols. The greatest of the integers $[\mu_1], [\mu_2], \dots [\mu_r]$ we shall represent by the letter $M$.



Each of the $r$ integers just mentioned is equal to or greater than the corresponding member of the set of adjoint numbers

$$\mu_1 - 1 + \frac{1}{\nu_1},\ \mu_2 - 1 + \frac{1}{\nu_2},\ \dots\ \mu_r - 1 + \frac{1}{\nu_r}\,.$$

It follows that the integer $M$ is less than none of these numbers and it is therefore possible to construct an integral rational function of $(z, v)$ having precisely $M$ as its order of coincidence with each of the $n$ branches of the curve, for as we have seen in Chapter V we can construct such a function having any given set of adjoint numbers as its orders of coincidence with the branches of the several cycles.

The number of the conditions which must be satisfied by the coefficients of an otherwise arbitrary integral rational function of $(z, v)$ in order that it may have $M$ as its order of coincidence with each of the $n$ branches, will evidently be obtained from (V, 8) on substituting $M$ for each of the symbols $\mu'_1, \dots \mu'_r$ appearing in that formula. The number of these conditions will therefore be

$$(2) \qquad A + \left(M - \mu_1 + 1 - \frac{1}{\nu_1}\right)\nu_1 + \cdots + \left(M - \mu_r + 1 - \frac{1}{\nu_r}\right)\nu_r.$$

Now these conditions only affect the coefficients of terms involving $z-a$ to a power lower than $(z-a)^M$. The expression just written therefore gives the number of the conditions which must be satisfied by the functions $\varphi^{(M)}(z, v)$ corresponding to the system of integral rational functions $\varphi(z, v)$, whose orders of coincidence with the $n$ branches of the curve are none of them less than $M$.

The number of the functions $\varphi^{(M)}(z, v)$ which are linearly independent we shall indicate by $l_A$. This number plus the number of the conditions to which the coefficients of the functions are subjected, must be equal to the total number $nM$ of the coefficients in a function of the form $\varphi^{(M)}(z,v)$.  It follows that

$$l_A + A + (M-\mu_1+1-\frac{1}{\nu_1})\nu_1 + \cdots + (M-\mu_r+1-\frac{1}{\nu_r})\nu_r = nM$$

whence

$$(3) \qquad l_A + A = (\mu_1 - 1 + \frac{1}{\nu_1})\nu_1 + \cdots + (\mu_r - 1 + \frac{1}{\nu_r})\nu_r$$

for we have $\nu_1 + \cdots + \nu_r = n$.

From what we have proved a little earlier in the chapter, we know that $l_A$ is just equal to the number of the conditions which must be satisfied by the coefficients of an integral rational function $\psi(z,v)$, in order that it may be complementary adjoint to the order $M$ to the system of functions $\varphi(z,v)$ here in question. In order that $\psi(z,v)$ should be complementary adjoint to the order $M$ to these functions $\varphi(z,v)$ however, it is evidently necessary and sufficient that its orders of coincidence with the branches of the several cycles should not be less than the adjoint numbers

$$\mu_1 - 1 + \frac{1}{\nu_1}, \ldots \mu_r - 1 + \frac{1}{\nu_r}$$

respectively. The conditions which the coefficients of $\psi(z,v)$ must satisfy are therefore just those conditions which are necessary to adjointness and whose number has been indicated by $A$. We must therefore have $l_A = A$.

From (3.) we derive the formula

$$(4) \qquad 2l_A = 2A = (\mu_1 - 1 + \frac{1}{\nu_1})\nu_1 + \cdots + (\mu_r - 1 + \frac{1}{\nu_r})\nu_r.$$

Since $A$, by virtue of its definition, is an integer, it follows that an even integer is represented by the sum on the right of this equation. For the number of the independent conditions then, which must be satisfied by the coefficients of the general integral rational function of $(z,v)$ in order that it may be adjoint to the fundamental curve for the finite value $z=a$, we obtain the formula

(5) $$A = \frac{1}{2}\sum_{s=1}^{r}(\mu_s - 1 + \frac{1}{\nu_s})\nu_s = \sum \mu_{\rho,\sigma} - \frac{1}{2}(n - r)$$

where the summation $\sum \mu_{\rho,\sigma}$ is supposed to extend itself to the $\frac{1}{2}n(n-1)$ combinations of the $n$ branches taken two at a time.

On substituting its value for $A$ in formula (V, 8.), we obtain an expression for the number of the independent conditions which must be satisfied by the coefficients of the general integral rational function of $(z, v)$, in order that it may have as its orders of coincidence with the branches of the several cycles a certain set of adjoint numbers $\mu'_1, \ldots \mu'_r$. The expression so obtained may be written in the form

(6) $$\sum_{s=1}^{r}\mu'_s\nu_s - \frac{1}{2}\sum_{s=1}^{r}(\mu_s - 1 + \frac{1}{\nu_s})\nu_s = \sum_{s=1}^{r}\mu'_s\nu_s - A$$

and is equal to the number of the coincidences imposed on the function less the number of the conditions requisite to adjointness for the value of the variable $z$ in question.

We now desire to deduce for $z = \infty$ the results which correspond to those which we have just obtained for finite values of the variable. To attain our end more expeditiously, we effect a simple transformation of the fundamental equation $F(z, v) = 0$ which reduces the case here in question to that already treated. Namely on substituting $z = \xi^{-1}$, $v = z^m\eta = \xi^{-m}\eta$ in $F(z, v) = 0$ and on multiplying by $\xi^{mn}$, the fundamental equation will be replaced by an equation in $(\xi, \eta)$ of the form

(7) $$G(\xi, \eta) = \eta^n + G_{n-1}\eta^{n-1} + \cdots + G_0 = 0,$$

where the coefficients $G$ will certainly be integral rational functions of $\xi$ in case $m$ has been chosen integral and large enough.* This in particular will be the case if $m$ has been taken equal to the largest exponent of $z$ which makes its appearance in the expression of the function $F(z, v)$. Since this

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* It is readily seen that the smallest eligible value for $m$ is the least integer which is not less than any of the numbers $m_1, \frac{1}{2}m_2, \frac{1}{3}m_3 \ldots \frac{1}{n}m_n$, where $m_k$ is the greatest exponent of $z$ which presents itself in the coefficient of $v^{n-k}$ in the function $F(z, v)$.

suffices for our purpose we assume once for all that $m$ has been so chosen.

The algebraic equation (7) is then an integral algebraic equation and the theory which has already been developed for functions of the variables $(z, v)$ relatively to the integral algebraic equation $F(z, v)=0$, will evidently also hold good for functions of the variables $(\xi, \eta)$ relatively to the integral algebraic equation $G(\xi, \eta)=0$. The theorems which we have obtained for finite values of the variable $z$ will then have their counterpart for finite values of the variable $\xi$, and in particular for the value $\xi=0$.

The $n$ branches of the equation $G(\xi, \eta)=0$ corresponding to the value $\xi=0$ will be given by $n$ equations of the form

(8) $$\eta-P_1(\xi)=0,\ \eta-P_2(\xi)=0,\ \ldots\ \eta-P_n(\xi)=0$$

where the series $P(\xi)$ involve no negative powers of $\xi$. From these $n$ equations, on multiplying by $z^m$, we evidently obtain the equations to the branches of the original equation $F(z, v)=0$ corresponding to the value $z=\infty$, in the form

(9) $$v-z^m P_1\left(\frac{1}{z}\right)=0,\ v-z^m P_2\left(\frac{1}{z}\right)=0,\ \ldots\ v-z^m P_n\left(\frac{1}{z}\right)=0\,.$$

These branches will group thenselves into some number $r$ of cycles, whose orders as heretofore we shall indicate by $\nu_1, \nu_2, \ldots \nu_r$ respectively. The cyclical characters of the branches in (8) will be the same as those of the corresponding branches in (9), and the order of coincidence of any two of the branches in (8) with each other will evidently be greater by $m$ than the order of coincidence with each other of the corresponding branches in (9). If then we indicate the orders of coincidence of the branches of the several cycles which present themselves in (9.), each with the product of the other $n-1$ branches, by

(10) $$\mu_1, \mu_2, \ldots \mu_r$$

respectively, the orders of coincidence of the branches of the corresponding cycles in (8.), each with the product of the other $n-1$ branches, will be equal to

(11) $$m(n-1)+\mu_1,\ m(n-1)+\mu_2,\ \ldots\ m(n-1)+\mu_r$$

respectively, numbers which by virtue of their signification can none of them be negative.

From the theory given in Chapter IV we immediately derive that the orders of coincidence of an integral rational function of $(\xi, \eta)$ which is not divisible by $\xi$, with the branches of the several cycles of the equation $G(\xi, \eta) = 0$ corresponding to the value $\xi = 0$, cannot simultaneously be greater than the respective members of the set of numbers (11.), but that it is always possible to construct such a function whose orders of coincidence with the branches of an arbitrarily chosen set of $r-1$ out of the $r$ cycles are as large as we may please, while its order of coincidence with the branches of the other cycle is equal to $m(n-1)+\mu_s-\frac{\sigma}{\nu_s}$, in case this cycle be the one of order $\nu_s$, where $\sigma$ may have any one of the $\nu_s$ values $0, 1, \ldots (\nu_s - 1)$. We also derive that an integral rational function of $(\xi, \eta)$ must be divisible by $\xi^i$, if its orders of coincidence with the branches of the several cycles be simultaneously equal to or greater than the numbers

$$m(n-1)+\mu_1+i-1+\frac{1}{\nu_1},\ \ldots\ m(n-1)+\mu_r+i-1+\frac{1}{\nu_r}$$

respectively, where $i$ is any positive integer. It furthermore follows that the greatest value of the exponent $j$ in the denominator $\xi^j$ of a rational fractional function $\xi^{-j}\Gamma(\xi, \eta)$, which is infinite for none of the branches corresponding to the value $\xi = 0$ and in which the numerator $\Gamma(\xi, \eta)$ is an integral rational function of $(\xi, \eta)$ which is not divisible by $\xi$, is the greatest of the integers

$$m(n-1)+[\mu_1],\ m(n-1)+[\mu_2],\ \ldots\ m(n-1)+[\mu_r].$$

In accord with the definition of the word adjoint given in Chapter V, a function of $(\xi, \eta)$ will be adjoint to the curve $G(\xi, \eta) = 0$ for the value $\xi = 0$, when its orders of coincidence with the branches of the several cycles are not less than the numbers

(12) $$m(n-1)+\mu_1-1+\frac{1}{\nu_1},\ \ldots\ m(n-1)+\mu_r-1+\frac{1}{\nu_r}$$

respectively. On employing the symbol $\bar{A}$ to indicate the number of the conditions necessary to the adjointness of an integral rational function of $(\xi, \eta)$ relative to the curve $G(\xi, \eta) = 0$ for the value $\xi = 0$, we shall have in analogy with (5), the formula



$$(13) \quad \bar{A} = \frac{1}{2}\sum_{s=1}^{r} \{m(n-1) + \mu_s - 1 + \frac{1}{\nu_s}\} \nu_s = \frac{1}{2} m n (n-1) + \frac{1}{2}\sum_{s=1}^{r} (\mu_s - 1 + \frac{1}{\nu_s}) \nu_s .$$

Furthermore from Chapter V we derive that it is possible to construct an integral rational function of $(\xi, \eta)$ having any arbitrarily assigned set of adjoint numbers as its orders of coincidence with the branches of the several cycles corresponding to the value $\xi = 0$, that is to say having as its orders of coincidence with the branches of the several cycles any integral multiples of $\frac{1}{\nu_1}, \frac{1}{\nu_2}, \ldots \frac{1}{\nu_r}$ respectively, which are equal to or greater than the corresponding members of the set of numbers (12). Writing any such set of adjoint numbers for convenience in the form

$$(14) \qquad m(n-1) + \mu'_1,\ m(n-1) + \mu'_2,\ \ldots\ m(n-1) + \mu'_r$$

the numbers $\mu'_1, \mu'_2, \ldots \mu'_r$ will be integral multiples, positive or negative, of $\frac{1}{\nu_1}, \frac{1}{\nu_2}, \ldots \frac{1}{\nu_r}$ respectively, and will evidently be equal to or greater than the numbers

$$\mu_1 - 1 + \frac{1}{\nu_1},\ \mu_2 - 1 + \frac{1}{\nu_2},\ \ldots\ \mu_r - 1 + \frac{1}{\nu_r}$$

respectively. The numbers $\mu'_1, \mu'_2, \ldots \mu'_r$ will therefore constitute a set of adjoint numbers relatively to the curve $F(z, v) = 0$ for the value $z = \infty$.

The number of the conditions which must be satisfied by the coefficients of the general integral rational function of $(\xi, \eta)$, in order that it may have as its orders of coincidence with the branches of the several cycles corresponding to $\xi = 0$ the set of numbers (14), will then be given by a formula analogous to (6) and will be equal to

(15) $$\sum_{s=1}^{r}\{m(n-1)+\mu'_s\}\nu_s-\frac{1}{2}\sum_{s=1}^{r}\{m(n-1)+\mu_s-1+\frac{1}{\nu_s}\}\nu_s$$

$$=\frac{1}{2}mn(n-1)+\sum_{s=1}^{r}\mu'_s\nu_s-\frac{1}{2}\sum_{s=1}^{r}(\mu_s-1+\frac{1}{\nu_s})\nu_s.$$

In regard to what some of these conditions are, it will be convenient to be more precise. Representing an integral rational function of $(\xi,\eta)$ by the expression

(16) $$\rho_{n-1}\eta^{n-1}+\rho_{n-2}\eta^{n-2}+\cdots+\rho_0$$

where the coefficients $\rho$ may be supposed to involve powers of $\xi$ up to as high an order as we please, we shall require that this function have as its orders of coincidence with the branches of the several cycles of the curve $G(\xi,\eta)=0$ corresponding to the value $\xi=0$, the set of numbers (12), that is we shall require that the function be adjoint to the curve $G(\xi,\eta)=0$ for the value of the variable in question. Certain of the coefficients in the function, as we shall see, must vanish.

Multiplying the expression (16) by $\xi^{-m(n-1)}$ we obtain a function

(17) $$\xi^{-m(n-1)}(\rho_{n-1}\eta^{n-1}+\rho_{n-2}\eta^{n-2}+\cdots+\rho_0)$$

whose orders of coincidence with the branches of the several cycles will be less by $m(n-1)$ than the numbers given in (12). The orders of coincidence of this product with the branches of the several cycles will then be equal to the numbers

$$\mu_1-1+\frac{1}{\nu_1},\ \mu_2-1+\frac{1}{\nu_2},\ \ldots\ \mu_r-1+\frac{1}{\nu_r}$$

respectively. Now these are the orders of coincidence necessary to the adjointness of a rational function of the variables $(z,v)$ with the curve $F(z,v)=0$ for the value $z=\infty$, and they will evidently also be the orders of coincidence of the function (17), transformed to terms of $(z,v)$, with the branches of the several cycles of the curve $F(z,v)=0$ corresponding to the value $z=\infty$. The function (17), expressed in terms of $(z,v)$, will then be adjoint to the curve $F(z,v)=0$ for the value $z=\infty$. We have shown however in Chapter VI, that a rational function of $(z, v)$ which is adjoint for the value $z=\infty$, must have a degree in $(z, v)$ which is $\leqq N-1$. The product (17) therefore, expressed in terms of $(z, v)$, must have a degree which is $\leqq N-1$.

Substituting $z^{-1}$ and $z^{-m}v$ respectively for $\xi$ and $\eta$ in (17), we obtain the expression

$$(18) \qquad \rho_{n-1}\left(\frac{1}{z}\right) v^{n-1} + \rho_{n-2}\left(\frac{1}{z}\right) z^{m} v^{n-2} + \rho_{n-3}\left(\frac{1}{z}\right) z^{2m} v^{n-3} + \cdots + \rho_{0}\left(\frac{1}{z}\right) z^{m(n-1)}$$

whose degree in $(z, v)$ must be $\leqq N-1$. The first element in the sum satisfies this condition as it stands, and evidently also satisfies the further condition required in Chapter VI from the element $h_{n-1}v^{n-1}$ in a rational function which is to be adjoint for the value $z=\infty$ — namely that its degree be $\leqq n-1$. In the case of the other elements $\rho_{n-s}\left(\frac{1}{z}\right) z^{(s-1)m} v^{n-s}$, the limitation that their degree be $\leqq N-1$, will require that the coefficients of a number of the lower powers of $\frac{1}{z}$ in the functions $\rho_{n-s}\left(\frac{1}{z}\right)$ be equal to 0. If namely $\left(\frac{1}{z}\right)^{\beta}$ is the lowest power of $\frac{1}{z}$ which appears in $\rho_{n-s}\left(\frac{1}{z}\right)$, we must have $-\beta + (s-1)m + n - s \leqq N-1$ and therefore $\beta \geqq (s-1)(m-1) + n - N$. It follows then that in a function $\rho_{n-s}\left(\frac{1}{z}\right)$, terms involving exponents which are less than $(s-1)(m-1)+n-N$ cannot appear. That the number $(s-1)(m-1)+n-N$ cannot be negative is evident, since for the case here in question we have $s \geqq 2$ and by our choice of the integer $m$ we have $N \leqq m+n-1$.

The number of terms which must be missing in a function $\rho_{n-s}\left(\frac{1}{z}\right)$, $(s \geqq 2)$, as a consequence of the limitation on the degree of the function (18), is then $(m-1)(s-1)+n-N$, and the total number of such terms lacking in the $n-1$ functions corresponding to the values $s=2, 3, \cdots n$ will therefore be given by the sum

$$B = \sum_{s=2}^{n} \{(m-1)(s-1) + n - N\} = \frac{1}{2} n(n-1)(m+1) - (n-1)N.$$

This then is the number of the conditions which must be imposed on the coefficients of the general function of the form (18) in order that it may have the degree $N-1$.

Under the $\bar{A}$ conditions necessary to the adjointness of a function of the form (18) are included the $B$ conditions here in question. The number of the conditions then that must be imposed on the coefficients of a function of the form (18) and of the degree $N-1$, in order that it may be adjoint to the curve $F(z,v)=0$ relatively to the value $z=\infty$, is $\bar{A}-B$. Under the form (18) however is included the most general rational function of $(z,v)$ of degree $N-1$, in which the coefficients of the several powers of $v$ are represented by series in $\frac{1}{z}$ involving a finite number of powers, positive and negative, and where the degree of the element of the function involving $v^{n-1}$ is not greater than $n-1$. It follows that the number of the independent conditions which must be satisfied by the coefficients in the most general rational function of $(z,v)$ of degree $N-1$ — that is to say in the most general function of the form (V, 10) — in which the element involving $v^{n-1}$ is of degree $n-1$, in order that the function may be adjoint to the curve $F(z,v)=0$ for the value $z=\infty$, is given by the formula

$$\bar{A}-B=(n-1)N-\frac{1}{2}n(n-1)+\frac{1}{2}\sum_{s=1}^{r}\left(\mu_s-1+\frac{1}{\nu_s}\right)\nu_s.$$

The limitation here made on the degree of the element involving $v^{n-1}$ in the function (V, 10) is — as we have proved in Chapter VI — a necessary one, in order that the function may be adjoint for the value $z=\infty$. The number of conditions imposed on the coefficients of the function by this limitation is evidently $N-n$, for it requires that the coefficients of the terms in $z^{N-n}v^{n-1}, z^{N-n-1}v^{n-1}, \ldots zv^{n-1}$ be all equal to 0. The number of conditions then which must be satisfied by the coefficients of the general rational function of degree $N-1$

$$h_{n-1}v^{n-1}+h_{n-2}v^{n-2}+\cdots+h_0,$$

in order that it may be adjoint to the curve $F(z,v)=0$ for the value

$z=\infty$, will be equal to $\overline{A}-B+N-n$. Indicating this number by $A$, as has already been done in Chapter V, we shall have



(19) $$A=n(N-1)-\frac{1}{2}n(n-1)+\frac{1}{2}\sum_{s=1}^{r}(\mu_s-1+\frac{1}{\nu_s})\nu_s.$$

This then is the number of independent conditions which are just sufficient to the adjointness of a rational function of degree $N-1$ for the value $z=\infty$.

We have seen in Chapter V that the expression

$$A+\sum_{s=1}^{r}(\mu'_s-\mu_s+1-\frac{1}{\nu_s})\nu_s$$

represents the number of conditions which must be satisfied by the coefficients in the general function of the form (V, 10), in order that it may have as its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$, a certain set of adjoint numbers $\mu'_1, \mu'_2, \ldots \mu'_r$. On substituting in this expression the value just determined for $A$ we obtain

(20) $$n(N-1)-\frac{1}{2}n(n-1)+\sum_{s=1}^{r}\mu'_s\nu_s-\frac{1}{2}\sum_{s=1}^{r}(\mu_s-1+\frac{1}{\nu_s})\nu_s.$$

This is therefore the number of independent conditions which must be satisfied by the coefficients in the general rational function of degree $N-1$, in order that it may have as its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$, a certain set of adjoint numbers $\mu'_1, \mu'_2, \ldots \mu'_r$.

# CHAPTER IX.

## Modified formulation of conditions.

The function $R(z, v)$. Introduction of the functions $\zeta^{(i)}(z, v)$. Modified formulation of the conditions satisfied by the coefficients of the function $\psi(z, v)$.

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In order that an integral rational function $\psi(z, v)$ may be complementary adjoint to the order $i$ to the system of functions $\varphi(z, v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1, \ldots \mu'_r$ respectively, it is necessary and sufficient, as we have seen in the chapter preceding, that the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_s^{(i)}(z, v)$ which appear in the system of identities (VIII, 1) should be equal to 0. We have also seen that the $l$ conditions so imposed on the coefficients of the function $\psi(z, v)$ are linearly independent of one another.

Let us now consider an identity of the type in question

(1) $$\varphi^{(i)}(z, v) \,.\, \psi(z, v) = \vartheta(z, v) \,.\, F(z, v) + \chi^{(i)}(z, v) + (z-a)^i \Omega(z, v).$$

Here it is to be borne in mind that $v^{n-2}$ is the highest power of $v$ which can appear in $\vartheta(z, v)$, and that $F_n$ the coefficient of $v^n$ in $F(z, v)$ is equal to 1.

Multiplying both sides of the identity by a function

$$R(z, v) = R_{n-1} v^{n-1} + R_{n-2} v^{n-2} + \cdots + R_0$$

in which the coefficients $R_{n-1}, \ldots R_0$ are as yet undetermined integral rational functions of $z$, we obtain

(2) $\psi(z,v) \cdot \varphi^{(i)}(z,v) \cdot R(z,v) = \vartheta(z,v) \cdot F(z,v) \cdot R(z,v)$



$$+ \chi^{(i)}(z,v) \cdot R(z,v) + (z-a)^i \Omega(z,v) \cdot R(z,v).$$

The unreduced product $F(z,v) \cdot R(z,v)$ will have the form

$$F(z,v) \cdot R(z,v) = S_{2n-1} v^{2n-1} + S_{2n-2} v^{2n-2} + \cdots + S_0$$

where the coefficients $S$ are integral rational functions of $z$. In particular the first $n$ coefficients will be given by the equations

$$S_{2n-r} = F_{n-r+1} R_{n-1} + F_{n-r+2} R_{n-2} + \cdots + F_n R_{n-r}, \ (r=1, \ldots n).$$

The as yet undetermined functions $R$ may now be so chosen that the $n-1$ expressions $S_{2n-2}, S_{2n-3}, \ldots S_n$ are all identically equal to 0. Namely on equating these expressions to 0 we obtain the $n-1$ equations

$$F_n R_{n-r} + F_{n-1} R_{n-r+1} + \cdots + F_{n-r+1} R_{n-1} = 0, \ (r=2, \ldots n).$$

These equations determine the ratios of the functions $R_0, R_1, \ldots R_{n-2}$ to $R_{n-1}$, and on taking $R_{n-1}=1$ and remembering that we have $F_n=1$ we obtain for the determination of the functions $R_t$ the formula

$$R_t = (-1)^{n-t-1} \begin{vmatrix} F_{n-1} & F_{n-2} & \cdots & \cdots & F_{t+1} \\ F_n & F_{n-1} & \cdots & \cdots & F_{t+2} \\ 0 & F_n & \cdots & \cdots & F_{t+3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & F_n & F_{n-1} \end{vmatrix}$$

where $t$ takes the values $0, 1, \ldots n-2$. The functions $R_t$ so determined are integral rational functions of $z$ and evidently give us for $R(z,v)$ a function which may be written in the form

(3) $$R(z,v) = (-1)^{n-1} \begin{vmatrix} 1 & v & v^2 & \cdots & v^{n-2} & v^{n-1} \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_2 & F_1 \\ 0 & F_n & F_{n-1} & \cdots & F_3 & F_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & F_n & F_{n-1} \end{vmatrix}$$

Our determination of the functions $R_0, \dots R_{n-1}$ make the coefficients $S_{2n-2}, \dots S_n$ in the expression for the product $F(z,v) \,.\, R(z,v)$ vanish identically and gives us at the same time $S_{2n-1} = F_n R_{n-1} = 1$, so that we may write

$$(4) \qquad F(z,v) \,.\, R(z,v) = v^{2n-1} + S_{n-1} v^{n-1} + \cdots + S_0$$

where the coefficients $S_{n-1}, \dots S_0$ are evidently integral rational functions of $z$. For the first element on the right-hand side of the identity (2) we shall then have

$$\vartheta(z,v) \,.\, F(z,v) \,.\, R(z,v) = \vartheta(z,v)\, v^{2n-1} + \vartheta(z,v)\,(S_{n-1} v^{n-1} + \cdots + S_0).$$

On effecting the multiplications here indicated, we note in the expression on the right that no term involving the power $v^{2n-2}$ can present itself, for a term will either be divisible by $v^{2n-1}$ or it will involve at most the power $v^{2n-3}$, since $\vartheta(z,v)$ contains no power of $v$ higher than $v^{n-2}$.

Returning now to the identity (2) with the function $R(z,v)$ determined as above, it is evident that any terms involving $v^{2n-2}$ in the expression on the right-hand side of the identity must be sought in the portion

$$\chi^{(i)}(z,v) \,.\, R(z,v) + (z-a)^i \,\Omega(z,v) \,.\, R(z,v)$$

of this expression. If in particular we would find the terms involving $(z-a)^{i-1} v^{2n-2}$, it plainly suffices to confine our attention to the product $\chi^{(i)}(z,v) \,.\, R(z,v)$. In this product furthermore $v^{2n-2}$ is the highest power of $v$ which can occur, since $v^{n-1}$ is the highest power which appears in either of the two factors. Also since the coefficient $R_{n-1}$ of $v^{n-1}$ in the second factor is equal to 1, the coefficient of $v^{2n-2}$ in the product must be equal to the coefficient of $v^{n-1}$ in $\chi^{(i)}(z,v)$, and in particular the term in $(z-a)^{i-1} v^{2n-2}$ in the product $\chi^{(i)}(z,v) \,.\, R(z,v)$ must have as its coefficient the constant coefficient of $(z-a)^{i-1} v^{n-1}$ in the function $\chi^{(i)}(z,v)$. The coefficient of $(z-a)^{i-1} v^{n-1}$ in $\chi^{(i)}(z,v)$ is then identical with the coefficient of $(z-a)^{i-1} v^{2n-2}$ in the expression on the right of (2), and therefore the same as the coefficient of $(z-a)^{i-1} v^{2n-2}$ in the product on the left of this identity.

The coefficients of $(z-a)^{i-1} v^{n-1}$ in the $l$ functions $\chi_s^{(i)}(z,v)$ appearing in the identities (VIII, 1) are then just the same as the coefficients of $(z-a)^{i-1} v^{2n-2}$ in the $l$ products

(5) $$\psi(z,v)\,.\,\varphi_s^{(i)}(z,v)\,.\,R(z,v), \qquad (s=1,\ldots l)$$



obtained on multiplying the expressions on the left of these identities by $R(z,v)$, and the vanishing of the coefficients of $(z-a)^{i-1}v^{2n-2}$ in these $l$ products will impose just the same conditions on the coefficients of $\psi(z,v)$, as would the vanishing of the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_s^{(i)}(z,v)$. In order that an integral rational function $\psi(z,v)$ should be complementary adjoint to the order $i$ to the system of functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1,\ldots\mu'_r$ respectively, it is therefore necessary and sufficient that the coefficients of $(z-a)^{i-1}v^{2n-2}$ in the $l$ products (5) should all vanish.

We shall now write the $l$ products $\varphi_s^{(i)}(z,v)\,.\,R(z,v)$ in the form

(6) $$\varphi_s^{(i)}(z,v)\,.\,R(z,v)=\theta_s(z,v)+v^{n-1}\zeta_s^{(i)}(z,v), \qquad (s=1,\ldots l)$$

where on the right the element $v^{n-1}\zeta_s^{(i)}(z,v)$ is made up of all terms of the product on the left which are divisible by $v^{n-1}$ and which at the same time are *not* divisible by $(z-a)^i$. In the element $\theta_s(z,v)$ then any term containing $v$ to a power higher than $v^{n-2}$ must be divisible by $(z-a)^i$.

On representing the $l$ products (5) in the form

$$\psi(z,v)\,(\theta_s(z,v)+v^{n-1}\zeta_s^{(i)}(z,v)), \qquad (s=1,\ldots l)$$

we see that the terms in the products $\psi(z,v)\,.\,\theta_s(z,v)$ which involve powers of $v$ higher than $v^{2n-3}$ must be divisible by $(z-a)^i$, and that therefore the coefficients of $(z-a)^{i-1}v^{2n-2}$ in the products

$$\psi(z,v)\,.\,v^{n-1}\zeta_s^{(i)}(z,v) \qquad (s=1,\ldots l)$$

must be the same as in the products (5). It follows that the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ products

(7) $$\psi(z,v)\,.\,\zeta_s^{(i)}(z,v) \qquad (s=1,\ldots,l)$$

are the same as the coefficients of $(z-a)^{i-1}v^{2n-2}$ in the $l$ products (5), and consequently the same as the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ functions $\chi_s^{(i)}(z,v)$ which appear in the identities (VIII, 1).

The necessary and sufficient conditions in order that the integral rational function $\psi(z,v)$ may be complementary adjoint to the order $i$ to the system of functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles are not less than the adjoint numbers $\mu'_1, \ldots \mu'_r$ respectively, are then obtained on equating to 0 the coefficients of $(z-a)^{i-1}v^{n-1}$ in the $l$ products (7), and the $l$ equations in the coefficients of $\psi(z,v)$ so obtained are linearly independent of one another.



The functions $\zeta_s^{(i)}(z,v)$ corresponding to the $l$ functions $\varphi_s^{(i)}(z,v)$ can be readily derived from the identities (6). Namely on effecting the multiplications indicated on the left of these identities, and discarding in the results terms divisible by $(z-a)^i$ and terms involving $v$ to a power lower than $v^{n-1}$, there remain the $l$ products $v^{n-1}\zeta_s^{(i)}(z,v)$, from which on division by $v^{n-1}$ we obtain the $l$ functions $\zeta_s^{(i)}(z,v)$.

We shall find it convenient conversely, to have the means of determining the functions $\varphi_s^{(i)}(z,v)$ in terms of the corresponding functions $\zeta_s^{(i)}(z,v)$. — On multiplying both sides of the identities (6) by $F(z,v)$ and substituting for $F(z,v).R(z,v)$ the expression given in (4), we obtain $l$ identities

$$\varphi_s^{(i)}(z,v).(v^{2n-1}+S_{n-1}v^{n-1}+\cdots+S_0)=\theta_s(z,v).F(z,v)+v^{n-1}\zeta_s^{(i)}(z,v).F(z,v). \quad (s=1,\ldots l)$$

Here the only terms in the product on the left which involve $v$ to a power higher than $v^{2n-2}$ are contained in the element $\varphi_s^{(i)}(z,v).v^{2n-1}$, none of whose terms is divisible by $(z-a)^i$. Furthermore, the only terms on the right which are not divisible by $(z-a)^i$ and which involve a power of $v$ higher than $v^{2n-2}$ are evidently contained in the element $v^{n-1}\zeta_s^{(i)}(z,v).F(z,v)$.

It follows that $v^{2n-1}\varphi_s^{(i)}(z,v)$ represents the aggregate of terms in the product $v^{n-1}\zeta_s^{(i)}(z,v).F(z,v)$, which are divisible by $v^{2n-1}$ but not by $(z-a)^i$. The aggregate of terms in the product $\zeta_s^{(i)}(zv).F(z,v)$ which are divisible by $v^n$ but not by $(z-a)^i$ are therefore represented by $v^n\varphi_s^{(i)}(z,v)$. This is equivalent to saying, that to the set of $l$ products $\zeta_s^{(i)}(z,v).F(z,v)$ corresponds a set of $l$ identities —

$$(8) \qquad \zeta_s^{(i)}(z,v).F(z,v)=\Pi_s(z,v)+v^n\varphi_s^{(i)}(z,v), \qquad (s=1,\ldots l)$$

where the functions $\Pi_s(z,v)$ are integral rational functions of $(z,v)$, in which any term which is divisible by $v^n$ is also divisible by $(z-a)^i$.

To determine the $l$ functions $\varphi_s^{(i)}(z,v)$ corresponding to $l$ given functions $\zeta_s^{(i)}(z,v)$ then, it is only necessary to effect the multiplications indicated on the left of the identities (8) and discard in the products terms which are divisible by $(z-a)^i$ and terms involving $v$ to a power lower than $v^n$. There remain the $l$ products $v^n\varphi_s^{(i)}(z,v)$, from which on dividing by $v^n$ we obtain the $l$ functions $\varphi_s^{(i)}(z,v)$.

On expressing the functions $\varphi_s^{(i)}(z,v)$ and $\zeta_s^{(i)}(z,v)$ more fully in the forms

$$(9)\qquad \begin{cases} \varphi_s^{(i)}(z,v)=P_{s,n-1}^{(i)}v^{n-1}+P_{s,n-2}^{(i)}v^{n-2}+\cdots+P_{s,0}^{(i)} \\ \zeta_s^{(i)}(z,v)=\top_{s,n-1}^{(i)}v^{n-1}+\top_{s,n-2}^{(i)}v^{n-2}+\cdots+\top_{s,0}^{(i)} \end{cases} \qquad (s=1,\dots l)$$

the process just described will evidently give for the determination of the functions $\varphi_s^{(i)}(z,v)$ in terms of the coefficients of the functions $\zeta_s^{(i)}(z,v)$, the set of $l$ congruences

$$(10)\qquad \varphi_s^{(i)}(z,v)\equiv\sum_{t=0}^{n-1}\left(F_n\top_{s,t}^{(i)}+F_{n-1}\top_{s,t+1}^{(i)}+\cdots+F_{t+1}\top_{s,n-1}^{(i)}\right).v^t,\quad [\text{mod.}\ (z-a)^i],$$
$$s=1,2,\dots l$$

whence also

$$(11)\qquad P_{s,t}^{(i)}\equiv F_n\top_{s,t}^{(i)}+F_{n-1}\top_{s,t+1}^{(i)}+\cdots+F_{t+1}\top_{s,n-1}^{(i)},\qquad [\text{mod.}\ (z-a)^i].$$

For the value $z=\infty$ we shall now derive formulae analogous to those which we have just obtained for finite values of the variable. To do this it is only necessary to transform to the variables $\xi=z^{-1}, \eta=z^{-m}v$, replacing our equation $F(z,v)=0$ by the equation $G(\xi,\eta)=0$ with which we have already had to do in (VIII, 7). On considering rational functions of $(\xi,\eta)$ relatively to the equation $G(\xi,\eta)=0$, we can then appropriate for finite values of the variable $\xi$, and in particular for the value $\xi=0$, the theory which we have developed in the present chapter for finite values of the variable $z$ and for rational functions of $(z,v)$ considered with reference to the equation $F(z,v)=0$.

Making use of the notation employed in (VIII, 14) to designate a set of adjoint numbers relatively to the cycles of the equation $G(\xi,\eta)=0$, corresponding to the value $\xi=0$, we shall consider the conditions which must be satisfied by the coefficients of an integral rational function $\psi(\xi, \eta)$ in order that it may be complementary adjoint to the order $i$ to the system of integral rational functions $\varphi(\xi, \eta)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $\xi = 0$ are not less than the adjoint numbers

$$m(n-1) + \mu'_1, m(n-1) + \mu'_2, \ldots m(n-1) + \mu'_r$$

respectively. If $l$ is the number of the linearly independent functions $\varphi^{(i)}(\xi, \eta)$ obtained on dropping terms divisible by $\xi^i$ in the functions $\varphi(\xi, \eta)$, it follows from Chapter VIII that $l$ is also the number of linearly independent conditions which must be satisfied by the coefficients of the function $\psi(\xi, \eta)$. To formulate these conditions more explicitly, we select a set of $l$ linearly independent functions $\varphi_1^{(i)}(\xi, \eta), \varphi_2^{(i)}(\xi, \eta), \ldots \varphi_l^{(i)}(\xi, \eta)$, and construct, after the analogy of the function $R(z, v)$ in formula (3), the function

$$(12) \qquad \overline{R}(\xi, \eta) = (-1)^{n-1} \begin{vmatrix} 1 & \eta & \eta^2 & \ldots & \eta^{n-2} & \eta^{n-1} \\ G_n & G_{n-1} & G_{n-2} & \ldots & G_2 & G_1 \\ 0 & G_n & G_{n-1} & \ldots & G_3 & G_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \ldots & G_n & G_{n-1} \end{vmatrix}$$

After the analogy of the identities in formula (6) then we shall have $l$ identities of the form

$$(13) \qquad \varphi_s^{(i)}(\xi, \eta) \,.\, \overline{R}(\xi\ \eta) = \theta_s(\xi, \eta) + \eta^{n-1} \zeta_s^{(i)}(\xi, \eta), \qquad (s = 1, \ldots l)$$

which define $l$ functions $\zeta_s^{(i)}(\xi, \eta)$. Namely the products $\eta^{n-1} \zeta_s^{(i)}(\xi, \eta)$ are obtained on discarding all terms in the products $\varphi_s^{(i)}(\xi, \eta) \,.\, \overline{R}(\xi, \eta)$ which are divisible by $\xi^i$ or which involve $\eta$ to a power lower than $\eta^{n-1}$; — we then arrive at the functions $\zeta_s^{(i)}(\xi, \eta)$ on dividing by $\eta^{n-1}$.

In analogy with what has been proved in connection with the products (7) in the present chapter, we see that the $l$ linearly independent conditions which must be satisfied by the coefficients of the function $\psi(\xi, \eta)$, in order that it may be complementary adjoint to the order $i$ to the system of functions $\varphi(\xi, \eta)$, are obtained on equating to 0 the coefficients of $\xi^{i-1} \eta^{n-1}$ in the $l$ products

$$\psi(\xi,\eta)\,.\,\zeta_s^{(i)}(\xi,\eta), \qquad (s=1,\ldots l) \tag{14}$$



The functions $\varphi_s^{(i)}(\xi,\eta)$ and $\zeta_s^{(i)}(\xi,\eta)$ can also be connected by identities constructed after the analogy of the identities (8). Namely we shall have $l$ identities of the form

$$\zeta_s^{(i)}(\xi,\eta)\,.\,G(\xi,\eta)=\Pi_s(\xi,\eta)+\eta^n\varphi_s^{(i)}(\xi,\eta), \qquad (s=1,\ldots l) \tag{15}$$

so that to determine the $l$ functions $\varphi_s^{(i)}(\xi,\eta)$ in terms of the corresponding functions $\zeta_s^{(i)}(\xi,\eta)$, it is only necessary to effect the multiplications indicated on the left of these identities and discard in the products terms which are divisible by $\xi^i$ and terms involving $\eta$ to a power lower than $\eta^n$. There remain the $l$ products $\eta^n\varphi_s^{(i)}(\xi,\eta)$, from which on dividing by $\eta^n$ we obtain the $l$ functions $\varphi_s^{(i)}(\xi,\eta)$.

On representing the functions $\varphi_s^{(i)}(\xi,\eta)$ and $\zeta_s^{(i)}(\xi,\eta)$ in the forms

$$\left.\begin{aligned}\varphi_s^{(i)}(\xi,\eta)&=P_{s,n-1}^{(i)}\eta^{n-1}+P_{s,n-2}^{(i)}\eta^{n-2}+\cdots+P_{s,0}^{(i)}\\ \zeta_s^{(i)}(\xi,\eta)&=\mathrm{T}_{s,n-1}^{(i)}\eta^{n-1}+\mathrm{T}_{s,n-2}^{(i)}\eta^{n-2}+\cdots+\mathrm{T}_{s,0}^{(i)}\end{aligned}\right\} \qquad (s=1,\ldots l) \tag{16}$$

the process just described will give for the determination of the functions $\varphi_s^{(i)}(\xi,\eta)$ in terms of the coefficients of the functions $\zeta_s^{(i)}(\xi,\eta)$, the set of $l$ congruences

$$\varphi_s^{(i)}(\xi,\eta)\equiv\sum_{t=0}^{n-1}\left(G_n\mathrm{T}_{s,t}^{(i)}+G_{n-1}\mathrm{T}_{s,t+1}^{(i)}+\ldots+G_{t+1}\mathrm{T}_{s,n-1}^{(i)}\right).\eta^t, \qquad [\text{mod. } \xi^i], \tag{17}$$

$$s=1,2,\ldots l$$

whence also

$$F_{s,t}^{(i)}\equiv G_n\mathrm{T}_{s,t}^{(i)}+G_{n-1}\mathrm{T}_{s,t+1}^{(i)}+\cdots+G_{t+1}\mathrm{T}_{s,n-1}^{(i)}, \qquad [\text{mod. } \xi^i], \tag{18}$$

The connection between the functions $\bar{R}(\xi,\eta)$ and $R(z,v)$, it may be remarked, is readily obtainable. Noting namely that we have $G_t=\xi^{(n-t)m}F_t$, it suffices to multiply the 3rd, 4th, ... $n$th rows of the determinant in (12) by $\xi^m, \xi^{2m}, \ldots \xi^{(n-2)m}$ respectively, dividing at the same time the 2nd, 3rd, ... $(n-1)$th columns by $\xi^m, \xi^{2m}, \ldots \xi^{(n-2)m}$ respectively, in order to obtain the relation

$$\bar{R}(\xi,\eta)=\xi^{(n-1)m}R(z,v). \tag{19}$$

This relation could also be immediately derived from the properties which define the functions $R(z, v)$ and $R(\xi, \eta)$. Namely the property that the coefficients of $v^{2n-2}, \ldots v^n$ in the product $F(z, v) . R(z, v)$ must vanish, and the property that the coefficients of $\eta^{2n-2}, \ldots \eta^n$ in the product $G(\xi, \eta) . \overline{R}(\xi, \eta) = \xi^{mn} F(z, v) . \overline{R}(\xi, \eta)$ must vanish, are one and the same, and, since this property determines each of the functions $R(z, v)$ and $\overline{R}(\xi, \eta)$ to a factor in $\xi$, it follows that these two functions can only differ by such a factor. From the further property that the coefficient of $v^{n-1}$ in $R(z, v)$ and the coefficient of $\eta^{n-1}$ in $\overline{R}(\xi, \eta)$ must be unity, it follows that the latter function is obtained on multiplying the former function by the factor $\xi^{(n-1)m}$.

## CHAPTER X.

## Rational functions of unrestricted character for $z = \infty$.

Form of the general rational function of $(z, v)$ which becomes infinite only for the value $z = \infty$. General form of a rational function of $(z, v)$ which, in addition to infinities at $\infty$, may possess an assigned set of infinities corresponding to finite values of the variable $z$.

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We shall now consider the form of a rational function of $(z, v)$ as related to the values of the variables for which it becomes infinite.

We have seen that any rational function of $(z, v)$ can be reduced to the form

$$h_{n-1} v^{n-1} + h_{n-2} v^{n-2} + \cdots + h_0$$

where the coefficients $h$ are rational functions of $z$. By application of the principle of partial fractions, this again may be written in the form

$$(1) \qquad H(z, v) = P(z, v) + \sum_k \frac{\varphi^{(i_k)}(z, v)}{(z - a_k)^{i_k}}$$

where the functions $P(z, v)$ and $\varphi^{(i_k)}(z, v)$ are integral rational functions of $(z, v)$, supposed as usual to be expressed in their reduced forms, and where, as indicated by the notation, the function $\varphi^{(i_k)}(z, v)$ does not involve $z$ to a power higher than $z^{i_k - 1}$.

The number of elements in the summation on the right-hand side of (1) is finite, and to any given factor $z - a_k$ of which a power appears as a denominator, we may suppose that only one element corresponds, for if there were several such elements they could evidently be combined in a single element with the highest power of $z - a_k$ which makes its appearance as denominator. Furthermore we may assume that the numerator of an element  is not divisible by the factor $z-a_k$ which appears in its denominator.

As a rule a function of the form (1) will become infinite for finite values of the variable $z$, as well as for the value $z=\infty$. For the moment we shall occupy ourselves with the determination of the most general form of the function which becomes infinite only for the value $z=\infty$.

The function $v$ as defined by the equation (I, 3) is an integral algebraic function of $z$, so that the integral rational function $P(z,v)$ will always be finite for finite values of $z$. A finite value of $z$ for which the function $H(z,v)$ becomes infinite can then only be one for which some one of the elements in the summation becomes infinite, and this can only be the value $z=a_k$ for which the denominator of the element in question vanishes. Also if an element of the summation becomes infinite for the value $z=a_k$ the function $H(z,v)$ too will become infinite, for the remaining elements of the summation will be finite for the value of the variable in question.

As regards finite values of the variable $z$ then, a function of the form (1) can only become infinite for a value $z=a_k$ corresponding to an element of the summation, and for such value it will or will not become infinite according as the corresponding element does or does not become infinite for this value. To determine under what circumstances the form (1) will represent an integral algebraic function of $z$, it will therefore suffice to determine the conditions under which the individual elements

$$\frac{\varphi^{(i)}(z, v)}{(z-a)^i}$$

will represent such functions.

In order that the element (2) may not become infinite for any one of the branches of the fundamental curve corresponding to the value $z=a$, it is necessary and sufficient that the orders of coincidence of these branches with the numerator of the element should each be equal to or greater than $i$. Now it has already been pointed out in Chapter VII that the largest value of an integer $i$, such that the orders of coincidence of an integral rational function $\varphi(z,v)$ which is not divisible by $z-a$ with the branches of the several cycles corresponding to the value $z=a$ are all equal to or greater than $i$, is the largest of the integers $[\mu_1], \ldots [\mu_r]$ corresponding to this value of the variable. All of these integers however are equal to 0 except in the case where the value $z=a$ belongs to the category (C), in which case the $r$ integers include among them ones which are $\geqq 1$. In the representation of an integral algebraic function of $z$ in the form (1) therefore, the summation can only involve elements corresponding to values $z=a_k$ which belong to the category (C). In other words, to the values $z=a_k$ with which we here have to do, must correspond, among other points on the fundamental curve, at least one multiple point.

Let us now suppose that the element (2) has reference to a value $z=a$ which belongs to the category (C). Employing for the moment $M$ to represent the greatest of the integers $[\mu_1], \ldots [\mu_r]$ this will be the greatest value which the integral exponent $i$ may have, consistent with the finiteness of the element for all the branches of the curve corresponding to $z=a$, on the assumption that the numerator is not divisible by $z-a$. If in the element (2) we have $i<M$ we can, by multiplying numerator and denominator by a power of $z-a$, express the element in a fractional form with $(z-a)^M$ as denominator, the numerator in this case then being divisible by a power of $z-a$. In the case where (2) is supposed to represent an element of an integral algebraic function of $z$, we may always assume therefore that we have $i=M$.

In order that an element

$$\frac{\varphi^{(M)}(z, v)}{(z-a)^M} \tag{3}$$

may be finite for all the branches of the curve corresponding to the value $z=a$, it is necessary and sufficient that the order of coincidence of the numerator with each one of these branches be $\geqq M$. Now in Chapter VIII we have employed the notation $l_A$ to indicate the number of the linearly independent functions $\varphi^{(M)}(z,v)$ corresponding to the system of integral rational functions $\varphi(z,v)$, whose orders of coincidence with the $n$ branches of the curve are none of them less than $M$. This then is also the number of the linearly independent functions of the form $\varphi^{(M)}(z,v)$, whose orders of coincidence with the $n$ branches of the curve are none of them less than $M$, since the general integral rational function $\varphi(z, v)$ may be written in the form


$$\varphi(z, v) = \varphi^{(M)}(z, v) + (z-a)^M((z-a, v)),$$

where the function $\varphi^{(M)}(z, v)$ alone on the right-hand side of the identity is affected by an order of coincidence $M$ relative to a branch of the curve corresponding to the value $z=a$. In Chapter VIII we have further seen that the number $l_A$ is equal to $A$, the number of the adjoint conditions corresponding to the value $z=a$. The number of the linearly independent functions $\varphi^{(M)}(z, v)$ which can serve as numerator in the element (3), supposed to be finite for all the branches of the curve corresponding to the value $z=a$, is then given by

(4) $$A = \frac{1}{2}\sum_{s=1}^{r}(\mu_s - 1 + \frac{1}{\nu_s})\nu_s .$$

Representing by $\varphi_1^{(M)}(z, v), \varphi_2^{(M)}(z, v), \ldots \varphi_A^{(M)}(z, v)$ a complete system of linearly independent functions of the form $\varphi^{(M)}(z, v)$, whose orders of coincidence with the $n$ branches of the curve are none of them less than $M$, the most general function which can serve as numerator in the element (3) may be written in the form

(5) $$\varphi^{(M)}(z, v) = \delta_1\varphi_1^{(M)}(z, v) + \delta_2\varphi_2^{(M)}(z, v) + \cdots + \delta_A\varphi_A^{(M)}(z, v)$$

where the $A$ coefficients $\delta$ are arbitrary constants.

Employing an extra index $k$ to distinguish expressions corresponding to the value $z=a_k$, it is evident that the most general rational function of $(z, v)$ which is infinite only for the value $z=\infty$ can be written in the form

(6) $$P(z, v) + \sum_k \frac{\varphi^{(M_k)}(z, v)}{(z-a_k)^{M_k}}$$

where $P(z, v)$ is an arbitrary integral rational function of $(z, v)$, where the summation extends to all values $z=a_k$ which belong to the category (C), and where the indices $M_k$ are equal to the greatest integers in the several sets of integers $[\mu_1^{(k)}], \ldots [\mu_r^{(k)}]$, a function $\varphi^{(M_k)}(z, v)$ being the most general function of the form implied in the index $M_k$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z=a_k$ are none of them less than $M_k$.

The number of arbitrary constants involved in the function $\varphi^{(M_k)}(z,v)$ is



$$A_k=\frac{1}{2}\sum_{s=1}^{r_k}(\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\nu_s^{(k)}$$

— the number of adjoint conditions corresponding to the value $z=a_k$ — and the total number of arbitrary coefficients involved in the essentially fractional portion of the form (6) is given by the sum

$$\sum_k A_k=\frac{1}{2}\sum_k\sum_{s=1}^{r_k}(\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\nu_s^{(k)} \tag{7}$$

where the summation with regard to $k$ is supposed to extend to all values $z=a_k$ which belong to the category (C).

It may be remarked that the limitation just made on the values of $z$ to which we extend the summation in (7), is superfluous — for it has already been pointed out in Chapter V, that the numbers $\mu_1-1+\frac{1}{\nu_1}, \ldots \mu_r-1+\frac{1}{\nu_r}$ are all equal to 0 for finite values of $z$ other than those belonging to the category (C). We might therefore say that the number of arbitrary coefficients $\delta$ involved in the essentially fractional portion of the form (6) is given by the double summation on the right of (7), where the summation may be conceived to extend to all finite values $a_k$ of the variable $z$.

We may also note that the number of arbitrary constants involved in the essentially fractional portion of the form (6), is just equal to the number of conditions which must be satisfied by the coefficients of the general integral rational function of $(z,v)$ in order that it may be adjoint for all finite values of the variable $z$. For the expression on the right-hand side of (7) is equal to the sum of the numbers of conditions requisite to the adjointness of an integral rational function of $(z,v)$ relative to the individual finite values of the variable, and in the case of an integral rational function of $(z,v)$ of degree $n-1$ in $v$ and of sufficiently high degree in $z$, it is evident that the conditions of adjointness relative to the individual finite values of the variable are independent of one another. Namely by imposition of a number of conditions, including those requisite to adjointness relative to the values $z=a_1$, $z=a_2$, ... $z=a_{k-1}$, we can reduce the general integral rational function of $(z,v)$ to the form $(z-a_1)^{M_1}\ldots$ $\ldots(z-a_{k-1})^{M_{k-1}}\varphi(z,v)$, where the coefficients of $\varphi(z,v)$ are arbitrary, and the requisition that our function be also adjoint relative to a further value $z=a_k$, demands the imposition on the coefficients of the factor $\varphi(z,v)$ of the full quota of conditions requisite to the adjointness of an integral rational function of $(z,v)$ relative to the individual value of the variable $z=a_k$ — consequently the conditions of adjointness relative to a given finite value of the variable $z$ are independent of the conditions of adjointness relative to any number of other finite values of the variable.

Turning now to the consideration of the case in which we have to do with a rational function of $(z,v)$ which may become infinite for finite values of the variable as well as for the value $z=\infty$, we shall first determine the general form of such a function which becomes infinite in a certain way for a given set of finite values of the variable, its conduct for the value $z=\infty$ being for the moment unrestricted.

We have seen that an element of the summation in the representation of a rational function $H(z,v)$ in the form (1), will not be finite for all the branches belonging to the corresponding value $z=a$ except in the case where this value belongs to the category (C), in which case it may happen that the element is finite for all $n$ branches. In the representation of a function which is to become infinite for certain finite values of the variable $z$ then, it will suffice to extend the summation in (1) so as to include these values as well as those belonging to the category (C). Representing the aggregate of these values by $a_1, a_2, a_3, \dots$ we shall employ the symbols $\sigma_1^{(k)}, \sigma_2^{(k)}, \dots \sigma_{r_k}^{(k)}$ to indicate the orders* to which the function which we propose to construct is to become infinite for the branches of the $r_k$ cycles corresponding to the value $z=a_k$.

In so far as the numbers $\sigma_1^{(k)}, \sigma_2^{(k)}, \dots \sigma_{r_k}^{(k)}$ are different from 0 they are assumed to be integral multiples of the fractions $\frac{1}{\nu_1^{(k)}}, \frac{1}{\nu_2^{(k)}}, \dots \frac{1}{\nu_{r_k}^{(k)}}$ respectively, where $\nu_1^{(k)}, \nu_2^{(k)}, \dots \nu_{r_k}^{(k)}$ are the orders of the $r_k$ cycles into which are grouped the $n$ branches of the curve corresponding to the value $z=a_k$.

* Where the development of a rational function of $(z,v)$ corresponding to a branch of a cycle of order $\nu$ commences with a term in $(z-a)^{-\sigma}$, we shall say that $\sigma$ is the order to which the function becomes infinite for the branches of the cycle, without regard to whether $\sigma$ is integral or fractional. At the same time however we shall also say that the function possesses $\nu\sigma$ infinities corresponding to the branches of the cycle in question.

The notation here employed, it is to be remarked, takes no account of whether the function in process of construction does or does not become equal to 0 for the branches of a cycle for which it does not become infinite, but simply takes 0 as the value of the number $\sigma^{(k)}$ corresponding to a cycle for which the function is not to become infinite. If the value $z=a_k$ belongs to the category (C), it may or may not happen that the numbers $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ in the case of the function proposed for construction are all equal to 0. When the value of the variable in question does not belong to this category however, one at least of the $r_k$ numbers $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ will be different from 0 if the summation in the form (1) is to contain a corresponding element.

The infinities of a rational function of $(z,v)$ for a value $z=a$ will be the same as those of the corresponding element in the representation of the function in the form (1). In order then to prove the existence of a rational function of $(z,v)$ having an assigned set of infinities for given finite values of $z$, and in order to construct the most general rational function having such infinities, it will be sufficient to prove the existence of elements having the required infinities for the corresponding values of the variable $z$ and to determine these elements in their most general form.

We shall now examine more closely the form of an element (2) supposed to become infinite to certain orders — which we shall indicate by $\sigma_1, \sigma_2, \ldots \sigma_r$ respectively — for the branches of the $r$ cycles corresponding to the value $z=a$. Such an element for example may be constructed with a numerator which is not divisible by $z-a$, and with a denominator in which the exponent $i$ is equal to the greatest of the integers

$$[\mu_1+\sigma_1], [\mu_2+\sigma_2], \ldots [\mu_r+\sigma_r],$$

as may readily be shown. Representing, for the moment, by $[\mu+\sigma]$ the greatest of these integers, we see that the $r$ differences

$$[\mu+\sigma]-\sigma_1, [\mu+\sigma]-\sigma_2, \ldots [\mu+\sigma]-\sigma_r$$

must constitute a set of adjoint numbers corresponding to the value $z=a$, for they are evidently not less than the respective numbers

$$[\mu_1+\sigma_1]-\sigma_1,\ [\mu_2+\sigma_2]-\sigma_2,\ \ldots\ [\mu_r+\sigma_r]-\sigma_r,$$


and these again are equal to or greater than the numbers

$$\mu_1-1+\frac{1}{\nu_1},\ \mu_2-1+\frac{1}{\nu_2},\ \ldots\ \mu_r-1+\frac{1}{\nu_r}$$

respectively. We can therefore construct an integral rational function of $(z, v)$ having $[\mu+\sigma]-\sigma_1, \ldots [\mu+\sigma]-\sigma_r$ respectively as its orders of coincidence with the branches of the several cycles. Also these orders of coincidence do not require divisibility of the function by the factor $z-a$, for the numbers $[\mu+\sigma]-\sigma_1, \ldots [\mu+\sigma]-\sigma_r$ are not simultaneously greater than the corresponding members of the set of numbers $\mu_1, \ldots \mu_r$. This follows namely from the fact, that one at least of the former set of numbers is equal to the corresponding number in the system $[\mu_1+\sigma_1]-\sigma_1, \ldots [\mu_r+\sigma_r]-\sigma_r$, and the numbers in this latter system are evidently equal to or less than the corresponding members of the set of numbers $\mu_1, \ldots \mu_r$.

We can therefore construct an integral rational function of $(z, v)$, which is not divisible by $z-a$ and whose orders of coincidence with the branches of the several cycles corresponding to the value $z=a$ are equal to the numbers $[\mu+\sigma]-\sigma_1, \ldots [\mu+\sigma]-\sigma_r$ respectively. The quotient of this function by $(z-a)^i$, where $i=[\mu+\sigma]$, will be infinite to the orders $\sigma_1, \ldots \sigma_r$ respectively for the branches of the several cycles, and the same will evidently also be true of the fraction obtained on dropping terms in the numerator which are divisible by $(z-a)^i$. The fraction so obtained will be an element of the form (2) — supposing the original numerator to have been arranged according to powers of $z-a$ and $v$.

It is possible then to construct an element of the form (2), which becomes infinite to the orders $\sigma_1, \ldots \sigma_r$ respectively for the branches of the several cycles corresponding to the value $z=a$, and in which the exponent $i$ is equal to $[\mu+\sigma]$, the greatest of the $r$ integers $[\mu_1+\sigma_1], \ldots [\mu_r+\sigma_r]$, while the numerator is not divisible by $z-a$. Furthermore, in an element of the form (2), which becomes infinite to the orders $\sigma_1, \ldots \sigma_r$ respectively for the branches of the several cycles, it is impossible that we should have $i>[\mu+\sigma]$ unless the numerator is divisible by $z-a$. For suppose that we have $i=[\mu+\sigma]+\lambda$, where $\lambda$ is a positive integer, and assume that the element does not become infinite to orders which are higher than $\sigma_1, \dots \sigma_r$ respectively for the branches of the several cycles. It follows that the numerator must have orders of coincidence with the branches of the several cycles which are not less than the numbers $[\mu+\sigma]+\lambda-\sigma_1, \dots [\mu+\sigma]+\lambda-\sigma_r$ respectively, and which are therefore also not less than the numbers $[\mu_1+\sigma_1]+\lambda-\sigma_1, \dots [\mu_r+\sigma_r]+\lambda-\sigma_r$. The members of the latter set of numbers however are evidently greater than the numbers $\mu_1+\lambda-1, \dots \mu_r+\lambda-1$ respectively and the numerator — in accord with the theory developed in Chapter V — must therefore be divisible by $(z-a)^\lambda$. On dividing numerator and denominator of the element in question by $(z-a)^\lambda$, it then reduces to a form in which the exponent appearing in the denominator has the value $[\mu+\sigma]$.

Also in the case where we have to do with an element in which the exponent $i$ is less than $[\mu+\sigma]$, it suffices to multiply numerator and denominator by a power of $z-a$ in order to bring it into a form in which the exponent in the denominator is equal to $[\mu+\sigma]$. In all cases then where we have to do with an element of the form (2), supposed to become infinite to the orders $\sigma_1, \sigma_2, \dots \sigma_r$ respectively, — or to orders which are not higher than these — for the branches of the several cycles corresponding to the value $z=a$, we may assume that the exponent $i$ is equal to the greatest of the $r$ integers $[\mu_1+\sigma_1], \dots [\mu_r+\sigma_r]$.

We shall then assume that the exponent $i$ in the element (2) has the value $[\mu+\sigma]$. In order that the element may not become infinite to orders which are higher than those indicated by the numbers $\sigma_1, \sigma_2, \dots \sigma_r$ respectively for the branches of the several cycles corresponding to the value $z=a$, it is necessary and sufficient that the orders of coincidence of the numerator with the branches of these cycles should not be less than the numbers $i-\sigma_1, i-\sigma_2, \dots i-\sigma_r$ respectively. Now these numbers, as has already been pointed out, constitute a set of adjoint numbers relative to the value $z=a$. The number of conditions then which must be satisfied by the coefficients of the general integral rational function $\varphi(z,v)$, in order that it may have the orders of coincidence $i-\sigma_1, i-\sigma_2, \dots i-\sigma_r$ with the branches of the several cycles, will be obtained on replacing by these numbers respectively, the numbers $\mu'_1, \mu'_2, \ldots \mu'_r$ which appear in the formula (VIII, 6). The number so obtained is



$$\sum_{s=1}^{r}(i-\sigma_s)\nu_s - A.$$

This then is evidently also the number of conditions which must be satisfied by the coefficients of the general function of the form $\varphi^{(i)}(z,v)$, in order that its orders of coincidence with the branches of the several cycles may not be less than the numbers $i-\sigma_1$, $i-\sigma_2$, $\ldots$ $i-\sigma_r$ respectively. Subtraction of this number from $in=\sum_{s=1}^{r} i\nu_s$ — the number of terms involved in the expression of the general function of the form $\varphi^{(i)}(z,v)$ — gives us the number of arbitrary constants involved in the expression of the most general function of the form $\varphi^{(i)}(z,v)$, whose orders of coincidence with the branches of the several cycles are equal to or greater than the numbers $i-\sigma_1$, $i-\sigma_2$, $\ldots$ $i-\sigma_r$ respectively. Indicating this number by the letter $l$ we have

$$l=\sum_{s=1}^{r}\sigma_s\nu_s + A. \tag{8}$$

This formula then gives us the number of arbitrary constants involved in the expression of the most general function $\varphi^{(i)}(z,v)$, which can serve as numerator in the element (2), supposed to become infinite to orders not exceeding $\sigma_1, \sigma_2, \ldots \sigma_r$ respectively for the branches of the several cycles. In other words $l$ is the number of linearly independent functions $\varphi^{(i)}(z,v)$ which can serve as numerator in the element (2). On representing by $\varphi_1^{(i)}(z,v)$, $\varphi_2^{(i)}(z,v)$, $\ldots$ $\varphi_l^{(i)}(z,v)$ a complete system of linearly independent functions $\varphi^{(i)}(z,v)$, whose orders of coincidence with the branches of the several cycles are equal to or greater than the numbers $i-\sigma_1$, $i-\sigma_2$, $\ldots$ $i-\sigma_r$ respectively, the most general numerator will have the form

$$\varphi^{(i)}(z,v)=\delta_1\varphi_1^{(i)}(z,v)+\delta_2\varphi_2^{(i)}(z,v)+\cdots+\delta_l\varphi_l^{(i)}(z,v)$$

where the coefficients $\delta$ are arbitrary constants.

Since, for special forms of the numerator at least — as we have already seen — the element actually becomes infinite to the orders $\sigma_1, \sigma_2, \ldots \sigma_r$ respectively for the branches of the several cycles, it will also become infinite to these orders excepting for conditioned values of the arbitrary constants $\delta$. It would be necessary, for example, to impose a condition on the constants $\delta$, in order that the order of coincidence of the numerator with the branches of the $s$th cycle may be greater than $i-\sigma_s$, and therewith that the order of infinity of the element for the branches of this cycle may be less than $\sigma_s$ — so long that is as we have $\sigma_s>0$.

In this connection a remark may be made in regard to the orders of coincidence of the numerator in an element, which actually becomes infinite to the orders $\sigma_1, \sigma_2, \ldots \sigma_r$ respectively for the branches of the several cycles. The order of coincidence of such numerator with the branches of the $s$th cycle must evidently be just equal to $i-\sigma_s$ in the case where we have $\sigma_s>0$, whereas this is not required in the case where we have $\sigma_s=0$, so that we may say of the orders of coincidence of the numerator with the branches of the several cycles, that they are not less than the numbers $i-\sigma_1, i-\sigma_2, \ldots i-\sigma_r$ respectively, but not in general that they are equal to these numbers, excepting in so far as the quantities $\sigma$ appearing in the expression of these numbers are different from 0. It may even happen that it would be impossible to construct a function of the form $\varphi^{(i)}(z,v)$ whose orders of coincidence with the branches of the several cycles are just equal to the numbers $i-\sigma_1, i-\sigma_2, \ldots i-\sigma_r$ respectively, in the case where certain of the $\sigma$'s have the value 0, so that in an element of the form (2), supposed to become infinite to the orders $\sigma_1, \sigma_2, \ldots \sigma_r$ respectively for the branches of the several cycles, the order of coincidence of the numerator with the branches of a certain cycle might necessarily be greater than $i$ in the case where the corresponding $\sigma$ has the value 0, in other words in the case where the element remains finite for the branches of the cycle in question.

This will best be illustrated by a simple example. — Let us suppose that the $n$ branches corresponding to the value $z=a$ are made up of a cycle of odd order $\nu_1>1$ and of $n-\nu_1$ isolated simple branches. Furthermore suppose that the equation to a branch of the cycle of order $\nu_1$ has the form

$$v-b=\alpha_2\varepsilon^2(z-a)^{\frac{2}{\nu_1}}+\alpha_3\varepsilon^3(z-a)^{\frac{3}{\nu_1}}+\ldots$$



where $\varepsilon$ is a $\nu_1$th root of unity and where we have $\alpha_2 \neq 0$. The number of the cycles with which we here have to do is $r=n-\nu_1+1$, where as usual each of the simple branches is regarded as constituting a cycle of order 1, and where the numbers $\mu_1, \mu_2, \ldots \mu_r$ corresponding to the branches of the several cycles evidently have the values $\frac{2(\nu_1-1)}{\nu_1}, 0, \ldots 0$ respectively.

If now we would construct an element of the form (2) corresponding to the value $z=a$ and becoming infinite to the orders $\sigma_1=0, \sigma_2=1, \ldots \sigma_r=1$ respectively for the branches of the several cycles, we should have 1 for the value of the exponent $i$, this being, by the general theory given above, equal to the greatest of the integers $[\mu_1+\sigma_1], \ldots [\mu_r+\sigma_r]$ — all of which, as it happens in the present case, have the common value 1. Our element then will have the form $(z-a)^{-1}\varphi^{(1)}(z,v)$, where the orders of coincidence of the numerator $\varphi^{(1)}(z,v)$ with the branches of the several cycles are not less than the numbers $i-\sigma_1, i-\sigma_2, \ldots i-\sigma_r$ respectively, and where in fact its orders of coincidence with the branches of the last $r-1$ cycles — otherwise said, with the $r-1$ simple branches — must all be equal to 0.

The order of coincidence of $\varphi^{(1)}(z,v)$ with the branches of the cycle of order $\nu_1$ however cannot be equal to 1, as may readily be shewn, and must therefore be greater than this value. A function of the form $\varphi^{(1)}(z,v)$ namely will, as indicated by the index (1), involve only terms in $v$ and may therefore be regarded as a polynomial in this variable of degree $n-1$. It may further be expressed in the form of a product $(v-b)^{\lambda}h(v)$ where $\lambda$ is 0 or a positive integer and where $h(v)$ is a polynomial in $v$ which is not divisible by $v-b$. The order of coincidence of $h(v)$ with the branches of the cycle of order $\nu_1$ will then be equal to 0 and the order of coincidence of the factor $v-b$ with these branches is evidently equal to $\frac{2}{\nu_1}$, so that the order of coincidence of $\varphi^{(1)}(z,v)=(v-b)^{\lambda}h(v)$ with the branches of the cycle will have the value $\frac{2\lambda}{\nu_1}$. The order of coincidence of $\varphi^{(1)}(z,v)$ with the branches of the cycle of order $\nu_1$ is then an even multiple of $\frac{1}{\nu_1}$ and cannot therefore have the value 1 since by hypothesis $\nu_1$ is odd.

The smallest value of $\lambda$ for which we have $\frac{2\lambda}{\nu_1}>1$ is $\lambda=\frac{\nu_1+1}{2}$, and this  then is the smallest value which $\lambda$ can have in the product $(v-b)^\lambda h(v)$, whose order of coincidence with the branches of the cycle of order $\nu_1$ is supposed to be $\overline{\gtrless} 1$. In an element $(z-a)^{-1}\varphi^{(1)}(z,v)$ corresponding to the value $z=a$ and becoming infinite to the orders 0, 1, ... 1 respectively for the branches of the several cycles, the numerator $\varphi^{(1)}(z,v)$ may therefore be expressed in the form $(v-b)^{\frac{\nu_1+1}{2}}h_1(v)$, where $h_1(v)$ is a polynomial in $v$ of degree $n-\frac{\nu_1+3}{2}$ which may or may not happen to be divisible by $v-b$, and the most general element of the character in question will evidently be represented by $(z-a)^{-1}(v-b)^{\frac{\nu_1+1}{2}}h_1(v)$, where $h_1(v)$ is a polynomial of the degree just stated, with arbitrary coefficients, excluding however such conditioned sets of values for the coefficients as would imply that the polynomial has an order of coincidence with a simple branch which exceeds 0. The number of arbitrary constants in the numerator of the element will evidently be equal to $n-\frac{\nu_1+1}{2}$, the number of terms in the arbitrary polynomial $h_1(v)$, and the order of coincidence of the numerator with the branches of the cycle of order $\nu_1$ is $\overline{\gtrless} 1+\frac{1}{\nu_1}$, while its orders of coincidence with the remaining $n-\nu_1$ branches are all equal to 0.

In an element of the form (2), corresponding to the value $z=a$ and becoming infinite to the orders $\sigma_1, \sigma_2, \ldots, \sigma_r$ respectively for the branches of the several cycles, we see then that it may happen to be impossible for the orders of coincidence of the numerator with the branches of the several cycles to be just equal to the numbers $i-\sigma_1$, $i-\sigma_2$, ... $i-\sigma_r$ respectively. Namely in the case where a number of the $\sigma$'s have the value 0, it may be that the orders of coincidence of the numerator with the branches of the corresponding cycles, or of certain of these cycles, are necessarily greater than $i$.

To shew how this is compatible with the fact that we can construct an integral rational function of $(z,v)$, whose orders of coincidence with the branches of the several cycles are just equal to the numbers $i-\sigma_1$, $i-\sigma_2$, ... $i-\sigma_r$ respectively, it is only necessary to write such a function in the form



$$\varphi(z,v) = \varphi^{(i)}(z,v) + (z-a)^i((z-a,v))$$

when we see that the order of coincidence of $\varphi^{(i)}(z,v)$, like that of $\varphi(z,v)$, with the branches of the $s$th cycle is just equal to $i-\sigma_s$ in the case where we have $\sigma_s > 0$, but that if $\sigma_s$ is equal to 0, the order of coincidence $i$ of $\varphi(z,v)$ with the branches of this cycle is compatible with a higher order of coincidence on the part of $\varphi^{(i)}(z,v)$ with these branches on account of the element $(z-a)^i((z-a,v))$, whose order of coincidence with the branches in question may happen to have the value $i$.

We shall now return to the consideration of the most general rational function $H(z,v)$ whose infinities — apart from those at $\infty$ — are included in a certain set of infinities corresponding to finite values of the variable $z$. Our function we shall suppose to be represented in the form (1) where, as already indicated, the values $a_1, a_2, a_3, \ldots$, to which elements in the summation correspond, are made up of all those values of $z$ for which the function is to become infinite, and of all those values of the variable which belong to the category (C) whether the function is to become infinite for them or not. If the function $H$ $(z,v)$ is to become infinite to orders not exceeding $\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots \sigma_{r_1}^{(1)}$ for the branches of the $r_1$ cycles corresponding to the value $z=a_1$, to orders not exceeding $\sigma_1^{(2)}, \sigma_2^{(2)}, \ldots \sigma_{r_2}^{(2)}$ for the branches of the $r_2$ cycles corresponding to the value $z=a_2, \ldots$ to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ for the branches of the $r_k$ cycles corresponding to the value $z=a_k$, ... the exponents $i_k$ — in accord with the theory developed in the present chapter — are to be taken equal to the greatest integers in the several sets of integers $[\mu_1^{(k)} + \sigma_1^{(k)}], \ldots [\mu_r^{(k)} + \sigma_{r_k}^{(k)}]$. Also the numerator in the $k$th element of the summation must have orders of coincidence with the branches of the $r_k$ cycles corresponding to the value $z=a_k$, which do not fall short of the numbers $i_k-\sigma_1^{(k)}, i_k-\sigma_2^{(k)}, \ldots i_k-\sigma_{r_k}^{(k)}$ respectively. On employing the symbol $l_k$ to indicate the number of the linearly independent functions $\varphi^{(i_k)}(z,v)$ which possess such orders of coincidence, and on representing by $\varphi_1^{(i_k)}(z,v), \varphi_2^{(i_k)}(z,v), \ldots \varphi_{l_k}^{(i_k)}(z,v)$ a complete system of linearly independent functions of the character in question, the most general function which can serve as numerator in the $k$th element will have the form


(9) $$\varphi^{(i_k)}(z,v)=\delta_1^{(k)}\varphi_1^{(i_k)}(z,v)+\delta_2^{(k)}\varphi_2^{(i_k)}(z,v)+\cdots+\delta_{l_k}^{(k)}\varphi_{l_k}^{(i_k)}(z,v)$$

where the coefficients $\delta^{(k)}$ are arbitrary constants.

In accord with formula (8) we shall have for $l_k$ the expression

(10) $$l_k=\sum_{s=1}^{r_k}\sigma_s^{(k)}\nu_s^{(k)}+A_k$$

and for the total number of the arbitrary coefficients $\delta$ involved in the essentially fractional portion of our representation of the function $H(z,v)$ in the form (1), we shall have

(11) $$\sum_k l_k=\sum_k\sum_{s=1}^{r_k}\sigma_s^{(k)}\nu_s^{(k)}+\sum_k A_k=\sum_k\sum_{s=1}^{r_k}\sigma_s^{(k)}\nu_s^{(k)}+\frac{1}{2}\sum_k\sum_{s=1}^{r_k}\left(\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)}$$

where the summations are extended to all those values of $z$ for which the function is supposed to become infinite, and to all those values of the variable which belong to the category (C). It evidently amounts to the same thing to say, that the former of the two double summations appearing on the right of this formula is extended to those values $z$ for which the function is supposed to become infinite, and the latter to those values of the variable which belong to the category (C).

In the special case where the summation in formula (1) is extended only to values of the variable $z$ which belong to the category (C), and where at the same time the numbers $\sigma$ corresponding to these values are all equal to 0, the formula in question represents the most general rational function of $(z,v)$ which becomes infinite only for $z=\infty$, and coincides with the formula (6), while the number of arbitrary constants involved in the fractional portion of the function and given by formula (11) identifies itself with the number given in formula (7).

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# CHAPTER XI.

## Rational functions which are restricted for all values of $z$.

Consideration of the character of a rational function of $(z, v)$ for the value $z=\infty$. Conditions which must be satisfied by the coefficients in a rational function of $(z, v)$ which may possess certain infinities corresponding to finite values of the variable $z$ and which at the same time must have a specified character for the value $z=\infty$. Reduction of the equations of condition to a form more convenient for interpretation.

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We shall now impose restrictions at infinity on the general rational function considered in the last chapter and there left unrestricted for the value $z=\infty$. To study the effect of such restrictions on the coefficients of the function, it will be necessary to modify the form of representation given in formula (X, 1). We shall begin by expressing in convenient form an individual element of the type (X, 2). Such element may evidently be expanded in the form

$$(1) \qquad \frac{\varphi^{(i)}(z, v)}{(z-a)^i}=\frac{B_1}{z}+\frac{B_2}{z^2}+\cdots+\frac{B_s}{z^s}+\ldots$$

where the coefficients $B$ are polynomials in $v$ of degree $\overline{\gtrless} n-1$. The forms of these polynomials may readily be obtained in terms of the coefficients of the function $\varphi^{(i)}(z, v)$. Namely on transferring to the left-hand side of the identity (1) the first $s-1$ terms on the right, and on multiplying through by $z^{s-1}(z-a)^i$, we arrive at the identity

$$(2) \quad z^{s-1}\varphi^{(i)}(z, v)-(z-a)^i(B_1 z^{s-2}+B_2 z^{s-3}+\cdots+B_{s-1})=(z-a)^i\left(\frac{B_s}{z}+\frac{B_{s+1}}{z^2}+\ldots\right).$$

The expression on the left-hand side of this identity is a polynomial in $z$ and the same must therefore hold good in regard to the product on the right, when the multiplication by the factor $(z-a)^i$ has been effected. The degree in $z$ of the expression on the right is evidently $i-1$ and the coefficient of the term of highest order in $z$ — that is of the term in $z^{i-1}$ — is $B_s$. This then will also be the coefficient of $(z-a)^{i-1}$ on arranging the expression according to powers of $z-a$. It follows that $B_s$ is the coefficient of $(z-a)^{i-1}$ in the expression on the left-hand side of (2), on arranging according to powers of $z-a$, and therefore also the coefficient of $(z-a)^{i-1}$ in the development of the function $z^{s-1}\varphi^{(i)}(z,v)$ in powers of $z-a$. We may therefore write

$$B_s=\frac{1}{i-1!}\left[\left(\frac{\partial}{\partial z}\right)^{i-1} z^{s-1}\varphi^{(i)}(z,v)\right]_{z=a}$$

and the element in (1) will now assume the form

$$\text{(3)}\qquad \frac{\varphi^{(i)}(z,v)}{(z-a)^i}=\sum_{s=1}^{\infty}\frac{1}{i-1!}\left[\left(\frac{\partial}{\partial z}\right)^{i-1} z^{s-1}\varphi^{(i)}(z,v)\right]_{z=a} z^{-s}.$$

Here we have found it convenient to make use of the notation of the differential calculus. We may remark in passing that formula (3) can also be very easily obtained by application of the methods of the differential calculus. Namely we evidently have

$$\frac{1}{i-1!}\left(\frac{\partial}{\partial a}\right)^{i-1}\frac{\varphi^{(i)}(z,v)-\varphi^{(i)}(a,v)}{z-a}=0$$

since the expression under the sign of differentiation, regarded as a function of $a$ alone, is a polynomial of degree $i-2$. We immediately derive therefrom

$$\frac{\varphi^{(i)}(z,v)}{(z-a)^i}=\frac{1}{i-1!}\left(\frac{\partial}{\partial a}\right)^{i-1}\frac{\varphi^{(i)}(a,v)}{z-a}=\sum_{s=1}^{\infty}\frac{1}{i-1!}\left(\frac{\partial}{\partial a}\right)^{i-1} a^{s-1}\varphi^{(i)}(a,v)\,z^{-s}$$

a formula which coincides with formula (3) above.

In the formula (X, 1) we shall now replace each of the elements in the summation on the right-hand side by a development of the type given in formula (3). We thus obtain for the rational function $H(z,v)$ a representation in the form

$$\text{(4)}\qquad H(z,v)=P(z,v)+\sum_k\sum_{s=1}^{\infty}\frac{1}{i_k-1!}\left[\left(\frac{\partial}{\partial z}\right)^{i_k-1} z^{s-1}\varphi^{(i_k)}(z,v)\right]_{z=a_k} z^{-s}.$$

We shall find it convenient for the moment to suppose the function $H(z,v)$ to be expressed in terms of the variables $\xi = z^{-}$, $\eta = z^{-m} v$ already introduced in Chapter VIII, where we saw that $\eta$ was an integral algebraic function of $\xi$ defined by the equation $G(\xi,\eta) = 0$. The function $H(z,v)$ will then be represented as a polynomial in $\eta$ of degree $n-1$, with coefficients which are rational functions of $\xi$. On developing these coefficients according to positive and negative powers of $\xi$ the number of terms involving negative exponents will be finite, and the function may evidently be represented in the form



$$(5) \qquad H(z,v) = -\frac{\varphi^{(i_\infty)}(\xi,\eta)}{\xi^{i_\infty}} + ((\xi,\eta))$$

where $\varphi^{(i_\infty)}(\xi,\eta)$ is a polynomial in $(\xi,\eta)$ of degrees $i_\infty - 1$ and $n-1$ respectively in these variables and where $((\xi,\eta))$ represents a polynomial in $\eta$ of degree $n-1$, in which the developments of the coefficients according to powers of $\xi$ involve no negative exponents. Here we have distinguished a number corresponding to the value $\xi = 0$ — *i.e* to the value $z = \infty$ — by attaching to it the symbol $\infty$, and this device we shall continue to employ in what follows. The introduction of the minus sign in formula (5) is merely intended to secure greater uniformity in the expression of certain formulae with which we shall have to do a little later on.

If the function $H(z,v)$ is to become infinite to orders not exceeding $\sigma_1^{(\infty)}, \sigma_2^{(\infty)}, \ldots \sigma_{r_\infty}^{(\infty)}$ respectively for the branches of the $r_\infty$ cycles of the equation $F(z,v) = 0$ corresponding to the value $z = \infty$, the same also must be true of the orders of infinity of the function, transformed to terms of $\xi$ and $\eta$, for the branches of the $r$ cycles of the equation $G(\xi,\eta) = 0$ corresponding to the value $\xi = 0$. Furthermore — in accord with the theory developed in the last chapter — the exponent $i_\infty$ in formula (5) may be assumed to be equal to the greatest of the integers

$$(6) \quad [m(n-1) + \mu_1^{(\infty)} + \sigma_1^{(\infty)}],\ [m(n-1) + \mu_2^{(\infty)} + \sigma_2^{(\infty)}],\ \ldots\ [m(n-1) + \mu_{r_\infty}^{(\infty)} + \sigma_{r_\infty}^{(\infty)}]$$

for the orders of coincidence of the branches of the several cycles of the equation $G(\xi,\eta) = 0$ corresponding to the value $\xi = 0$, each with the product of the other $n-1$ branches, are equal to the numbers

$$m(n-1)+\mu_1^{(\infty)},\ m(n-1)+\mu_2^{(\infty)},\ \dots\ m(n-1)+\mu_{r_\infty}^{(\infty)}$$



respectively — numbers with which we have already had to do in formula (VIII, 11) but to which we now attach the symbol $\infty$. These numbers from the very nature of their definition can none of them be negative. In fact they must evidently be equal to or greater than the numbers $1-\frac{1}{\nu_1^{(\infty)}},\ 1-\frac{1}{\nu_2^{(\infty)}},\ \dots\ 1-\frac{1}{\nu_{r_\infty}^{(\infty)}}$ respectively. Of the integers in (6) too one at least must evidently be $\geqq 1$ unless the $\sigma$'s are all equal to 0. If the $\sigma$'s all have the value 0 the exponent $i_\infty$ will be equal to the greatest of the integers

(7) $$m(n-1)+[\mu_1^{(\infty)}],\ m(n-1)+[\mu_2^{(\infty)}],\ \dots\ m(n-1)+[\mu_{r_\infty}^{(\infty)}].$$

These $r_\infty$ integers it may happen all take the value 0, in which case we have $i_\infty=0$ and the fractional element in (5) reduces to 0 — in fact the numerator $\varphi^{(0)}(\xi,\eta)$ by definition represents something non-existent, namely a polynomial in $(\xi,\eta)$ of degree less than 0 in $\xi$ and is therefore to be replaced by 0. This case will only present itself in connection with a class of curves $F(z,v)=0$ of a very special form.

In the fractional element on the right-hand side of formula (5), the orders of coincidence of the numerator $\varphi^{(i_\infty)}(\xi,\eta)$ with the branches of the several cycles will of course be equal to or greater than the numbers $i_\infty-\sigma_1^{(\infty)},\ i_\infty-\sigma_2^{(\infty)},\ \dots\ i_\infty-\sigma_{r_\infty}^{(\infty)}$ respectively — numbers which are adjoint relatively to the curve $G(\xi,\eta)=0$ for the value $\xi=0$.

The form (5) has been constructed with reference to the infinities only of the function $H(z,v)$ corresponding to the value $z=\infty$ and is not adapted as it stands to a discussion of the zeros which the function may possess for this value of the variable. We shall find it convenient however on occasion to consider both the zeros and the infinities of our function for the value $z=\infty$. To this end then we shall represent our function in the modified form

(8) $$H(z,v)=-\frac{\varphi^{(i_\infty+j)}(\xi,\eta)}{\xi^{i_\infty}}+\xi^j((\xi,\eta))$$

where for $j$ we may choose 0 or any positive integer, as it happens to suit our convenience. Here as before the exponent $i_\infty$ is supposed to be taken equal to the greatest integer in the set of integers (6). Also the orders of coincidence of the numerator $\varphi^{(i_\infty+j)}(\xi,\eta)$ with the branches of the several cycles will be equal to or greater than the numbers $i_\infty-\sigma_1^{(\infty)}$, $i_\infty-\sigma_2^{(\infty)}$, ... ... $i_\infty-\sigma_{r_\infty}^{(\infty)}$ respectively. That the function $H(z,v)$ can be represented in the form (8) follows immediately from the possibility of its representation in the form (5).



If now we were to characterize the function $H(z,v)$ for the value $z=\infty$, by saying that its orders of coincidence with the branches of the several cycles corresponding to this value of the variable are equal to or greater than the numbers $\tau_1^{(\infty)}$, $\tau_2^{(\infty)}$, ... $\tau_{r_\infty}^{(\infty)}$ respectively — where these numbers may be positive, negative or zero — we would choose for $j$ an integral number which is not negative and which also is not less than the greatest of the numbers $\tau^{(\infty)}$. If no one of the numbers $\tau^{(\infty)}$ happens to be positive we can put $j$ equal to 0; otherwise $j$ will be a positive integer. If $\tau_s^{(\infty)}$ is negative we have $\tau_s^{(\infty)}=-\sigma_s^{(\infty)}$; if $\tau_s^{(\infty)}$ is 0 or positive we have $\sigma_s^{(\infty)}=0$. In any case the orders of coincidence of the numerator $\varphi^{(i_\infty+j)}(\xi,\eta)$ of the fractional element in formula (8) with the branches of the several cycles corresponding to the value $z=\infty$ will be equal to or greater than the numbers $i_\infty+\tau_1^{(\infty)}$, $i_\infty+\tau_2^{(\infty)}$, ... $i_\infty+\tau_{r_\infty}^{(\infty)}$ respectively. Furthermore these numbers constitute an adjoint set of numbers relatively to the curve $G(\xi,\eta)=0$ for the value $\xi=0$, since this was true of the set of numbers $i_\infty-\sigma_1^{(\infty)}$, $i_\infty-\sigma_2^{(\infty)}$, ... $i_\infty-\sigma_{r_\infty}^{(\infty)}$.

Now the number of conditions which must be satisfied by the coefficients of the general integral rational function of $(\xi,\eta)$, in order that it may have the set of adjoint orders of coincidence $i_\infty+\tau_1^{(\infty)}$, $i_\infty+\tau_2^{(\infty)}$, .. $i_\infty+\tau_{r_\infty}^{(\infty)}$ relatively to the equation $G(\xi,\eta)=0$ for the value $\xi=0$, will be obtained — in accord with formula (VIII, 6) — on subtracting from the total number of the coincidences in question, the number of the conditions requisite to adjointness for the value $\xi=0$. The number so obtained is

(9)
$$\sum_{s=1}^{r_\infty}\left(i_\infty+\tau_s^{(\infty)}\right)\nu_s^{(\infty)}-\frac{1}{2}\sum_{s=1}^{r_\infty}\left(m(n-1)+\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}\right)\nu_s^{(\infty)}.$$

This then must evidently also be the number of conditions which must be satisfied by the coefficients of the general function of the form $\varphi^{(i_\infty+j)}(\xi,\eta)$, in order that it may have orders of coincidence with the branches of the several cycles corresponding to the value $\xi=0$, which do not fall short of the numbers $i_\infty+\tau_1^{(\infty)}$, $i_\infty+\tau_2^{(\infty)}$, ... $i_\infty+\tau_{r_\infty}^{(\infty)}$ respectively, for none of these numbers exceeds the index $i_\infty+j$. The number of arbitrary coefficients involved in the expression of the most general function $\varphi^{(i_\infty+j)}(\xi,\eta)$, whose orders of coincidence are as here required, will therefore be obtained on subtracting the number given in formula (9) from $(i_\infty+j)n$ — the total number of terms involved in the general function of the form here in question. On indicating the number so obtained by the symbol $l_\infty$ we have

$$(10)\qquad l_\infty=\sum_{s=1}^{r_\infty}(j-\tau_s^{(\infty)})\,\nu_s^{(\infty)}+\frac{1}{2}\sum_{s=1}^{r_\infty}\left(m\,(n-1)+\mu_s^{(\infty)}-1+\frac{1}{\nu_s^{(\infty)}}\right)\nu_s^{(\infty)}.$$

The number of the linearly independent functions of the form $\varphi^{(i_\infty+j)}(\xi,\eta)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $\xi=0$ are equal to or greater than the numbers $i_\infty+\tau_1^{(\infty)}$, $i_\infty+\tau_2^{(\infty)}$, ... $i_\infty+\tau_{r_\infty}^{(\infty)}$ respectively, is then $l_\infty$, and on representing by $\varphi_1^{(i_\infty+j)}(\xi,\eta)$, $\varphi_2^{(i_\infty+j)}(\xi,\eta)$ ... $\varphi_{l_\infty}^{(i_\infty+j)}(\xi,\eta)$ a complete system of such linearly independent functions, we have for the most general function which can serve as numerator in the fractional element in formula (8)

$$(11)\qquad \varphi^{(i_\infty+j)}(\xi,\eta)=\delta_1^{(\infty)}\varphi_1^{(i_\infty+j)}(\xi,\eta)+\delta_2^{(\infty)}\varphi_2^{(i_\infty+j)}(\xi,\eta)+\cdots+\delta_{l_\infty}^{(\infty)}\varphi_{l_\infty}^{(i_\infty+j)}(\xi,\eta)$$

where the coefficients $\delta^{(\infty)}$ are arbitrary constants — except in so far as they are limited by the character attributed to the function $H(z,v)$ for finite values of the variable $z$.

The most general rational function $H(z,v)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ do not fall short of the numbers $\tau_1^{(\infty)}$, $\tau_2^{(\infty)}$, ... $\tau_{r_\infty}^{(\infty)}$ respectively, must therefore in terms of $(\xi,\eta)$ be representable in the form (8), where the function $\varphi^{(i_\infty+j)}(\xi,\eta)$ has the form given in (11) with arbitrary coefficients $\delta^{(\infty)}$, while the most general rational function $H(z,v)$ whose conduct is unrestricted for the value $z=\infty$ and whose infinities corresponding to finite values of the variable are included in an assigned set of infinities may be represented in the form (4), where the double summation is extended to all the values $z=a_k$ for which the function may become infinite or which belong to the category (C) — the indices $i_k$ and the functions $\varphi^{(i_k)}(z,v)$ having the values and the forms respectively already assigned to them at the close of the preceding chapter.



If therefore a rational function $H(z,v)$ is to have orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$, which do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$ respectively, and if at the same time it is to become infinite to orders not exceeding $\sigma_1^{(1)}, \sigma_2^{(1)}, \ldots \sigma_{r_1}^{(1)}$ for the branches of the $r_1$ cycles corresponding to the value $z=a_1$, to orders not exceeding $\sigma_1^{(2)}, \sigma_2^{(2)}, \ldots \sigma_{r_2}^{(2)}$ for the branches of the $r_2$ cycles corresponding to the value $z=a$, to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ for the branches of the $r_k$ cycles corresponding to the value $z=a_k, \ldots$ where $a_1, a_2, \ldots a_k, \ldots$ include all finite values of the variable for which the function may become infinite or which belong to the category (C) — then must the function be simultaneously representable in the forms (4) and (8) and to determine the conditions which must be satisfied by the coefficients in either of them separately, we identify the two forms and investigate the resulting relations which involve the coefficients of both forms conjointly.

Before identifying the forms (4) and (8), we shall find it convenient to arrange the fractional element in the latter form according to powers of $\xi$. The coefficient of $\xi^{i_\infty+s}$ in the function $\varphi^{(i_\infty+j)}(\xi,\eta)$ is the same as the coefficient of $\xi^{i_\infty+j-1}$ in the function $\xi^{j-s-1}\varphi^{(i_\infty+j)}(\xi,\eta)$, and since for $-i_\infty \leqq s \leqq j-1$ this function involves no negative power of $\xi$, we can evidently write

$$(12)\qquad \varphi^{(i_\infty+j)}(\xi,\eta)=\sum_{s=-i_\infty}^{j-1}\frac{1}{i_\infty+j-1\,!}\left[\left(\frac{\partial}{\partial\xi}\right)^{i_\infty+j-1}\xi^{j-s-1}\varphi^{(i_\infty+j)}(\xi,\eta)\right]_{\xi=0}\xi^{i_\infty+s}$$

and for the formula (8) we have

$$(13)\qquad H(z,v)=-\sum_{s=-i_\infty}^{j-1}\frac{1}{i_\infty+j-1\,!}\left[\left(\frac{\partial}{\partial\xi}\right)^{i_\infty+j-1}\xi^{j-s-1}\varphi^{(i_\infty+j)}(\xi,\eta)\right]_{\xi=0}\xi^{s}+\xi^{j}((\xi,\eta))$$

Identifying this expression with that which has been given for the function $H(z,v)$ in formula (4), we obtain

$$(14)\quad \sum_k \sum_{s=1}^{\infty} \frac{1}{i_k - 1\,!} \left[ \left(\frac{\partial}{\partial z}\right)^{i_k - 1} z^{s-1} \varphi^{(i_k)}(z, v) \right]_{z=a_k} z^{-s}$$
$$+ \sum_{s=-i_\infty}^{j-1} \frac{1}{i_\infty + j - 1\,!} \left[ \left(\frac{\partial}{\partial \xi}\right)^{i_\infty + j - 1} \xi^{j-s-1} \varphi^{(i_\infty + j)}(\xi, \eta) \right]_{\xi=0} \xi^s = -P(z, v) + \xi^j\, ((\xi, \eta)).$$

Now to bring into evidence the relations between the coefficients of the functions $\varphi^{(i_k)}(z, v)$, $\varphi^{(i_\infty + j)}(\xi, \eta)$ and $P(z, v)$ implied in this identity, we shall find it convenient to represent these functions more explicitly in the forms

$$(15)\quad \varphi^{(i_k)}(z, v) = \sum_{t=0}^{n-1} P_t^{(i_k)}(z)\, v^t, \quad \varphi^{(i_\infty + j)}(\xi, \eta) = \sum_{t=0}^{n-1} P_t^{(i_\infty + j)}(\xi)\, \eta^t, \quad P(z, v) = \sum_{t=0}^{n-1} P_t(z)\, v^t.$$

Substituting these forms in (14) and replacing $\eta$ by $\xi^m v$, terms involving the same power of $v$ in the resulting identity must evidently cancel, so that the identity (14) is equivalent to the $n$ identities

$$(16)\quad \sum_k \sum_{s=1}^{\infty} \frac{1}{i_k - 1\,!} \left[ \left(\frac{d}{dz}\right)^{i_k - 1} z^{s-1} P_t^{(i_k)}(z) \right]_{z=a_k} z^{-s}$$
$$+ \sum_{s=-i_\infty}^{j-1} \frac{1}{i_\infty + j - 1\,!} \left[ \left(\frac{d}{d\xi}\right)^{i_\infty + j - 1} \xi^{j-s-1} P_t^{(i_\infty + j)}(\xi) \right]_{\xi=0} \xi^{s+mt} = -P_t(z) + \xi^{j+mt}\, ((\xi))_t$$
$$t = 0, 1, \ldots (n-1)$$

where $((\xi))_t$ represents a series in positive integral powers of $\xi$.

Our assumptions with regard to the integer $j$ have heretofore been that it is not negative and that at the same time it is not less than the greatest of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$. The latter assumption evidently includes the former save when none of the numbers $\tau^{(\infty)}$ is greater than $-1$. From this on we shall find it convenient to assume that the integer $j$ is in no case less than 2 and at the same time that it is never less than the greatest of the numbers $\tau^{(\infty)}$. We shall also find it convenient to replace $-i_\infty$, where it appears as lower limit of a summation in the identities (16), by $-(i_\infty + mt)$. This is admissible since each additional term so introduced into the summation evidently vanishes identically. It implies therefore only a convenient change in our notation of summation when we write the identities (16) in the form

$$(17)\quad \sum_k \sum_{s=1}^{\infty} \frac{1}{i_k - 1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} z^{-s}$$



$$+ \sum_{s=-(i_\infty+mt)}^{j-1} \frac{1}{i_\infty + j - 1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} \xi^{s+mt} = -P_t(z) + \xi^{j+mt}((\xi))_t$$

$$t = 0, 1, \ldots (n-1).$$

On replacing $\xi$ by $z^{-1}$ and separating terms involving negative powers of $z$ from the remaining terms, these $n$ identities evidently split up into the two sets of identities

$$(18)\quad \sum_k \sum_{s=1}^{\infty} \frac{1}{i_k - 1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} z^{-s}$$

$$+ \sum_{s=-mt+1}^{j-1} \frac{1}{i_\infty + j - 1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} z^{-(s+mt)} = \xi^{j+mt}((\xi))_t$$

$$(19)\quad P_t(z) + \sum_{s=-(i_\infty+mt)}^{-mt} \frac{1}{i_\infty + j - 1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} z^{-(s+mt)} = 0$$

$$t = 0, 1, \ldots (n-1).$$

Multiplying each of the identities in the latter set by the corresponding power $v^t$ and adding, these $n$ identities may be embodied in a single identity which can evidently be written in the form

$$(20)\quad P(z,v) = -\sum_{s=0}^{n-1} \sum_{r=0}^{i_\infty} \frac{1}{i_\infty + j - 1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+r+mt-1} P_t^{(i_\infty+j)}(\xi)\right] z^r v^t.$$

This of course is nothing more than the statement that the function $P(z,v)$ constitutes that portion of the element $-\xi^{-i_\infty}\varphi^{(i_\infty+j)}(\xi,\eta)$ transformed to terms of $(z,v)$, which does not involve negative powers of $z$ — in other words $P(z,v)$ is the integral portion of the function $-z^{i_\infty}\varphi^{(i_\infty+j)}(z^{-1}, z^{-m}v)$. To indicate this fact we might conveniently make use of the notation

$$(21)\quad P(z,v) = -[z^{i_\infty}\varphi^{(i_\infty+j)}(z^{-1}, z^{-m}v)].$$

That $P(z,v)$ must be identical with the integral portion of the element $-\xi^{-i_\infty}\varphi^{(i_\infty+j)}(\xi,\eta)$, transformed to terms of $(z,v)$, is of course immediately evident, on comparing the representations of the function $H(z,v)$ in the forms (4) and (8).

We see that the degree of $P(z, v)$ in the variable $z$ is equal to or less than $i_\infty$, and that its coefficients can be expressed in terms of the coefficients $\delta^{(\infty)}$ in $\varphi^{(i_\infty+j)}(\xi, \eta)$ as represented in formula (11), no matter what the values of these coefficients may be. No condition then is imposed on the $l_\infty$ constants $\delta^{(\infty)}$ because of the existence of the identity (21), and therefore also none by virtue of the equivalent set of identities (19). It follows that all the conditions to which the constants $\delta^{(\infty)}$ are subject, as well as all those which must be satisfied by the constants $\delta^{(k)}$ in the functions $\varphi^{(i_k)}(z, v)$ — as represented in formula (X, 9) — are involved in the set of $n$ identities (18). With these identities therefore we shall now occupy ourselves.

An identity of the set (18) in which the right-hand side is $\xi^{j+mt}((\xi))_t$, implies simply that the left-hand side must be representable by a series in $\xi$ beginning with the power $\xi^{j+mt}$ — that is, by a series in $\frac{1}{z}$ beginning with the power $\left(\frac{1}{z}\right)^{j+mt}$. In each of the identities (18) then, equating to 0 the aggregate of terms in which the exponents of $\frac{1}{z}$ are less than $j+mt$, we obtain the set of $n$ identities

$$\sum_k \sum_{s=1}^{j+mt-1} \frac{1}{i_k - 1\,!} \left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} z^{-s} + \sum_{s=-mt+1}^{j-1} \frac{1}{i_\infty + j - 1\,!} \left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} z^{-(s+mt)} = 0$$
$$t = 0, 1, \ldots (n-1)$$

which we may evidently also write in the form

$$(22)\quad \sum_k \sum_{s=1}^{j+mt-1} \frac{1}{i_k - 1\,!} \left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} z^{-s} + \sum_{s=1}^{j+mt-1} \frac{1}{i_\infty + j - 1\,!} \left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} z^{-s} = 0$$
$$t = 0, 1, \ldots (n-1).$$

Equating to 0 the aggregate coefficients of the individual powers of $z$ in each of these identities, we obtain the equivalent system of equations

$$(23)\qquad \sum_k \frac{1}{i_k - 1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} + \frac{1}{i_\infty + j - 1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} = 0$$

$$s = 1, 2, \ldots j + mt - 1;\ t = 0, 1, \ldots (n-1).$$



In these $n(j-1) + \frac{1}{2} mn(n-1)$ equations then are involved all the conditions which must be satisfied by the constants $\delta^{(k)}$ and $\delta^{(\infty)}$ in the functions $\varphi^{(i_k)}(z, v)$ and $\varphi^{(i_\infty+j)}(\xi, \eta)$ respectively. To interpret the equations however we shall find it convenient to write them in a modified form.

Introducing the $\zeta$-functions corresponding to the functions $\varphi^{(i_k)}(z, v)$ and $\varphi^{(i_\infty+j)}(\xi, \eta)$ and defined in accord with the formulae (IX, 6) and (IX, 13), we write

$$(24)\qquad \left\{\begin{aligned} \zeta^{(i_k)}(z, v) &= \top_{n-1}^{(i_k)} v^{n-1} + \top_{n-2}^{(i_k)} v^{n-2} + \cdots + \top_0^{(i_k)} \\ \zeta^{(i_\infty+j)}(\xi, \eta) &= \top_{n-1}^{(i_\infty+j)} \eta^{n-1} + \top_{n-2}^{(i_\infty+j)} \eta^{n-2} + \cdots + \top_0^{(i_\infty+j)} \end{aligned}\right\}$$

After the analogy of the congruences (IX, 10) and (IX, 17), we then have

$$(25)\qquad \left\{\begin{aligned} \varphi^{(i_k)}(z, v) &\equiv \sum_{t=0}^{n-1} (F_n \top_t^{(i_k)} + F_{n-1} \top_{t+1}^{(i_k)} + \cdots + F_{t+1} \top_{n-1}^{(i_k)})\, v^t, \ [\text{mod.}\ (z - a_k)^{i_k}] \\ \varphi^{(i_\infty+j)}(\xi, \eta) &\equiv \sum_{t=0}^{n-1} (G_n \top_t^{(i_\infty+j)} + G_{n-1} \top_{t+1}^{(i_\infty+j)} + \cdots + G_{t+1} \top_{n-1}^{(i_\infty+j)})\, \eta^t, \ [\text{mod.}\ \xi^{i_\infty+j}] \end{aligned}\right.$$

and consequently also

$$(26)\qquad \begin{aligned} P_t^{(i_k)}(z) &\equiv F_n \top_t^{(i_k)} + F_{n-1} \top_{t+1}^{(i\ )} + \cdots + F_{t+1} \top_{n-1}^{(i_k)}, \ [\text{mod.}\ (z - a_k)^{i_k}] \\ P_t^{(i_\infty+j)}(\xi) &\equiv G_n \top_t^{(i_\infty+j)} + G_{n-1} \top_{t+1}^{(i_\infty+j)} + \cdots + G_{t+1} \top_{n-1}^{(i_\infty+j)}, \ [\text{mod.}\ \xi^{i_\infty+j}]. \end{aligned}$$

From these latter congruences we derive

$$\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} P_t^{(i_k)}(z)\right]_{z=a_k} = \left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} (F_n \top_t^{(i_k)}(z) + F_{n-1} \top_{t+1}^{(i_k)}(z) + \cdots + F_{t+1} \top_{n-1}^{(i_k)}(z))\right]_{z=a_k}$$

$$\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1} P_t^{(i_\infty+j)}(\xi)\right]_{\xi=0}$$



$$= \left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1}\left(G_n \Upsilon_t^{(i_\infty+j)}(\xi) + G_{n-1}\Upsilon_{t+1}^{(i_\infty+j)}(\xi) + \cdots + G_{t+1}\Upsilon_{n-1}^{(i_\infty+j)}(\xi)\right)\right]_{\xi=0}$$

and as a consequence we can replace the $n(j-1)+\frac{1}{2}mn(n-1)$ equations (23) by the system of equations

$$(27)\quad \begin{cases} \sum_k \frac{1}{i_k-1\,!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1}\left(F_n\Upsilon_t^{(i_k)}(z) + F_{n-1}\Upsilon_{t+1}^{(i_k)}(z) + \cdots + F_{t+1}\Upsilon_{n-1}^{(i_k)}(z)\right)\right]_{z=a_k} \\ + \frac{1}{i_\infty+j-1\,!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1}\left(G_n\Upsilon_t^{(i_\infty+j)}(\xi)\right.\right. \\ \left.\left. + G_{n-1}\Upsilon_{t+1}^{(i_\infty+j)}(\xi) + \cdots + G_{t+1}\Upsilon_{n-1}^{(i_\infty+j)}(\xi)\right)\right]_{\xi=0} = 0 \\ \qquad s=1,2,\ldots j+mt-1;\ t=0,1,\ldots(n-1). \end{cases}$$

These equations again may readily be shewn to be equivalent to the system of equations

$$(28)\quad \sum_k \frac{1}{i_k-1\,!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1}\Upsilon_t^{(i_k)}(z)\right]_{z=a_k} + \frac{1}{i_\infty+j-1\,!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+mt-s-1}\Upsilon_t^{(i_\infty+j)}(\xi)\right]_{\xi=0} = 0$$

$$s=1,2,\ldots j+mt-1;\ t=0,1,\ldots(n-1).$$

To shew this we shall make use of the method of induction. We shall first suppose the equations of the former system corresponding to the pairs of values

$$s=1,2,\ldots j+mt-1;\ t=t_1+1,\ldots(n-1)$$

to be equivalent to the equations of the latter system corresponding to these pairs of values, and shall then prove that the equations of the former system corresponding to the pairs of values

$$s=1,2,\ldots j+mt-1;\ t=t_1,\ t_1+1,\ldots(n-1)$$

may be replaced by the equations of the latter system corresponding to these same pairs of values.

For brevity we shall employ the notation $B_{s,t}$, $C_{s,t}$ to designate the expressions on the left-hand side of the equations (27) and (28) respectively. Let us then assume the equivalence of the sets of equations

$$\text{(29)} \qquad \left.\begin{array}{l} B_{s,t}=0 \\ C_{s,t}=0 \end{array}\right\} s=1,2,\ldots j+mt-1;\ t=t_1+1,\ldots(n-1)$$

and on the assumption that these equations hold good consider the equations

$$\text{(30)} \qquad B_{s,t}=0 \quad s=1,2,\ldots j+mt-1;\ t=t_1.$$

Bearing in mind that we have $F_n=G_n=1$, these latter equations may evidently be written in the form

$$\text{(31)} \quad C_{s,t_1}+\sum_k \frac{1}{i_k-1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1}\left(F_{n-1}\mathrm{T}^{(i_k)}_{t_1+1}(z)+\cdots+F_{t_1+1}\mathrm{T}^{(i_k)}_{n-1}(z)\right)\right]_{z=a_k}$$
$$+\frac{1}{i_\infty+j-1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1}\xi^{j+mt_1-s-1}\left(G_{n-1}\mathrm{T}^{(i_\infty+j)}_{t_1+1}(\xi)+\cdots+G_{t_1+1}\mathrm{T}^{(i_\infty+j)}_{n-1}(\xi)\right)\right]_{\xi=0}=0$$

$$s=1,2,\ldots j+mt_1-1.$$

Recurring to the notation employed for the coefficients of $F(z,v)$ in (I, 4) and remembering that we have $G_{n-\gamma}=\xi^{m\gamma}F_{n-\gamma}$, we shall write

$$F_{n-\gamma}=\sum_{\beta,\gamma} a_{\beta,n-\gamma}z^{\beta},\quad G_{n-\gamma}=\xi^{m\gamma}\sum_{\beta,\gamma} a_{\beta,n-\gamma}\xi^{-\beta}$$

where it is to be noted that the exponent $\beta$ will in no case exceed $m$. The equations (31) will then assume a form which is linear in the constants $a_{\beta,n-\gamma}$, the aggregate multiplier of a constant $a_{\beta,n-\gamma}$, in any one of these equations where it presents itself, being evidently of the type

$$C_{s+\beta,t_1+\gamma}=\frac{1}{i_k-1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s+\beta-1}\mathrm{T}^{(i_k)}_{t_1+\gamma}(z)\right]_{z=a_k}$$
$$+\frac{1}{i_\infty+j-1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1}\xi^{j+m(t_1+\gamma)-s-\beta-1}\mathrm{T}^{(i_\infty+j)}_{t_1+\gamma}(\xi)\right]_{\xi=0}$$

where we have to note that $t_1+\gamma$ has one of the values $t_1+1,\ldots(n-1)$ and that $s+\beta$ must then certainly be included among the numbers $1,2,\ldots j+m(t_1+\gamma)-1$, since $s$ is included among the numbers $1,2,\ldots j+mt_1-1$ and $\beta$ does not exceed $m$. The expressions $C_{s+\beta,t_1+\gamma}$ here in question then, are included among those which are equated to zero in the equations (29).

By aid of the equations $C_{s,t}=0$ in (29) the equations (31), and therefore also the equations (30), reduce to the set of equations

$$(32) \qquad C_{s,t}=0 \quad s=1,\,2,\,\ldots\, j+mt-1;\; t=t_1.$$

If then the two sets of equations in (29) are equivalent to each other, it follows that the equations $B_{s,t}=0$ in (29) combined with the equations (30) are equivalent to the equations $C_{s,t}=0$ in (29) combined with the equations (32). From the equivalence of the two sets of equations in (29) therefore, we conclude also the equivalence of the two sets of equations

$$\left\{\begin{matrix} B_{s,t}=0 \\ C_{s,t}=0 \end{matrix}\right\} s=1,\,2,\,\ldots\, j+mt-1;\; t=t_1,\, t_1+1,\,\ldots\,(n-1).$$

On comparing the equations corresponding to the value $t=n-1$ however in the two sets of equations (27) and (28), we see that they are the same. The two sets of equations in (29) are therefore equivalent in the case where we have $t_1+1=n-1$, and by successive induction we can then also arrive at the equivalence of the two sets of equations

$$\left\{\begin{matrix} B_{s,t}=0 \\ C_{s,t}=0 \end{matrix}\right\} s=1,\,2,\,\ldots\, j+mt-1;\; t=0,\,1,\,\ldots\,(n-1)$$

that is the complete system of equations (27) must be equivalent to the complete system of equations (28). These equations are evidently linear and homogeneous in the constants $\delta$ and give the aggregate of conditions which must be satisfied by these constants in order that the function $H(z,v)$ may be simultaneously representable in the forms (4) and (8), where the functions $\varphi^{(ik)}(z,v)$ and $\varphi^{(i\infty+j)}(\xi,\eta)$ have the forms given in formula (X, 9) and in formula (11) of the present chapter.

On representing by $\zeta_1^{(ik)}(z,v),\, \zeta_2^{(ik)}(z,v),\,\ldots\, \zeta_{l_k}^{(ik)}(z,v)$ and $\zeta_1^{(i\infty+j)}(\xi,\eta),\, \zeta_2^{(i\infty+j)}(\xi,\eta)\,\ldots$ $\ldots\, \zeta_{l_\infty}^{(i\infty+j)}(\xi,\eta)$ the sets of $\zeta$-functions corresponding to the sets of functions $\varphi_1^{(ik)}(z,v),\, \varphi_2^{(ik)}(z,v),\,\ldots\, \varphi_{l_k}^{(ik)}(z,v)$ and $\varphi_1^{(i\infty+j)}(\xi,\eta),\, \varphi_2^{(i\infty+j)}(\xi,\eta),\,\ldots\, \varphi_{l_\infty}^{(i\infty+j)}(\xi,\eta)$ respectively, and on noting the expressions for the functions $\varphi^{(ik)}(z,v)$ and $\varphi^{(i\infty+j)}(\xi,\eta)$ given in the two formulae just referred to, we see that for the corresponding $\zeta$-functions we must have

$$(33)\quad \begin{cases} \zeta^{(i_k)}(z, v) = \delta_1^{(k)} \zeta_1^{(i_k)}(z, v) + \delta_2^{(k)} \zeta_2^{(i_k)}(z, v) + \cdots + \delta_{l_k}^{(k)} \zeta_{l_k}^{(i_k)}(z, v) \\ \zeta^{(i_\infty + j)}(\xi, \eta) = \delta_1^{(\infty)} \zeta_1^{(i_\infty + j)}(\xi, \eta) + \delta_2^{(\infty)} \zeta_2^{(i_\infty + j)}(\xi, \eta) + \cdots + \delta_{l_\infty}^{(\infty)} \zeta_{l_\infty}^{(i_\infty + j)}(\xi, \eta). \end{cases}$$

Also on writing

$$(34)\quad \begin{cases} \zeta_\lambda^{(i_k)}(z, v) = \top_{\lambda, n-1}^{(i_k)} v^{n-1} + \top_{\lambda, n-2}^{(i_k)} v^{n-2} + \cdots + \top_{\lambda, 0}^{(i_k)} \\ \zeta_\lambda^{(i_\infty + j)}(\xi, \eta) = \top_{\lambda, n-1}^{(i_\infty + j)} \eta^{n-1} + \top_{\lambda, n-2}^{(i_\infty + j)} \eta^{n-2} + \cdots + \top_{\lambda, 0}^{(i_\infty + j)} \end{cases}$$

we evidently have

$$(35)\quad \begin{cases} \top_t^{(i_k)}(z) = \delta_1^{(k)} \top_{1,t}^{(i_k)}(z) + \delta_2^{(k)} \top_{2,t}^{(i_k)}(z) + \cdots + \delta_{l_k}^{(k)} \top_{l_k, t}^{(i\ )}(z) \\ \top_t^{(i_\infty + j)}(\xi) = \delta_1^{(\infty)} \top_{1,t}^{(i_\infty + j)}(\xi) + \delta_2^{(\infty)} \top_{2,t}^{(i_\infty + j)}(\xi) + \cdots + \delta_{l_\infty}^{(\infty)} \top_{l_\infty, t}^{(i_\infty + j)}(\xi). \end{cases}$$

# CHAPTER XII.

## The equations of condition. The complementary theorem.

The question of the linear dependence of the equations of condition on one another? The function $\psi(z,v)$. The number of the dependent equations is equal to the number of the arbitrary coefficients in $\psi(z,v)$. Expression for the number of the arbitrary constants involved in the general solution of the equations of condition. Bases of coincidences. The complementary theorem.

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We shall now endeavor to interpret the equations (XI, 28), and to that end we shall first ask whether they are or are not all independent of one another. Our notation in what follows will shape itself more neatly if we replace $t$ by $n-t$ in these equations. We shall therefore represent the system of equations (XI, 28) in the form

$$(1)\quad \sum_k \frac{1}{i_k-1!}\left[\left(\frac{d}{dz}\right)^{i_k-1} z^{s-1} \top_{n-t}^{(i_k)}(z)\right]_{z=a_k} + \frac{1}{i_\infty+j-1!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \xi^{j+m(n-t)-s-1} \top_{n-t}^{(i_\infty+j)}(\xi)\right]_{\xi=0} = 0$$

$$s=1,\,2,\,\ldots\, j+m(n-t)-1;\; t=1,\,2,\,\ldots\, n.$$

Now these equations in the undetermined constants $\delta$ are not or are linearly independent of one another, according as it is or is not possible to find a system of multipliers — not all of which are 0 — such that the sum of the products of the expressions on the left-hand side of the equations, each by the corresponding multiplier, is equal to 0 independently of the values of the $\delta$'s — and therefore such that the coefficients of the individual $\delta$'s in the sum in question are all equal to 0. On employing the notation



$$(2) \qquad \alpha_{s-1,\,t-1} \qquad s=1,\,2,\,\ldots\, j+m(n-t)-1\,;\; t=1,\,2,\,\ldots\, n$$

to designate a system of multipliers for the expressions in the several equations, a statement equivalent to that of the preceding sentence evidently is — that the equations (1) are not or are linearly independent of one another, according as it is or is not possible to simultaneously satisfy all the equations

$$(3) \qquad \sum_{s,t} \frac{1}{i_k-1\,!}\left[\left(\frac{d}{dz}\right)^{i_k-1} \alpha_{s-1,\,t-1}\, z^{s-1}\, \mathrm{T}^{(i_k)}_{n-t}(z)\right]_{z=a_k}=0 \qquad (k=1,\,2,\,\ldots)$$

$$(4) \qquad \sum_{s,t} \frac{1}{i_\infty+j-1\,!}\left[\left(\frac{d}{d\xi}\right)^{i_\infty+j-1} \alpha_{s-1,\,t-1}\, \xi^{j+m(n-t)-s-1}\, \mathrm{T}^{(i_\infty+j)}_{n-t}(\xi)\right]_{\xi=0}=0$$

for arbitrary values of the constants $\delta$, and by values of the quantities $\alpha_{s-1,\,t-1}$ not all of which are equal to 0.

Constructing the integral rational function

$$(5) \qquad \psi(z,v)=\sum_{s,t} \alpha_{s-1,\,t-1}\, z^{s-1}\, v^{t-1},$$

multiply it by the function

$$\zeta^{(i_k)}(z,v)=\sum_{t=1}^{n} \mathrm{T}^{(i_k)}_{n-t}(z)\, v^{n-t}$$

and arrange the product according to powers of $z-a_k$ and $v$. The coefficient of $(z-a_k)^{i_k-1}v^{n-1}$ in the product is evidently an expression such as appears on the left-hand side of (3).

The equation

$$\sum_{s,t} \frac{1}{i_k-1\,!}\left[\left(\frac{d}{dz}\right)^{i_k-1} \alpha_{s-1,\,t-1}\, z^{s-1}\, \mathrm{T}^{(i_k)}_{n-t}(z)\right]_{z=a_k}=0$$

then simply states that the coefficient of $(z-a_k)^{i_k-1}v^{n-1}$ in the product

$$\psi(z,v)\,.\,\zeta^{(i_k)}(z,v)$$

arranged according to powers of $z-a_k$ and $v$, is equal to 0. Since the equation is to hold good for arbitrary values of the constants $\delta_1^{(k)}, \delta_2^{(k)}, \dots \delta_{l_k}^{(k)}$ involved in the expression of the function $\zeta^{(i_k)}(z,v)$, it follows that the vanishing of the coefficient in question is equivalent to the vanishing of the coefficients of $(z-a_k)^{i_k-1}v^{n-1}$ in the $l_k$ products

$$\psi(z,v)\,\zeta_\lambda^{(i_k)}(z,v) \qquad (\lambda=1,2,\dots l_k).$$

In the vanishing of these $l_k$ coefficients however, we have the necessary and sufficient conditions in order that the function $\psi(z,v)$ may be complementary adjoint to the order $i_k$ to the system of functions $\varphi(z,v)$, whose orders of coincidence with the branches of the several cycles corresponding to the value $z=a_k$ are not less than the adjoint numbers $i_k-\sigma_1^{(k)}, i_k-\sigma_2^{(k)}, \dots i_k-\sigma_{r_k}^{(k)}$ respectively. This follows namely from the theory which has been developed in Chapter IX. The satisfaction of the above equation then for arbitrary values of the constants $\delta_1^{(k)}, \delta_2^{(k)}, \dots \delta_{l_k}^{(k)}$ expresses the necessary and sufficient conditions that the function $\psi(z,v)$ may have orders of coincidence with the branches of the several cycles corresponding to the value $z=a_k$ which are complementary adjoint to the numbers $i_k-\sigma_1^{(k)}, i_k-\sigma_2^{(k)}, \dots i_k-\sigma_{r_k}^{(k)}$, and the satisfaction of the system of equations (3) therefore expresses the necessary and sufficient conditions that the function $\psi(z,v)$ may have, for the various values $z=a_k$ and for the several branches of the cycles corresponding to these values, orders of coincidence which are not less than the corresponding members of the sets of numbers

$$(6)\qquad \sigma_1^{(k)}+\mu_1^{(k)}-1+\frac{1}{\nu_1^{(k)}},\ \sigma_2^{(k)}+\mu_2^{(k)}-1+\frac{1}{\nu_2^{(k)}},\ \dots\ \sigma_{r_k}^{(k)}+\mu_{r_k}^{(k)}-1+\frac{1}{\nu_{r_k}^{(k)}},\ (k=1,2,3\dots).$$

Constructing the integral rational function of $(\xi,\eta)$

$$(7)\qquad \bar{\psi}(\xi,\eta)=\sum_{s,t}\alpha_{s-1,t-1}\,\xi^{j+m(n-t)-s-1}\,\eta^{t-1}=\xi^{j+m(n-1)-2}\,\psi(z,v)$$

multiply it by the function

$$\zeta^{(i_\infty+j)}(\xi,\eta)=\sum_{t=1}^{n}\mathrm{T}_{n-t}^{(i_\infty+j)}(\xi)\,\eta^{n-t}.$$

In the product arranged according to powers of $\xi$ and $\eta$, the term in $\xi^{i_\infty+j-1}\eta^{n-1}$ evidently has as coefficient the expression which appears on the left-hand side of the equation (4). This equation is then equivalent to the statement that the coefficient of $\xi^{i_\infty+j-1}\eta^{n-1}$ in the product

$$\bar{\psi}(\xi,\eta) \,.\, \zeta^{(i_\infty+j)}(\xi,\eta)$$

is equal to 0. The equation however is supposed to hold good for arbitrary values of the constants $\delta_1^{(\infty)}, \delta_2^{(\infty)}, \ldots \delta_{l_\infty}^{(\infty)}$ involved in the expression of the function $\zeta^{(i_\infty+j)}(\xi,\eta)$. It follows therefore that the vanishing of the coefficient in question is equivalent to the vanishing of the coefficients of $\xi^{i_\infty+j-1}\eta^{n-1}$ in the $l_\infty$ products

$$\bar{\psi}(\xi,\eta)\,\zeta_\lambda^{(i_\infty+j)}(\xi,\eta) \qquad (\lambda = 1, 2, \ldots l_\infty).$$

By the theory developed in Chapter IX however, the vanishing of these $l_\infty$ coefficients furnishes just those conditions which are necessary and sufficient in order that the function $\bar{\psi}(\xi,\eta)$ may — with reference to the equation $G(\xi,\eta)=0$ — be complementary adjoint to the order $i_\infty + j$ for the value $\xi = 0$, to the system of functions $\varphi(\xi,\eta)$ whose orders of coincidence with the branches of the several cycles are not less than the numbers $i_\infty + \tau_1^{(\infty)}, i_\infty + \tau_2^{(\infty)}, \ldots i_\infty + \tau_{r_\infty}^{(\infty)}$ respectively, for the functions $\zeta_1^{(i_\infty+j)}(\xi,\eta), \zeta_2^{(i_\infty+j)}(\xi,\eta), \ldots \zeta_{l_\infty}^{(i_\infty+j)}(\xi,\eta)$ are the $\zeta$-functions corresponding to the functions $\varphi_1^{(i_\infty+j)}(\xi,\eta), \varphi_2^{(i_\infty+j)}(\xi,\eta), \ldots \varphi_{l_\infty}^{(i_\infty+j)}(\xi,\eta)$, obtained on dropping terms divisible by $\xi^{i_\infty+j}$ in the system of functions $\varphi(\xi,\eta)$ in question. The satisfaction of the equation (4) then for arbitrary values of the constants $\delta_1^{(\infty)}, \delta_2^{(\infty)}, \ldots \delta_{l_\infty}^{(\infty)}$, expresses the necessary and sufficient conditions that the function $\bar{\psi}(\xi,\eta)$ may have orders of coincidence with the branches of the several cycles corresponding to the value $\xi = 0$, which are complementary adjoint to the order $i_\infty + j$ to the numbers $i_\infty + \tau_1^{(\infty)}, i_\infty + \tau_2^{(\infty)}, \ldots i_\infty + \tau_{r_\infty}^{(\infty)}$. In other words, the equation (4) represents the necessary and sufficient conditions which must be satisfied by the coefficients of the function $\bar{\psi}(\xi,\eta)$, in order that its orders of coincidence with the branches of the several cycles corresponding to the value $\xi = 0$, may not be less than the numbers

$$(8)\quad j-\tau_1^{(\infty)}+m(n-1)+\mu_1^{(\infty)}-1+\frac{1}{\nu_1^{(\infty)}},\ \ldots\ j-\tau_{r_\infty}^{(\infty)}+m(n-1)+\mu_{r_\infty}^{(\infty)}-1+\frac{1}{\nu_{r_\infty}^{(\infty)}}$$



respectively. This however is evidently equivalent to saying, that the equation (4) represents the necessary and sufficient conditions which must be satisfied by the coefficients of the function $\psi(z,v)$, in order that its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$, may not be less than the numbers

$$(9)\qquad -\tau_1^{(\infty)}+2+\mu_1^{(\infty)}-1+\frac{1}{\nu_1^{(\infty)}},\ \ldots\ -\tau_{r_\infty}^{(\infty)}+2+\mu_{r_\infty}^{(\infty)}-1+\frac{1}{\nu_{r_\infty}^{(\infty)}}$$

respectively.

The aggregate conditions imposed on the coefficients of the function $\psi(z,v)$ by the simultaneous satisfaction of the equations (3) and (4) — supposed to hold good for arbitrary values of the $\delta$'s — are then stated, when we say that the function must have as its orders of coincidence with the branches of the several cycles corresponding to a finite value $z=a_k$, numbers which are not exceeded by the respective members of the corresponding set of numbers (6), and as its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$, numbers which are not exceeded by the respective members of the set of numbers (9). The finite values $z=a_k$ here referred to, are those for which the function $H(z,v)$ may become infinite and those belonging to the category (C). We may however, if we will, suppose our statement to be made with reference to all finite values of the variable, since for a value of the variable other than one of those just mentioned, the members of the corresponding set of numbers (6) would evidently all be equal to 0.

If then the quantities $\alpha_{s-1,t-1}$ in (2) are to serve as multipliers for the equations of the system (1) — supposed to be linearly connected — the necessary and sufficient conditions thereto are stated, when we say that the quantities $\alpha_{s-1,t-1}$ in question constitute the coefficients of an integral rational function $\psi(z,v)$, whose orders of coincidence with the branches of the several cycles corresponding to a finite value $z=a_k$ are not less than the respective members of the corresponding set of numbers (6), while its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ are not exceeded by the respective members of the set of numbers (9).



Since the equations (1) are not or are linearly independent of one another according as it is or is not possible to find a system of multipliers $\alpha_{s-1,t-1}$, not all of which are 0, they will not or will be linearly independent of one another according as a function $\psi(z,v)$ of the character just described does or does not exist. Let us now suppose a certain number of the equations (1) to be linearly independent of one another, while each one of the remaining equations is connected with these by a linear relation. To each such linear relation will correspond a distinct function $\psi(z,v)$, and the number of the dependent equations in the system (1) will evidently be just equal to the number of the linearly independent functions $\psi(z,v)$ of the description here in question. In other words, the number of the dependent equations in the system (1) will be just equal to the number of arbitrary constants involved in the expression of the most general function $\psi(z,v)$ of the character described in the foregoing. To find how many of the equations (1) are linearly independent of one another then, we should first determine the number of the arbitrary constants involved in the expression of the most general function $\psi(z,v)$ of the character here in question, and this number we should subtract from the total number of the equations.

We might here take notice of the limitation on the degree of the function $\psi(z,v)$ implied in the fact that its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ do not fall short of the respective numbers in (9), and must therefore, as we readily see, be greater than the numbers

$$(10) \qquad \mu_1^{(\infty)}-\lambda,\ \mu_2^{(\infty)}-\lambda,\ \ldots\ \mu_{r_\infty}^{(\infty)}-\lambda$$

respectively, where we employ the letter $\lambda$ to designate the greatest integer which is less than the greatest of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$. — If, for example, $\tau_s^{(\infty)}$ be the greatest of these numbers we shall have $\lambda=\left[\tau_s^{(\infty)}-\frac{1}{\nu_s^{(\infty)}}\right]$. — On referring to the results obtained in Chapter VI namely, we see that the degree of our function must be $\leqq N+\lambda-2$ and that the degree of the element involving $v^{n-1}$ must be $\leqq n+\lambda-2$. The function can therefore be represented in the form



$$(11) \qquad \psi(z, v) = \sum_{s,t} a_{s-1,t-1} z^{s-1} v^{t-1}, \quad s+t \leqq N+\lambda,\ s>0,\ t=1, 2, \dots n$$

where at the same time we also have $s+t \leqq n+\lambda$ for $t=n$. Every term in the form (11) may readily be shewn to be included in our earlier form (5) where the summation was extended to all terms for which

$$s = 1, 2, \dots j + m(n-t) - 1;\ t = 1, 2, \dots n.$$

Remarking the inequality $N \leqq m+n-1$ and noting that we have $j > \lambda$ — since by hypothesis $j$ is not less than the greatest of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \dots \tau_{r_\infty}^{(\infty)}$ — we will compare the greatest value which $s$ may take in the form (5), for a given value of $t$, with that which it may take in the form (11), on subtracting the latter greatest value from the former. For $t \leqq n-1$ the subtraction gives us $j + m(n-t) - 1 - (N+\lambda-t) \geqq j + m(n-t) - 1 - (m+n-1) - \lambda + t \geqq m(n-t) - (m+n-1) + t = (m-1)(n-t-1) \geqq 0$ and for $t=n$ we have $s \leqq \lambda \leqq j-1$. All the terms in the form (11) are therefore included in the earlier form.

The number of the equations (1) which are linearly independent of one another is then obtained, on subtracting from the total number of these equations the number of the arbitrary constants involved in the expression of the most general function $\psi(z, v)$ of the form (11), whose orders of coincidence with the branches of the several cycles corresponding to the finite values $z = a_k$ are not less than the respective members of the sets of numbers (6), while its orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ are equal to or greater than the numbers (9) respectively. Our reference in this statement to the form of the function $\psi(z, v)$ is superfluous, inasmuch as a rational function of $(z, v)$ possessing orders of coincidence such as those here in question, must necessarily be of the form (11). Because of its orders of coincidence for finite values of the variable $z$ namely, the function is adjoint for all such values of the variable and must therefore be integral, and because of its orders of coincidence for the value $z=\infty$ the integral function must, as we have seen, be of the restricted form (11).

Since in any linear relation which may happen to exist between the equations (1), the multipliers of the several equations must be the coefficients $\alpha_{s-1,t-1}$ in a function $\psi(z,v)$ of the form (11), it follows that those equations, at least, corresponding to pairs of values $(s,t)$ which do not appear in the summation in (11), are linearly independent of one another and of the remaining equations of the system, for their multipliers must have the value 0. Unless the number of the equations (1) which are linearly independent of one another is less than the number of the constants $\delta$ involved, these constants must all have the value 0 and the rational function $H(z,v)$ to which the system of equations is supposed to correspond must itself be 0 identically. If however the number of the equations (1) which are linearly independent of one another is less than the number of the $\delta$'s, the number of arbitrary constants involved in the solutions of the equations will be obtained on subtracting from the total number of the $\delta$'s the number of the linearly independent equations. The number of arbitrary constants so obtained will also be the number of the independent arbitrary constants involved in the expression of the most general rational function $H(z,v)$, which does not become infinite to orders exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ respectively for the branches of the several cycles corresponding to the various finite values $z=a_k$, and whose orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ are equal to or greater than the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$ respectively.

To assure ourselves of the truth of the last statement, we shall suppose the number of the arbitrary constants involved in the solution of the system of equations (1) to be $r$, employing at the same time the notation $\delta_1, \delta_2, \ldots \delta_r$ to indicate a complete system of such arbitrary constants. The constants $\delta^{(\infty)}$ involved in the form (XI, 8), as also the coefficients of $P(z,v)$ and the constants $\delta^{(k)}$ involved in the expression of the functions $\varphi^{(i_k)}(z,v)$ in the form (X, 1), will then all be linearly expressible in terms of the constants $\delta_1, \delta_2, \ldots \delta_r$.

The function $H(z,v)$ may then be represented in the form



$$\delta_1 U_1 + \delta_2 U_2 + \cdots + \delta_r U_r \tag{12}$$

where $U_1, U_2, \ldots U_r$ are rational functions of $(z,v)$, which we have to prove are linearly independent of one another. — If these functions are not linearly independent we can find a set of multipliers $d_1, d_2, \ldots d_r$, not all of which are 0, and such that the function

$$d_1 U_1 + d_2 U_2 + \cdots + d_r U_r$$

is identically equal to 0. On giving to the constants $\delta_1, \delta_2, \ldots \delta_r$ then the values $d_1, d_2, \ldots d_r$ respectively, and on solving the equations (1) for the remaining constants $\delta$, we should have a system of values for the constants $\delta^{(k)}$ and $\delta^{(\infty)}$, not all of which are 0, and such that the corresponding function $H(z,v)$ is identically equal to 0. This however is impossible, for if the function $H(z,v)$ is to vanish identically, the element $\xi^{-i_\infty}\varphi^{(i_\infty+j)}(\xi,\eta)$ in the form (XI, 8) and therewith also the constants $\delta^{(\infty)}$ involved in the expression of this element must vanish identically, and at the same time the individual elements of the summation in the form (X, 1), and therewith the constants $\delta^{(k)}$ involved in the numerators of these elements, must evidently also vanish identically. We conclude therefore, that the $r$ arbitrary constants $\delta_1, \delta_2, \ldots \delta_r$ which appear in the solution of the system of equations (1), must likewise present themselves as independent arbitrary constants in the representation of the general function $H(z,v)$ in the form (X, 1).

On reducing the several elements in the form (X, 1) to a common denominator, the function $H(z,v)$ may evidently also be represented in the form

$$H(z,v) = \frac{G(z,v)}{g(z)} \tag{13}$$

where the numerator is an integral rational function of $(z,v)$, and where the denominator is an integral rational function of $z$ alone which may be represented as a product in the form

$$g(z) = \prod_k (z - a_k)^{i_k}. \tag{14}$$

Here factors of the product correspond to all those values $z=a_k$ for which the function $H(z,v)$ may become infinite, as also to all those values of the variable which belong to the category (C). The numerator $G(z,v)$ is evidently linear and homogeneous in the arbitrary constants $\delta_1, \delta_2, \ldots \delta_r$. Its orders of coincidence with the branches of the several cycles corresponding to the various values $z=a_k$ are equal to or greater than the numbers $i_k - \sigma_1^{(k)}, \ldots i_k - \sigma_{r_k}^{(k)}$ respectively, and its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ are not less than the numbers $-i+\tau_1^{(\infty)}, \ldots -i+\tau_{r_\infty}^{(\infty)}$ respectively, where the letter $i$ is employed to indicate the degree of the denominator $g(z)$. Since an exponent $i_k$ is equal to the greatest integer in the corresponding set of integers $[\mu_1^{(k)}+\sigma_1^{(k)}], \ldots [\mu_{r_k}^{(k)}+\sigma_{r_k}^{(k)}]$, we see that $G(z, v)$ and $g(z)$ are both adjoint for all finite values of the variable $z$. On employing the symbol $\bar{\lambda}$ to designate the greatest of the integers

$$[\mu_1^{(\infty)} - \tau_1^{(\infty)}] + i + 1, \ldots [\mu_{r_\infty}^{(\infty)} - \tau_{r_\infty}^{(\infty)}] + i + 1$$

the orders of coincidence of $G(z,v)$ with the branches of the several cycles corresponding to the value $z=\infty$, will evidently be greater than the numbers

$$\mu_1^{(\infty)} - \bar{\lambda}_1, \ldots \mu_{r_\infty}^{(\infty)} - \bar{\lambda}$$

respectively and its degree, in accord with Chapter VI, must therefore be $\overline{\gtrless} N + \bar{\lambda} - 2$.

On adding the numbers given in formulae (X, 11) and (XI, 10), we obtain for the total number of the constants $\delta$ involved in the equations (1) of the present chapter, the expression

$$(15)\quad \sum_k l_k = \frac{1}{2} m n(n-1) + jn + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + + \frac{1}{\nu_s^{(k)}}\right) + {\sum_k}' \sum_{s=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)}$$

where the accented summation with regard to $k$ is supposed to extend only to finite values $z=a_k$, whereas the summation without the accent extends also to the value $z=\infty$. The number of the equations which are linearly independent of one another is obtained, on subtracting from their total number $(j-1)n + \frac{1}{2} m n(n-1)$ the number of the arbitrary constants involved in the expression of the most general function $\psi(z,v)$, whose orders of coincidence with the branches of the several cycles corresponding to any finite value $z=a_k$ are not less than the respective members of the corresponding set of numbers (6), while its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ are not exceeded by the respective members of the set of numbers (9). Employing the symbol $N_\psi$ to designate the number of the arbitrary constants here in question, we have for the number of the linearly independent equations the expression

$$(16) \qquad (j-1)\,n + \frac{1}{2} m\,n\,(n-1) - N_\psi\,.$$

Now in a system of equations, linear and homogeneous in a number of undetermined constants, the number of the equations which are linearly independent of one another cannot exceed the total number of the constants, and the number of arbitrary constants involved in the solution of the system will be obtained on subtracting the number of the linearly independent equations from the total number of the constants. The number of arbitrary constants involved in the solution of the system of equations (1), will then be obtained on subtracting the number (16) from the number (15). This gives us the expression

$$(17) \qquad N_\psi + n + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)} + \sum_k{}' \sum_{s=1}^{r_k} \sigma_s^{(k)} \nu_s^{(k)} - \sum_{s=1}^{r_\infty} \tau_s^{(\infty)} \nu_s^{(\infty)}$$

for the number of the arbitrary constants involved in the solution of the system of equations (1), and the same expression therefore also represents the number of independent arbitrary constants involved in the coefficients of the most general rational function $H(z,v)$, which becomes infinite to orders not exceeding $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ respectively for the branches of the several cycles corresponding to the various finite values $z=a_k$, while its orders of coincidence with the branches of the several cycles corresponding to the value $z=\infty$ do not fall short of the numbers $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$ respectively.

We may, if we will, define $N_\psi$ in the foregoing as the number of the arbitrary constants involved in the expression of the most general rational function of $(z, v)$, whose orders of coincidence with the branches of the several cycles corresponding to the various finite values $z = a_k$ exceed by $\sigma_1^{(k)}, \sigma_2^{(k)}, \ldots \sigma_{r_k}^{(k)}$ respectively the orders of coincidence requisite to adjointness for the values of the variable in question, while its orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ exceed by $-\tau_1^{(\infty)} + 2, -\tau_2^{(\infty)} + 2, \ldots -\tau_{r_\infty}^{(\infty)} + 2$ respectively the orders of coincidence requisite to adjointness for this value of the variable — for the orders of coincidence here in question already imply that the function must have the form given in (11).



Instead of speaking of the zeros and infinities of a rational function of $(z, v)$ we might speak of its positive and negative orders of coincidence. On replacing the symbols $-\sigma_1^{(k)}, -\sigma_2^{(k)}, \ldots -\sigma_{r_k}^{(k)}$ in the preceding by the symbols $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$ respectively, the expression (17) takes the form

(18) $$N_\psi + n + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}$$

where the summations with regard to $k$ may be supposed to extend to all values of the variable $z$, the value $z = \infty$ included, and where the numbers $\tau_s^{(k)}$ corresponding to a finite value $z = a_k$ are zero or negative, whereas the numbers $\tau_s^{(\infty)}$ may be positive, zero or negative. The expression (18) then represents the number of arbitrary constants involved in the expression of the most general rational function $H(z, v)$, whose orders of coincidence with the branches of the several cycles corresponding to the values $z = a_k$ are equal to or greater than the corresponding members of the sets of numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$.

The symbol $N_\psi$ in terms of the new notation represents the number of arbitrary constants involved in the expression of the most general rational function, whose orders of coincidence with the branches of the several cycles corresponding to the various finite values $z = a_k$ exceed by $-\tau_1^{(k)}, -\tau_2^{(k)}, \ldots -\tau_{r_k}^{(k)}$ respectively the orders of coincidence requisite to adjointness for the values of the variable in question, while its orders of coincidence with the branches of the several cycles corresponding to the value $z = \infty$ exceed by $-\tau_1^{(\infty)} + 2, \ldots -\tau_{r_\infty}^{(\infty)} + 2$ respectively the orders of coincidence requisite to adjointness for this value of the variable. Otherwise said, the symbol $N_\psi$ represents the number of arbitrary constants involved in the expression of the most general rational function $\psi(z, v)$, whose orders of coincidence for every value $z = a_k$ are complementary adjoint to the corresponding orders of coincidence $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$ which we have attributed to the function $H(z, v)$, and whose orders of coincidence for the value $z = \infty$ are complementary adjoint to the order 2 to the orders of coincidence $\tau_1^{(\infty)}, \tau_2^{(\infty)}, \ldots \tau_{r_\infty}^{(\infty)}$.

For brevity we will name a system of numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$ associated with the different values of the variable $z = a_k$ a *Basis of Coincidences* for the building of a rational function. We may conceive a set of numbers to be associated with each value of the variable, the numbers however being all 0 save in the sets associated with a finite number of values of $z$. When we specify the sets associated with a finite number of values of the variable only, it is then to be understood that the sets associated with the remaining values of the variable are all made up of zeros. By the most general rational function built on a given basis of coincidences, we shall mean the most general rational function whose orders of coincidence with the branches of the several cycles corresponding to the various values of $z$ do not fall short of the corresponding numbers mentioned in the basis. A proposed basis might of course be an impossible one, or in so far impossible that the function built on it would have to be identically 0.

In the foregoing we have assumed the numbers $\tau_s^{(k)}$ corresponding to finite values of $z$ to be zero or negative, and incident thereto we saw that the function $\psi(z, v)$ had to be an integral rational function. We shall now remove this limitation. — Consider any arbitrary basis of coincidences in which the numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$ corresponding to a value $z = a_k$ may be positive, zero or negative, We shall briefly refer to this basis of coincidences as the basis $(\tau)$. A basis $(\bar{\tau})$ defined by a system of numbers $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \ldots \bar{\tau}_{r_k}^{(k)}$ which are connected with the numbers $\tau_1^{(k)}, \tau_2^{(k)}, \ldots \tau_{r_k}^{(k)}$ by the equations

(19) $$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \ (s = 1, 2, \ldots r_k)$$



for all finite values of the variable $z = a_k$, and for the value $z = \infty$ by the equations

(20) $$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}, \ (s = 1, 2, \ldots r_\infty)$$

we shall call the basis *complementary* to the basis $(\tau)$. The basis $(\tau)$ will then evidently also be the basis complementary to the basis $(\bar{\tau})$. Otherwise said, two bases of coincidences are complementary to each other when the orders of coincidence which define them are complementary adjoint for all finite values of the variable $z$ and over and above this are complementary adjoint to the order 2 for the value $z = \infty$.

Representing by $H(z,v)$ and $\bar{H}(z,v)$ the most general rational functions which can be built on the bases $(\tau)$ and $(\bar{\tau})$ respectively, we shall employ the symbols $N_H$ and $N_{\bar{H}}$ to designate the numbers of arbitrary constants involved in these respective functions. Any pair of functions respectively included in the forms $H(z,v)$ and $\bar{H}(z,v)$ we shall say are complementary to each other.

If for every value of the variable $z$ we subtract the actual orders of coincidence of a given rational function $R(z,v)$ from the corresponding numbers in the basis $(\tau)$, we obtain a new basis on which the most general rational function which can be constructed is evidently the quotient $\frac{H(z,v)}{R(z,v)}$. Also the most general rational function complementary to this quotient we readily see to be the product $R(z,v) . \bar{H}(z,v)$. Now let us choose some rational function of the variable $z$ alone, a definite polynomial $P(z)$ say, such that on subtracting its actual orders of coincidence for the different values of $z$ from the corresponding numbers in the basis $(\tau)$, we obtain a new basis $(t)$ in which all the numbers corresponding to finite values of the variable are zero or negative. Since the aggregate sum of the orders of coincidence subtracted from the numbers of the basis $(\tau)$ in order to obtain the basis $(t)$ is 0, we evidently have

(21) $$\sum_k \sum_{s=1}^{r_k} t_s^{(k)} \nu_s^{(k)} = \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}.$$



The general rational function built on the basis $(t)$ will be the quotient $\frac{H(z,v)}{P(z)}$ and the general rational function built on the complementary basis will be the product $P(z).\bar{H}(z,v)$. This product we shall designate as the function $\psi(z,v)$ — it is adjoint for all finite values of the variable $z$ and must therefore be an integral rational function of $(z,v)$.

The case here in question is that already considered, in which we obtained the expression (18) for the number of the arbitrary constants involved in the representation of the most general rational function built on a basis, all of whose numbers corresponding to finite values of the variable $z$ are zero or negative. The number of arbitrary constants involved in the quotient $\frac{H(z,v)}{P(z)}$ — and therefore the number of arbitrary constants involved in the function $H(z,v)$ — will then be given by the expression

$$N_\psi + n + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} t_s^{(k)} \nu_s^{(k)}$$

where we evidently have $N_\psi = N_{\bar{H}}$, since the number of arbitrary constants involved in the function $\psi(z,v) = P(z).\bar{H}(z,v)$ is the same as the number of arbitrary constants involved in the factor $\bar{H}(z,v)$. On noting the equality (21) we shall have, for the number of arbitrary constants involved in the general function $H(z,v)$ built on the basis $(\tau)$, the expression

(22) $$N_H = N_{\bar{H}} + n + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}.$$

In like manner, on interchanging the complementary bases $(\tau)$ and $(\bar{\tau})$, we obtain, for the number of the arbitrary constants involved in the general function $\bar{H}(z,v)$ built on the basis $(\bar{\tau})$, the expression

(23) $$N_{\bar{H}} = N_H + n + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}.$$

The formulae (22) and (23) are equivalent, for on adding the corresponding sides of the two formulae and taking account of the relations (19) and (20) existing between the numbers of the complementary bases, we evidently obtain an identity. On equating the differences of the corresponding sides of the formulae (22) and (23), we obtain a relation to which each of these formulae is equivalent and which may evidently be written in the form

$$(24) \qquad 2(N_H - N_{\bar{H}}) = \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}.$$

This formula again may be written in the more symmetrical form

$$(25) \qquad N_H + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N_{\bar{H}} + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}.$$

In words this formula states, that the number of arbitrary constants involved in the expression of the most general rational function built on a given basis of coincidences, added to half the sum of the orders of coincidence explicitly required by the basis, is equal to the like number constructed with reference to the complementary basis. We shall refer to the theorem here stated as the *Complementary Theorem* — the formula itself we shall call the *Complementary Formula.*

We might regard the content of the complementary theorem from a somewhat more general standpoint. Employing the notation $m_1^{(k)}, m_2^{(k)}, \ldots m_{r_k}^{(k)}$ to indicate the actual orders of coincidence of any definite but arbitrarily selected rational function $R(z,v)$ with the branches of the several cycles corresponding to a value $z = a_k$, we shall designate the aggregate system of numbers

$$(26) \qquad m_1^{(k)} - 1 + \frac{1}{\nu_1^{(k)}},\; m_2^{(k)} - 1 + \frac{1}{\nu_2^{(k)}},\; \ldots\; m_{r_k}^{(k)} - 1 + \frac{1}{\nu_{r_k}^{(k)}}$$

for all finite values of the variable $z$, together with the numbers

$$(27) \qquad m_1^{(\infty)} + 1 + \frac{1}{\nu_1^{(\infty)}},\; m_2^{(\infty)} + 1 + \frac{1}{\nu_2^{(\infty)}},\; \ldots\; m_{r_\infty}^{(\infty)} + 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}}$$

corresponding to the value $z=\infty$, as the *level* furnished by the function $R(z, v)$. Employing furthermore the respective notations $\tau_1^{(k)}, \tau_2^{(k)}, \dots \tau_{r_k}^{(k)}$ and $\bar{\tau}_1^{(k)}, \bar{\tau}_2^{(k)}, \dots \bar{\tau}_{r_k}^{(k)}$ to indicate those numbers of two bases $(\tau)$ and $(\bar{\tau})$ which correspond to the value $z=a_k$, we shall say of these two bases that they are *complementary* with regard to the level furnished by the function $R(z,v)$, in case their numbers satisfy the system of equalities



$$(28) \qquad \tau_s^{(k)} + \bar{\tau}_s^{(k)} = m_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \quad (s=1, 2, \dots r_k)$$

for all finite values of the variable $z=a_k$, and for the value $z=\infty$ the equality

$$(29) \qquad \tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = m_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}, \quad (s=1, 2, \dots r_\infty).$$

Now on referring to the equalities (19) and (20), and remembering that the orders of coincidence of the function $F'_v(z,v)$ with the branches of the several cycles corresponding to a value $z=a_k$ are given by the numbers $\mu_1^{(k)}, \mu_2^{(k)}, \dots \mu_{r_k}^{(k)}$ respectively, we see that the bases $(\tau)$ and $(\bar{\tau})$ which we have heretofore considered and for which we have proved the complementary theorem, are complementary with regard to the level furnished by the function $F'_v(z,v)$. The complementary theorem however may readily be shewn to hold good also for a pair of bases $(\tau)$ and $(\bar{\tau})$ which are complementary with regard to the level furnished by the rational function $R(z,v)$. If we suppose $(t)$ to represent the basis complementary to the basis $(\tau)$ with regard to the level furnished by the function $F'_v(z,v)$, we evidently have

$$\bar{\tau}_s^{(k)} - t_s^{(k)} = m_s^{(k)} - \mu_s^{(k)}, \quad (s=1, 2, \dots r_k)$$

for all values of the variable $z$, and therefore

$$(30) \qquad \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)} = \sum_k \sum_{s=1}^{r_k} t_s^{(k)} \nu_s^{(k)}$$

since the aggregate sum of the numbers $m_s^{(k)} - \mu_s^{(k)}$ must be 0 — representing as they do the orders of coincidence of the rational function $\frac{R(z,v)}{F'_v(z,v)}$

with the branches of the several cycles corresponding to the different values of $z$. Furthermore, on representing by $h(z,v)$, $H(z,v)$ and $\bar{H}(z,v)$ respectively the most general functions built on the several bases $(t)$, $(\tau)$ and $(\bar{\tau})$, we plainly have

(31) $$\bar{H}(z,v)=\frac{R(z,v)}{F'_v(z,v)}\cdot h(z,v)$$

and consequently also $N_{\bar{H}}=N_h$. By the complementary theorem proved in the case of the level furnished by the function $F'_v(z,v)$ however, we have

$$N_H+\frac{1}{2}\sum_k\sum_{s=1}^{rk}\tau_s^{(k)}\nu_s^{(k)}=N_h+\frac{1}{2}\sum_k\sum_{s=1}^{rk}t_s^{(k)}\nu_s^{(k)}$$

and on replacing in this formula $N_h$ by $N_{\bar{H}}$ and taking account of the equality (29) we arrive at the formula (25). The complementary formula (25) then holds good where $H(z,v)$ and $\bar{H}(z,v)$ are the most general rational functions built respectively on bases $(\tau)$ and $(\bar{\tau})$, which are complementary with regard to a level furnished by any rational function $R(z,v)$.

The form for the general Complementary Formula corresponding to the form given in (22) will evidently be

(32) $$N_H=N_{\bar{H}}+n+\frac{1}{2}\sum_k\sum_{s=1}^{rk}\left(m_s^{(k)}-1+\frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)}-\sum_k\sum_{s=1}^{rk}\tau_s^{(k)}\nu_s^{(k)}.$$

The aggregate sum of the orders of coincidence of a rational function is equal to 0. We therefore have $\sum_k\sum_{s=1}^{rk}m_s^{(k)}=0$ and can consequently also write the general Complementary Formula in the form

(33) $$N_H=N_{\bar{H}}+n-\frac{1}{2}\sum_k\sum_{s=1}^{rk}(\nu_s^{(k)}-1)-\sum_k\sum_{s=1}^{rk}\tau_s^{(k)}\nu_s^{(k)}.$$

Now reverting to the fundamental integral algebraic equation $F(z,v)=0$ with which our argument has concerned itself throughout, and to the algebraic equation $f(z,u)=0$ from which the integral equation was derived by the transformation $v = gu$ at the beginning of the first chapter, we see that the general Complementary Theorem as embodied in the formulae (25), (32) and (33) has as much reference to one equation as to the other. For the two equations have the like cycles for any given value of the variable $z$ and the orders of coincidence of a rational function with the branches of a cycle are the same whether the function be expressed in terms of $(z, v)$ or in terms of $(z, u)$. We shall therefore regard the more general equation $f(z, u) = 0$ as our fundamental equation and shall suppose the general complementary theorem as stated in the above formulae to have reference to this equation. The statement of the theorem contained in formula (25) for example, will then read as follows: — If $(\tau)$ and $(\bar{\tau})$ be two bases of coincidences which are complementary with regard to the level furnished by any rational function $R(z, u)$, then will the number of arbitrary constants involved in the expression of the most general rational function built on the basis $(\tau)$, added to half the sum of the orders of coincidence explicitly required by this basis, be equal to the like number constructed with reference to the complementary basis $(\bar{\tau})$.

If we select the level furnished by the rational function $f'_u(z, u)$, the symbols $m_1^{(k)}, m_2^{(k)}, \ldots m_{r_k}^{(k)}$ in formula (32) will represent the orders of coincidence of this function with the branches of the several cycles corresponding to the value $z = a_k$, or — what evidently amounts to the same thing — these symbols will represent the orders of coincidence of the branches of the several cycles corresponding to the value $z = a_k$, each with the product of the remaining $n - 1$ branches corresponding to this value of the variable. If instead of employing the symbols $\mu_1^{(k)}, \mu_2^{(k)}, \ldots \mu_{r_k}^{(k)}$ to designate the orders of coincidence of the function $F'_v(z, v)$ with the branches of the several cycles corresponding to the value $z = a_k$, we should make use of the symbols to designate the orders of coincidence of the function $f'_u(z, u)$ with the branches of these cycles, the formulae (22) and (23) would then have reference to the equation $f(z, u) = 0$. Also the numbers $\mu_1^{(k)} - 1 + \frac{1}{\nu_1^{(k)}}, \ldots \mu_{r_k}^{(k)} - 1 + \frac{1}{\nu_{r_k}^{(k)}}$ would define the orders of coincidence requisite to adjointness for the branches of the several cycles of the equation $f(z, u) = 0$ corresponding to the value $z = a_k$.

Though our argument in the preceding has implicitly assumed a multiplicity of branches corresponding to a given value of the variable $z$, thereby implying on the part of the fundamental algebraic equation a degree in the dependent variable which is greater than 1, we can readily verify that the complementary formula still holds good in the case where we have $n = 1$. In this case the fundamental equation has the form $u - P(z) = 0$ and rational functions of $(z, u)$ are simply rational functions of $z$. Here we have $f'_u(z, u) = 1$ and the formulae (22) and (23) evidently assume the forms

$$(34) \qquad N_H = N_{\bar H} + 1 - \sum_k \tau_1^{(k)},\ N_{\bar H} = N_H + 1 - \sum_k \bar\tau_1^{(k)}$$

where $(\tau)$ and $(\bar\tau)$ are complementary bases with regard to the level furnished by the constant $f'_u(z, u) = 1$ — in other words, where the numbers $\tau$ and $\bar\tau$ satisfy the equalities

$$\tau_1^{(k)} + \bar\tau_1^{(k)} = 0$$

for finite values $z = a_k$, and for the value $z = \infty$ the equality

$$\tau_1^{(\infty)} + \bar\tau_1^{(\infty)} = 2.$$

Now we know that we can construct a rational function of $z$ possessing any arbitrary combination of zeros and infinities, so long as the number of the former is equal to that of the latter. We can therefore construct a rational function $H(z)$ on the basis $(\tau)$, so long as we have $\sum_k \tau^{(k)} \leqq 0$. At the same time however we shall evidently have $\sum_k \bar\tau^{(k)} \geqq 2$ and consequently $\bar H(z) = 0$. So long then as we have $\sum_k \tau^{(k)} \leqq 0$ the formulae (34) take the form

$$(35) \qquad N_H = 1 - \sum_k \tau_1^{(k)} = -1 + \sum_k \bar\tau_1^{(k)}.$$

Otherwise said — the number of arbitrary constants involved in the expression of the most general rational function of $z$ which possesses a certain system of infinities and whose zeros include among them certain specified zeros, is greater by 1 than the difference between the total number of the infinities and the number of the specified zeros. This however is the statement of a well-known theorem in the elementary theory of the rational functions of a single variable $z$. The formulae (34) are then equivalent to this elementary theorem, for in these formulae we must evidently have either $\sum_k \tau_1^{(k)} \geqq 1$ or $\sum_k \bar{\tau}^{(k)} \geqq 1$ and consequently either $N_H = 0$ or $N_{\bar{H}} = 0$. We shall have both $N_H = 0$ and $N_{\bar{H}} = 0$ when, and only when, $\sum_k \tau_1^{(k)} = 1 = \sum_k \bar{\tau}^{(k)}$.

If, instead of employing the level furnished by the constant $f'_u(z, u) = 1$, we should make use of the level furnished by any arbitrarily chosen rational function $R(z)$ in constructing the basis $(\bar{\tau})$ complementary to the basis $(\tau)$, we readily see that the complementary formula in any of its forms (25), (32) or (33) is still equivalent to the elementary theorem just stated. The general complementary formula is then true in the case $n = 1$ as well as in the cases where we have $n > 1$. The general complementary theorem as embodied in the various forms of the complementary formula then holds good in all cases without exception. In deriving the complementary formula we have put no restrictions on the basis of coincidences. It may be that the most general rational function which can be built on a given basis is identically 0 — in such case we shall call the basis in question an *impossible basis*. According as our fundamental algebraic equation is reducible or irreducible we shall say that a corresponding basis is reducible or irreducible. A reducible basis is evidently made up of a number of irreducible bases corresponding respectively to the several irreducible equations whose aggregate constitutes the reducible equation in question. A reducible basis is evidently impossible if its several constituent irreducible bases are impossible for the corresponding irreducible equations, and conversely the constituent irreducible bases are severally impossible for the corresponding irreducible equations if the reducible basis is impossible for our fundamental algebraic equation.

If the aggregate sum of the orders of coincidence required by a given basis be zero or negative the basis may or may not be an impossible basis. In the case of an irreducible basis however if such sum be positive the basis in question will certainly be an impossible one. For by an elementary theorem in the theory of the algebraic functions we know that the number of the infinities of a rational function of $(z, u)$ is equal to the number of its zeros — or what amounts to the same thing that the sum of its orders of coincidence for all values of the variable $z$ is 0 — where $u$ is defined as an algebraic function of $z$ by an irreducible algebraic equation. If then the aggregate sum of the orders of coincidence mentioned in a given irreducible basis be greater than zero, it follows that no rational function can be built on the basis, for the sum of the actual orders of coincidence of any rational function built on a given basis cannot be less than the aggregate sum of the orders of coincidence mentioned in the basis. With reference to an impossible basis we may express ourselves, as we find it convenient, by saying either that no rational function can be built on the basis, or that the most general rational function which can be built on it is 0.

An irreducible basis $(\tau)$, in which the aggregate sum of the orders of coincidence mentioned is equal to zero, we shall call a *complete* irreducible basis. A reducible basis we shall say is complete if its several constituent irreducible bases are all complete for the corresponding irreducible equations. A complete basis may be a possible basis or it may be an impossible basis. If our fundamental algebraic equation is irreducible the most general rational function which can be built on a basis which is at the same time complete and possible involves one arbitrary constant, for by a well-known theorem in the theory of the algebraic functions the function is determined to a constant factor by its zeros and infinities.

CHAPTER XIII.

## The genus and the $\varphi$-functions.

The representation of rational functions corresponding to a reducible algebraic equation. The *genus* of an algebraic equation. The *$\varphi$-functions.* The adjoint functions. The independence of the conditions of adjointness. The $\rho$ dependent coincidences in the definition of the $\varphi$-function. Complete sets of $2p-2$ $\varphi$-coincidences. No coincidence common to all complete sets. No rational function possesses but one infinity in the case of an irreducible equation of genus other than 0. Criterion for the reducibility of an algebraic equation and determination of its genus.

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From the general complementary theorem we can immediately deduce a number of the more important theorems in the theory of the algebraic functions. Before considering these theorems however it will be convenient to make a remark or two with regard to the case in which the fundamental algebraic equation is reducible. We shall suppose the fundamental equation to be equivalent to $\rho$ irreducible equations. It will then have the form

(1) $$f(z,u) = f_1(z,u)\, f_2(z,u) \ldots f_\rho(z,u) = 0$$

where the factors $f_1(z,u)$, $f_2(z,u)$, ... $f_\rho(z,u)$ are irreducible polynomials in $u$ with coefficients which are rational functions of $z$. The exponents of the highest powers of $u$ which present themselves in the several polynomials we shall represent by $n_1, n_2, \ldots n_\rho$ respectively. Also we may assume that the coefficient of the highest power of $u$ in each one of the polynomials is unity.

We shall now define $\rho$ functions $Q_1(z,u), Q_2(z,u), \dots Q_\rho(z,u)$ by the $\rho$ identities


(2) $$f_1 Q_1 = f,\ f_2 Q_2 = f, \dots f_\rho Q_\rho = f.$$

It may then be readily shewn that the reduced form relative to the equation $f(z,u)=0$ of any rational function $H(z,u)$ may be expressed in the form

(3) $$H_1 Q_1 + H_2 Q_2 + \cdots + H_\rho Q_\rho$$

where the functions $H_1, H_2, \dots H_\rho$ are reduced forms relative to the equations $f_1(z,u)=0,\ f_2(z,u)=0,\ \dots\ f_\rho(z,u)=0$ respectively. If namely for $H_1, H_2, \dots H_\rho$ we substitute the reduced forms relative to these equations of the functions

(4) $$\frac{H(z,u)}{Q_1(z,u)}, \frac{H(z,u)}{Q_2(z,u)}, \dots \frac{H(z,u)}{Q_\rho(z,u)}$$

respectively, the expression (3) will represent the function $H(z,u)$ for each of the irreducible equations $f_1(z,u)=0, \dots f_\rho(z,u)=0$ and therefore also for the equation $f(z,u)=0$. For the irreducible equation $f_1(z,u)=0$, for example, all the elements in (3) save $H_1 Q_1$ reduce to 0 and this element is evidently equal to $H(z,u)$ for $f_1(z,u)=0$. The difference between the function $H(z,u)$ and the expression (3) is then equal to 0 for $f(z,u)=0$ and is consequently divisible by $f(z,u)$ on regarding it as a polynomial in $u$ with coefficients in $z$. The function $H(z,u)$ is therefore represented by the expression (3). Also the form of this expression relative to the equation $f(z,u)=0$ is a reduced form, for the degrees in $u$ of the polynomials $Q_1, Q_2, \dots Q_\rho$ are $n-n_1, n-n_2, \dots n-n_\rho$ respectively and the respective degrees of the functions $H_1, H_2, \dots H_\rho$ in the same variable are less than $n_1, n_2, \dots n_\rho$ by our original hypothesis in regard to these functions. The degree in $u$ of the expression (3) is then less than $n$ and this expression is therefore a reduced form relative to the equation $f(z,u)=0$.

It is to be remarked that any element $H_s Q_s$ which actually presents itself in the expression (3) is linearly independent of the remaining $\rho-1$

elements, for each of these $\rho-1$ elements is divisible by $f_s(z,u)$ which is not a factor of the element in question.

In the case of an irreducible equation $f(z,u)=0$ a rational function of $(z,u)$ which nowhere becomes infinite is, as we know, necessarily a constant. In the case of a reducible equation a rational function of $(z,u)$ which nowhere becomes infinite will have a constant value for each one of the constituent irreducible equations $f_1(z,u)=0$, $f_2(z,u)=0$, ... $f_\rho(z,u)=0$. The function however will in general have a different constant value for each one of the irreducible equations in question. Suppose the function to have the values $c_1, c_2, \ldots c_\rho$ respectively for the several irreducible equations — we may readily obtain its expression in the form (3). Representing namely by $P_1, P_2, \ldots P_\rho$ the reduced forms relative to the equations $f_1(z,u)=0$, $f_2(z,u)=0$, ... $f_\rho(z,u)=0$ respectively of the functions $Q_1^{-1}, Q_2^{-1}$, ... $Q_\rho^{-1}$ the expression

$$(5) \qquad c_1 P_1 Q_1 + c_2 P_2 Q_2 + \cdots + c_\rho P_\rho Q_\rho$$

will have the values $c_1, c_2, \ldots c_\rho$ for the respective equations here in question. The expression (5) is a reduced form relative to the equation $f(z,u)=0$ and for arbitrary values of the coefficients $c_1, c_2, \ldots c_\rho$ evidently represents the most general rational function of $(z,u)$ which possesses no infinities. The most general rational function of $(z,u)$ which possesses no infinities then involves $\rho$ independent arbitrary constants since the several elements in (5) are linearly independent of one another.

The basis of coincidences $(\tau)$ in which all the numbers $\tau$ have the value 0 we shall call the 0-basis of coincidences. The most general rational function which can be built on the 0-basis then involves $\rho$ arbitrary constants. The number $\rho$ is evidently also the number of the arbitrary constants involved in the expression of the most general rational function which can be built on a complete basis, which is at the same time a possible basis for each of the irreducible equations which go to make up our fundamental algebraic equation. For on representing such general function in the form (3) we see that each one of the $\rho$ elements is determined to a constant factor. This of course is equivalent to saying, that $\rho$ is the number of the arbitrary constants involved in the expression of the most general rational function, whose orders of coincidence are the same as those of a specific rational function which does not vanish identically for any one of the irreducible equations whose aggregate constitutes the fundamental algebraic equation. A basis such as that here in question, we shall briefly refer to as a basis furnished by an actually existent function — meaning thereby however that the function actually exists for each one of the irreducible equations which go to make up the fundamental algebraic equation. By a basis furnished by an existent function then we simply mean a complete basis, which is at the same time a possible basis for each one of the irreducible equations which are included in our fundamental algebraic equation.

Turning now to the general complementary formula as stated, for example, in (XII, 33), we shall suppose the basis $(\tau)$ there in question to be furnished by an existent function. We shall then have $\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = 0$, $N_H = \rho$ and the complementary formula furnishes us with an expression for $N_{\bar{H}}$, the number of the arbitrary constants involved in the representation of the most general rational function which can be built on the complementary basis $(\bar{\tau})$. The number so obtained is a fixed number with reference to the fundamental algebraic equation. We call it the *genus* of the fundamental algebraic equation and represent it by the letter $p$. For the genus of the fundamental algebraic equation then we have the expression

$$(6) \qquad p = -n + \rho + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1).$$

In like manner we should obtain from (XII, 32) for the genus $p$, the expression

$$(7) \qquad p = -n + \rho - \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(m_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)}$$

and in particular corresponding to the form (XII, 22) we should have

$$(8) \qquad p = -n + \rho - \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)}.$$

To recapitulate: — If the basis $(\tau)$ be furnished by any existent rational function and if the basis $(\bar{\tau})$ be complementary with regard to the level furnished by an arbitrarily chosen rational function, then is the number of the arbitrary constants involved in the expression of the most general rational function which can be built on the basis $(\bar{\tau})$ a fixed number with reference to the fundamental algebraic equation, which we call the genus of the equation and for which expressions are furnished as in (6), (7) and (8) above by the complementary formula.

From formula (6), for example, it is evident that the genus of a reducible algebraic equation is equal to the sum of the genera of its constituent irreducible equations. It follows also from this formula that the sum $\sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1)$ must be an even integer. Furthermore, if none of the cycles be of an order greater than 2 the number of the cycles of this order will evidently be $2p + 2n - 2\rho$, a number which is $\geqq 2n - 2\rho$ since $p$ from its signification cannot be a negative number.

On introducing the number $p$ we may evidently write the complementary formula in the form

$$N_{\bar{H}} + \rho = N_H + p + \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)}. \tag{9}$$

For the basis $(\tau)$ let us now select the 0-basis. The function $H$ is then the most general rational function which is nowhere infinite and therefore involves $\rho$ arbitrary constants. We consequently have $N_H = \rho$ and from the complementary formula we then have $N_{\bar{H}} = p$. Here $\bar{H}$ is the most general rational function which can be built on the basis $(\bar{\tau})$, which basis is complementary to the 0-basis with regard to the level furnished by any arbitrarily chosen rational function. Let us select the level furnished by the rational function $f'_u(z, u)$. The formulae (XII, 28) and (XII, 29) then give us for the orders of coincidence $\bar{\tau}$ required of the function $\bar{H}$

$$\bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \; (s = 1, 2, \ldots r_k)$$

for all finite values of the variable $z = a_k$, and for the value $z = \infty$

$$\bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}, \ (s = 1, 2, \dots r_\infty).$$



The orders of coincidence here in question are those which are requisite to adjointness relative to the equation $f(z, u) = 0$ for finite values of the variable $z$, while for the value $z = \infty$ they exceed by 2 the orders of coincidence requisite to adjointness for this value of the variable. A function possessing such orders of coincidence we call a $\varphi$-function. The general $\varphi$-function corresponding to our fundamental algebraic equation then involves $N_{\overline{H}} = p$ arbitrary constants — in other words the number of the linearly independent $\varphi$-functions is $p$.

The representation in the form (3) of the general $\varphi$-function corresponding to the equation $f(z, u) = 0$ may readily be shewn to be

(10) $$\Phi_1 Q_1 + \Phi_2 Q_2 + \cdots + \Phi_\rho Q_\rho$$

where $\Phi_1, \Phi_2, \dots \Phi_\rho$ represent the general $\varphi$-functions corresponding to the irreducible equations $f_1(z, u) = 0$, $f_2(z, u) = 0, \dots f_\rho(z, u) = 0$ respectively. That the expression (10) represents a $\varphi$-function is evident, for the order of coincidence of a branch of the equation $f_1(z, u) = 0$, for example, with the function represented by this expression is the same as its order of coincidence with the element $\Phi_1 Q_1$, since the $\rho - 1$ functions $Q_2, \dots Q_\rho$ contain $f_1(z, u)$ as factor. Now the order of coincidence of the branch in question with the function $\Phi_1$ is equal to the sum of its orders of coincidence with the remaining $n_1 - 1$ branches of the equation $f_1(z, u) = 0$, minus $1 - \frac{1}{\nu}$ or plus $1 + \frac{1}{\nu}$, according as the value of the variable in question is not or is $z = \infty$, where $\nu$ is the order of the cycle to which the branch belongs, while its order of coincidence with the function $Q_1$ is equal to the sum of its orders of coincidence with the $n - n_1$ branches of the $\rho - 1$ equations $f_2(z, u) = 0, \dots f_\rho(z, u) = 0$. The order of coincidence of the branch in question with the product $\Phi_1 Q_1$ is therefore equal to the sum of its orders of coincidence with the remaining $n - 1$ branches of the equation $f(z, u) = 0$, minus $1 - \frac{1}{\nu}$ or plus $1 + \frac{1}{\nu}$, according as we do not or do have to do with the value $z = \infty$. This order of coincidence however is that which is necessary to a $\varphi$-function corresponding to the equation $f(z,u)=0$. It follows therefore that the expression (10) represents a $\varphi$-function corresponding to the equation $f(z,u)=0$.



Conversely any $\varphi$-function corresponding to the equation $f(z,u)=0$ is included in the form (10). For on expressing the $\varphi$-function in the form (3) its order of coincidence with a branch of the equation $f_1(z,u)=0$, for example, will be the same as the order of coincidence of the element $H_1 Q_1$ with this branch. The order of coincidence of this element with the branch in question will therefore be equal to or greater than the sum of the orders of coincidence of this branch with the $n-1$ conjugate branches of the equation $f(z,u)=0$, minus $1-\frac{1}{\nu}$ or plus $1+\frac{1}{\nu}$, according as the value of the variable with which we have to do is not or is the value $z=\infty$. The order of coincidence of the factor $Q_1$ with the branch is however equal to the sum of the orders of coincidence of the branch with the $n-n_1$ corresponding branches of the $\rho-1$ equations $f_2(z,u)=0, \ldots f_\rho(z,u)=0$. The order of coincidence of the branch with the factor $H_1$ must therefore be equal to or greater than the sum of its orders of coincidence with the $n_1-1$ conjugate branches of the equation $f_1(z,u)=0$, minus $1-\frac{1}{\nu}$ or plus $1+\frac{1}{\nu}$, according as the value of the variable in question is not or is $z=\infty$. The order of coincidence of the function $H_1$ with a branch of the equation $f_1(z,u)=0$ is then that which is necessary to a $\varphi$-function corresponding to this equation. The function $H_1$ is therefore a $\varphi$-function corresponding to the equation $f_1(z,u)=0$, and in like manner it may be shewn that the functions $H_2, \ldots H_\rho$ are $\varphi$-functions corresponding to the equations $f_2(z,u)=0, \ldots f_\rho(z,u)=0$ respectively. It follows therefrom that the form (10) represents the most general $\varphi$-function corresponding to the equation $f(z,u)=0$.

Similarly it can be proved that the most general adjoint function corresponding to the equation $f(z,u)=0$ is represented by the form (3), when for $H_1, H_2, \ldots H_\rho$ we substitute the general adjoint functions corresponding to the $\rho$ irreducible equations $f_1(z,u)=0$, $f_2(z,u)=0$, $\ldots$ $f_\rho(z,u)=0$ respectively. — By *adjoint function*, in the case of the general algebraic equation $f(z,u)=0$, as in that of the integral algebraic equation $F(z,v)=0$, we mean of course a function which is adjoint relatively to the equation in question for all values of the variable $z$, the value $z=\infty$ included. From its representation in the form (3), we see that the number of arbitrary constants involved in the expression of the general adjoint function corresponding to the equation $f(z,u)=0$, is equal to the sum of the numbers of the arbitrary constants involved in the expressions for the general adjoint functions corresponding to the several irreducible equations $f_1(z,u)=0$, $f_2(z,u)=0, \ldots f_\rho(z,u)=0$.



From the complementary formula we may readily deduce an expression for the number of the arbitrary constants involved in the general adjoint function corresponding to our fundamental algebraic equation. To this end we select a basis $(\tau)$ in which all the numbers corresponding to finite values of the variable $z$ are equal to 0, while each of the $r_\infty$ numbers $\tau_s^{(\infty)}$ has the value 2. The function $H$ built on the basis $(\tau)$ must then evidently be identically 0, and in the formula (9) we shall have $\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = 2n$, $N_H=0$. We consequently have $N_{\bar{H}} = p+2n-\rho$, where $\bar{H}$ is the most general rational function which can be built on a basis $(\bar{\tau})$, which is complementary to the basis $(\tau)$ with regard to the level furnished by an arbitrarily chosen rational function. Selecting the level furnished by the function $f'_u(z,u)$ and substituting their values for the numbers of the basis $(\tau)$ in the formulae (XII, 28) and (XII, 29), we obtain for the numbers of the basis $(\bar{\tau})$ the values

$$\bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \quad (s=1,\,2,\,\ldots\,r_k)$$

for all values of the variable $z$, the value $z=\infty$ included. These numbers however give the orders of coincidence requisite to adjointness and the function $\bar{H}$ built on the basis $(\bar{\tau})$ is therefore the general adjoint function corresponding to the fundamental algebraic equation. From what we have just seen then, the general adjoint function involves $p+2n-\rho$ arbitrary constants — in other words the number of linearly independent adjoint functions is $p + 2n - \rho$. From this result it is evident that the number of arbitrary constants involved in the expression of the general adjoint function corresponding to a reducible algebraic equation, is equal to the sum of the numbers of the arbitrary constants involved in the expressions for the general adjoint functions corresponding to the several constituent irreducible equations, what we have already seen to follow also from the representation of the function in the form (3). Since $p + 2n - \rho$ is the number of the arbitrary constants involved in the expression of the general adjoint function, while $p$ is the number of the arbitrary constants involved in the expression of the general $\varphi$-function, we derive the general $\varphi$-function from the general adjoint function on subjecting the coefficients of the latter to $2n - \rho$ independent conditions.

In like manner if we increase by 2 the orders of coincidence of the general adjoint function with the several branches of the fundamental equation corresponding to an arbitrary value $z = a$, we subject the coefficients of the general adjoint function to $2n - \rho$ independent conditions, for the function so obtained, by virtue of its coincidences, must evidently be the product of the general $\varphi$-function by the factor $(z - a)^2$ and will therefore involve just $p$ arbitrary constants.

Suppose $(\bar{\tau})$ to be a basis of coincidences in which no number exceeds the corresponding order of coincidence required by the general $\varphi$-function and in which one number at least, corresponding to each of the irreducible equations which constitute the fundamental equation, falls short of the order of coincidence required by the general $\varphi$-function. The general rational function $\overline{H}$ built on the basis $(\bar{\tau})$ will then involve a certain number of arbitrary constants. If now we adjoin further coincidences to the basis in question we at the same time impose a number of conditions on the coefficients of the function $\overline{H}$, and for every extra coincidence required from the function we impose a further condition on its coefficients, *so long as the basis* $(\bar{\tau})$ *retains its character as described above* — that is so long as the orders of coincidence required of the function do not in any case exceed those involved in the definition of the general $\varphi$-function and so long as one at least of the orders of coincidence in question, for each of the irreducible equations constituting the fundamental equation, falls short of the corresponding order of coincidence involved in such definition. The proof of this theorem follows immediately from the complementary formula. —

Selecting namely the level furnished by the function $f'_u(z, u)$, the numbers of the basis $(\tau)$, which is complementary to the basis $(\bar{\tau})$ with regard to this level, will be given by the equations

$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \quad (s = 1, 2, \ldots r_k)$$

for finite values of the variable $z = a_k$, and for the value $z = \infty$ by the equations

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}, \quad (s = 1, 2, \ldots r_\infty).$$

From these equations then, by virtue of the values given to the numbers in the basis $(\bar{\tau})$, it follows that the numbers in the basis $(\tau)$ are none of them negative and that one of them at least is positive for each of the irreducible equations which constitute the fundamental equation. The basis $(\tau)$ is therefore an impossible basis and the most general function which can be built on it is identically 0. As a consequence we have $N_H = 0$, and the complementary formula as stated in (9) gives us for the number of the arbitrary constants involved in the general function $\bar{H}$ built on the basis $(\bar{\tau})$, the expression

$$(11) \qquad N_{\bar{H}} = p - \rho + \sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}.$$

This expression for $N_{\bar{H}}$ holds good so long as the basis $(\bar{\tau})$ retains the character indicated above. For every extra coincidence added to the basis $(\bar{\tau})$ however a coincidence is subtracted from the basis $(\tau)$ and the sum on the right-hand side of (11) is diminished by 1. For every extra coincidence required from the function $\bar{H}$ then a further condition is imposed on its coefficients, so long at least as the basis $(\bar{\tau})$ on which the function is built involves no numbers which are greater than the corresponding orders of coincidence required by our definition of the general $\varphi$-function, and so long as it involves one number at least for each of the irreducible equations constituting the fundamental equation, which falls short of the corresponding order of coincidence required by our definition of the general $\varphi$-function. In particular if $\bar{H}$ be the most general rational function built on a basis $(\bar{\tau})$ none of whose numbers exceeds the corresponding order of coincidence requisite to adjointness, we see that for every extra coincidence required from the function a further condition is imposed on its coefficients. In the case of such a function then the conditions requisite to adjointness for the various values of the variable $z$ are evidently independent of one another.

Reverting for the moment to the integral algebraic equation $F(z, v) = 0$ we shall prove the independence of the conditions of adjointness for a polynomial in $(z, v)$ of degree $N-1$. It would be possible to give a proof depending directly on the principles enunciated in the preceding paragraph. We shall find it more expeditious however to proceed as follows. —

In formula (VIII, 19) we have obtained an expression for $A$, the number of the conditions which must be satisfied by the coefficients in the general rational function of $(z, v)$ of degree $N-1$ in order that the function may be adjoint for the value $z = \infty$. The number of the conditions requisite to the adjointness of the general polynomial in $(z, v)$ of degree $N-1$ relative to the value $z = \infty$ cannot be greater than $A$. Nor can the number of conditions requisite to the adjointness of the general polynomial in $(z, v)$ of degree $N-1$ relative to any finite value $z = a_k$ be greater than

$$\frac{1}{2}\sum_{s=1}^{r_k}\left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right)\nu_s^{(k)},$$

the number of conditions to which the coefficients of the general integral rational function must be subjected in order that it may be adjoint relatively to the value of the variable in question.

For the number of conditions which must be satisfied by the coefficients of the general polynomial in $(z, v)$ of degree $N-1$ in order that it may be adjoint for all values of the variable $z$, the value $z=\infty$ included, we shall then evidently have an expression



$$A+\frac{1}{2}\sum_k{}' \sum_{s=1}^{r_k} (\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\,\nu_s^{(k)}-\varepsilon$$

where the summation with regard to $k$ is extended to all finite values $z=a_k$ and where $\varepsilon$ is 0 or some positive integer. Subtracting this expression from $nN-\frac{1}{2}n(n-1)$, the total number of coefficients in the general polynomial in $(z,v)$ of degree $N-1$ — we obtain, for the number of arbitrary constants involved in the general adjoint function, the expression

$$nN-\frac{1}{2}n(n-1)-A-\frac{1}{2}\sum_k{}' \sum_{s=1}^{r_k} (\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\,\nu_s^{(k)}+\varepsilon.$$

Replacing $A$ by its value given in (VIII, 19), we have

$$n-\frac{1}{2}\sum_k \sum_{s=1}^{r_k} (\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\,\nu_s^{(k)}+\varepsilon$$

for the number of the linearly independent adjoint functions, the summation with regard to $k$ being here supposed to extend to all values of the variable $z$, the value $z=\infty$ included. This gives us $p+2n-\rho+\varepsilon$ for the number of the linearly independent adjoint functions, as we see on referring to the expression for the genus furnished by formula (8) of the present chapter. We have proved however that the number of the linearly independent adjoint functions is $p+2n-\rho$. As a consequence we have $\varepsilon=0$, and for the number of conditions which must be satisfied by the coefficients of the general polynomial in $(z,v)$ of degree $N-1$ in order that it may be an adjoint function, we have the expression

$$A+\frac{1}{2}\sum_k \sum_{s=1}^{r_k} (\mu_s^{(k)}-1+\frac{1}{\nu_s^{(k)}})\,\nu_s^{(k)}.$$

Now this expression is equal to the sum of the number of conditions which must be satisfied by the coefficients of the general rational function of degree $N-1$ in order that it may be adjoint for the value $z=\infty$, and the numbers of the conditions which must be satisfied by the coefficients of the general integral rational function in order that it may be adjoint for the different finite values of the variable $z$. It follows therefore that the number of conditions requisite to adjointness relative to the value $z=\infty$ is the same in the case of the general polynomial in $(z, v)$ of degree $N-1$ as in that of the general rational function of this degree, and also that the number of conditions requisite to adjointness relative to any individual finite value of the variable $z$ is the same in the case of the general polynomial of degree $N-1$ as in that of an arbitrary polynomial of higher degree. It furthermore follows that the conditions of adjointness for the different values of the variable $z$ are all independent of one another in the case of the general polynomial in $(z, v)$ of degree $N-1$.

Coming back to the equation $f(z, u)=0$ and to rational functions of $(z, u)$ considered with reference to this equation, — we have seen that an increase of 2 in the orders of coincidence of the general adjoint function with the several branches of the equation corresponding to any arbitrary value $z=a$, subjects the coefficients of the general adjoint function to $2n-\rho$ independent conditions. Of the $2n$ extra coincidences so imposed on the general adjoint function then some $2n-\rho$ are independent of one another. The $\rho$ dependent coincidences consist of one coincidence for each of the irreducible equations included in the fundamental equation and we can readily shew that a coincidence may correspond to any one of the cycles of its irreducible equation. — Select for example a basis $(\bar{\tau})$ so that the coincidences which it requires are precisely those which are required by the definition of the general $\varphi$-function, save for $\rho$ coincidences which are lacking and one of which corresponds to each of the $\rho$ irreducible equations. The complementary basis $(\tau)$ then consists of $\rho$ coincidences, one for each of the irreducible equations. We consequently have $N_H=0$ and formula (11) gives us $N_{\bar{H}}=p$. The general rational function $\bar{H}$ built on the basis $(\bar{\tau})$, which lacks $\rho$ of the coincidences required by the definition of the general $\varphi$-function, then involves the same number of arbitrary constants as the general $\varphi$-function; from which it follows that the $\rho$ coincidences lacking in the basis $(\bar{\tau})$ are already implied in the coincidences explicitly required by the basis.

The aggregate sum of the orders of coincidence required by the definition of the general $\varphi$-function is evidently

$$2n + \sum_{k} \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)}.$$

This expression as we see from formula (8) is equal to $2\rho - 2p$, which is not in general a positive number. For the moment we shall suppose the fundamental algebraic equation to be irreducible. The sum of the orders of coincidence required by the definition of the general $\varphi$-function will in this case then be equal to $2 - 2p$ — a number which is positive only when the genus $p$ of the irreducible equation is 0.

A $\varphi$-function does not exist in the case of an equation of genus 0. In all other cases $\varphi$-functions exist. Confining our attention then to the case in which we have $p > 0$ and remembering that the aggregate sum of the orders of coincidence of any rational function must be equal to 0, we see that any particular $\varphi$-function must possess $2p - 2$ coincidences over and above those implied in the definition of the general $\varphi$-function. We shall call such a set of $2p - 2$ coincidences a *complete set of $\varphi$-coincidences.* A complete set of $\varphi$-coincidences evidently determines a $\varphi$-function to a constant factor. Since however the general $\varphi$-function involves just $p$ arbitrary constants, a complete set of $\varphi$-coincidences imposes $p - 1$ conditions on the coefficients of the general $\varphi$-function. It follows that a particular $\varphi$-function is determined by some $p - 1$ coincidences out of its complete set of $2p - 2$ coincidences. The $2p - 2$ coincidences in a complete set of $\varphi$-coincidences are then always determined by some $p - 1$ of their number.

Still confining our attention to the case of an irreducible equation whose genus is greater than 0, we shall prove that no coincidence is common to all the different sets of $2p - 2$ coincidences which define the various individual $\varphi$-functions within the general $\varphi$-function. In other words we shall prove that no further coincidence is implied in the system of coincidences which define the general $\varphi$-function. We shall find it convenient to employ the letter $c$ to indicate a coincidence corresponding to some cycle of our fundamental equation and shall suppose if possible that $s$ further coincidences $c_1, c_2, \dots c_s$ are implied in the possession of the coincidences which define the general $\varphi$-function. — We do not say that these $s$ coincidences are different from one another.

Selecting for basis $(\bar{\tau})$ the coincidences required by the definition of the general $\varphi$-function together with the additional coincidence $c_1$, the most general rational function $\bar{H}$ which can be built on this basis is evidently the general $\varphi$-function and in formula (9) we therefore have $N_{\bar{H}} = p$. Selecting at the same time for $(\tau)$ the basis which is complementary to the basis $(\bar{\tau})$ with regard to the level furnished by the function $f'_u(z, u)$, we see that the basis $(\tau)$ consists of a single infinity which we shall indicate by the notation $c_1^{-1}$. We also have $\rho = 1$ for the case here in question, and for the number of arbitrary constants involved in the most general function which can be built on the basis $(\tau)$ the formula (9) evidently furnishes us with the value $N_H = 2$.

Since the general rational function $H$ built on the basis $(\tau)$ involves two arbitrary constants, it would evidently be possible to construct a rational function which actually possesses the single infinity $c_1^{-1}$. Employing for the moment the notation $H(c_1^{-1})$ to indicate such function, while we make use of the notation $\varphi(c_1, c_2, \dots c_s)$ to indicate the general $\varphi$-function, we see that the product $H(c_1^{-1}).\varphi(c_1, c_2, \dots c_s)$ possesses the orders of coincidence required by the definition of the general $\varphi$-function. The product however evidently lacks the coincidence $c_1$, which was supposed to be one of the $s$ coincidences $c_1, c_2, \dots c_s$ common to all the $\varphi$-functions. The assumption that the $\varphi$-functions possess a certain number of coincidences in common over and above those mentioned in the definition of the general $\varphi$-function then leads to contradiction. It follows therefore that no coincidence is common to all the different sets of $2p-2$ coincidences which define the various individual $\varphi$-functions within the general $\varphi$-function.

From the complementary formula we can also prove, in the case of an irreducible equation of genus other than 0, that a rational function which is not a constant must possess at least two infinities. For suppose if possible, that a rational function actually possesses a single infinity $c_1^{-1}$. On multiplying such function by an arbitrary constant and on adding an arbitrary constant, we evidently obtain the most general rational function which involves no infinity other than the one in question. Selecting the single negative coincidence $c_1^{-1}$ for basis $(\tau)$ in the complementary formula (9), the number of arbitrary constants involved in the most general function $H$ which can be built on this basis is therefore $N_H = 2$. The formula (9) then gives us $N_{\bar{H}} = p$, where we may suppose $\bar{H}$ to represent the most general rational function built on the basis $(\bar{\tau})$, which is complementary to the basis $(\tau)$ with regard to the level furnished by the function $f'_u(z, u)$. Such basis $(\bar{\tau})$ however is made up of all the coincidences required by the definition of the general $\varphi$-function together with the extra coincidence $c_1$. The function $\bar{H}$ built on the basis $(\bar{\tau})$ must therefore be a $\varphi$-function, and since it involves $p$ arbitrary constants it must coincide with the general $\varphi$-function. Over and above the coincidences involved in the definition of the general $\varphi$-function then such function would further possess the coincidence $c_1$, contrary to what has been proved in the preceding. It follows therefore that a rational function does not exist which actually possesses but one infinity.

Earlier in the chapter we have seen that the general $\varphi$-function is obtained from the general adjoint function on imposing $2n - \rho$ conditions on the coefficients of the latter function. This should furnish us with a criterion for the reducibility or irreducibility of the fundamental algebraic equation. — Supposing the fundamental equation to be transformed to the integral form $F(z, v) = 0$, we shall write

$$U = \frac{A_0 v^{n-1} + A_1 v^{n-2} + \cdots + A_{n-1}}{F'_v(z, v)}$$

where the degree of the numerator in $(z, v)$ is $N - 1$. First impose on the coefficients of the numerator the conditions that are necessary in order that it may be adjoint for the various values of the variable $z$. In order that it may be adjoint for the value $z=a_k$, it must have with the branches of the several cycles corresponding to this value of the variable the orders of coincidence



$$\mu_1^{(k)}-1+\frac{1}{\nu_1^{(k)}}, \ldots \mu_{r_k}^{(k)}-1+\frac{1}{\nu_{r_k}^{(k)}}$$

respectively. Since the orders of coincidence of the function $F'_v(z,v)$ with the branches of the same cycles are $\mu_1^{(k)}, \ldots \mu_{r_k}^{(k)}$ respectively, the fraction $U$ can evidently only become infinite for a value of $z$ to which corresponds at least one cycle whose order is greater than 1. All such values of $z$ are of course included among those which make the discriminant of $F(z,v)$ vanish identically. On representing the product of all the distinct linear factors of the discriminant of $F(z,v)$ by $P(z)$, the conditions that the numerator of the function $U$ should be adjoint for all finite values of the variable $z$ are evidently all embodied in the requirement, that the product $P(z).U$ should become zero for every branch of the fundamental equation corresponding to every value of the variable $z$ which makes the discriminant vanish. Also the condition that the numerator in the function $U$ should be adjoint for the value $z=\infty$ is evidently embodied in the requirement, that the function $z^{-1}U$ should become zero for every branch of the fundamental equation corresponding to the value $z=\infty$.

Eliminating $v$ between the equation

$$U.F'_v(z,v)-(A_0v^{n-1}+A_1v^{n-2}+\cdots+A_{n-1})=0$$

and the equation $F(z,v)=0$, we obtain an equation of degree $n$ in $U$ which we shall indicate by the notation

$$f_1(U,z,A_0,A_1,\ldots A_{n-1})=0.$$

On writing $P(z).U=V$ this equation transforms to an equation

$$F_1(V,z,A_0,A_1,\ldots A_{n-1})=0$$

and subjecting the polynomials $A_0, \dots A_{n-1}$ to the conditions necessary in order that this equation may be not only an integral algebraic equation in $V$, but also in order that every value of $V$ which corresponds to a value $z$ which satisfies the equation $P(z)=0$ may be 0, we have the conditions necessary to the adjointness of the function



$$A_0 v^{n-1} + A_1 v^{n-2} + \cdots + A_{n-1}$$

for all finite values of the variable $z$. Again on writing $z^{-1}U = V_1$ the function $V_1$ is determined by the equation

$$f_1(z V_1, z, A_0, A_1, \dots A_{n-1}) = 0,$$

and on subjecting the polynomials $A_0, \dots A_{n-1}$ to the conditions necessary in order that all the values of $V_1$ corresponding to the value $z=\infty$ may be zero, these are also the conditions which are necessary in order that the numerator of the function $U$ may be adjoint for the value $z=\infty$.

Furthermore on writing $zU = V_2$ the function $V_2$ is determined by the equation

$$f_1(z^{-1} V_2, z, A_0, A_1, \dots A_{n-1}) = 0,$$

and on subjecting the polynomials $A_0, \dots A_{n-1}$ to the conditions necessary in order that all the values of $V_2$ corresponding to the value $z=\infty$ may be zero, these additional conditions are evidently also those which are necessary in order that the numerator of the function $U$ may be a $\varphi$-function. Noting the actual number of these conditions, which we otherwise know to be $2n-\rho$, we thus arrive at the value of $\rho$, determining therewith the reducibility or irreducibility of the fundamental equation and the number of its constituent irreducible equations.

At the same time the number of the arbitrary constants remaining in the expression of the function $U$ gives us the genus of the fundamental equation, the numerator represents the general $\varphi$-function and the fractional form itself the integrand of the general Abelian integral of the first kind.

# CHAPTER XIV.

## Riemann-Roch theorem. Related theorems. Plücker's formulae.

The reciprocity theorem of Brill and Nöther. The Riemann-Roch theorem. The Weierstrassian gap theorem. A theorem of Hurwitz. The genus of an algebraic equation remains unaltered by a birational transformation and a complete set of $\varphi$-coincidences transforms into a complete set of $\varphi$-coincidences. Generalization of Plücker's formulae. Theory of the coresiduals.

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In the case of a reducible fundamental algebraic equation any particular $\varphi$-function which actually exists for each of the factors of the fundamental equation must possess $2p-2\rho$ coincidences over and above those implied in the definition of the general $\varphi$-function. Such a set of $2p-2\rho$ coincidences we shall call a *complete set of $\varphi$-coincidences* relative to the reducible algebraic equation. A complete set of $\varphi$-coincidences relative to a reducible algebraic equation is evidently made up of complete sets of $\varphi$-coincidences corresponding to the several constituent irreducible algebraic equations. The most general $\varphi$-function corresponding to a given complete set of $2p-2\rho$ $\varphi$-coincidences involves $\rho$ arbitrary constants and a complete set of $2p-2\rho$ $\varphi$-coincidences is evidently determined by some $p-\rho$ of their number.

Let us now suppose the complete set of $2p-2\rho$ coincidences corresponding to a given function $\varphi$ to be arbitrarily subdivided into two sets of $Q$ and $Q'$ coincidences respectively. The coincidences of these sets we shall represent by $c_1, c_2, \ldots c_Q$ and $c'_1\, c'_2, \ldots c'_{Q'}$ respectively. On representing by $\varphi_c$ the most general $\varphi$-function which includes in its complete set of $\varphi$-coincidences the $Q$ coincidences $c$ and by $\varphi_{c'}$ the most general $\varphi$-function which includes in its complete set of $\varphi$-coincidences the $Q'$ coincidences $c'$, we see that $\varphi_c$ and $\varphi_{c'}$ are the most general rational functions built on certain bases $(\tau)$ and $(\bar{\tau})$ respectively. The basis $(\tau)$ is made up of the coincidences implied in the definition of the general $\varphi$-function and of the $Q$ coincidences $c$, and the basis $(\bar{\tau})$ is made up of the coincidences implied in the definition of the general $\varphi$-function and of the $Q'$ coincidences $c'$. Also the bases $(\tau)$ and $(\bar{\tau})$ are complementary with regard to the level furnished by the rational product $\varphi . f'_u$, for on representing by $m_1^{(k)}, m_2^{(k)}, \dots m_{r_k}^{(k)}$ respectively the orders of coincidence of the given function $\varphi$ with the branches of the several cycles corresponding to the value $z = a_k$, we evidently have



$$\tau_s^{(k)} + \bar{\tau}_s^{(k)} = m_s^{(k)} + \mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}, \quad (s = 1, 2, \dots r_k)$$

for all finite values of the variable $z$, while for the value $z = \infty$ we have

$$\tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = m_s^{(\infty)} + \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}}, \quad (s = 1, 2, \dots r_\infty).$$

The complementary formula, as given for example in the form (XII, 25), then holds good with regard to the bases $(\tau)$ and $(\bar{\tau})$ here in question, and we therefore have

$$N\varphi_c + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = N\varphi_{c'} + \frac{1}{2}\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)}$$

where $N\varphi_c$ and $N\varphi_{c'}$ represent the numbers of the arbitrary constants involved in the respective functions $\varphi_c$ and $\varphi_{c'}$. Also we evidently have

$$\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = Q + 2n + \sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)}$$

$$\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)} = Q' + 2n + \sum_k \sum_{s=1}^{r_k} \left(\mu_s^{(k)} - 1 + \frac{1}{\nu_s^{(k)}}\right) \nu_s^{(k)}$$

and from the formula immediately preceding we derive



(1) $$Q-Q'=2(N_{\varphi c'}-N_{\varphi c}).$$

This is the Reciprocity Theorem of Brill and Nöther*.

We shall employ the expression *φ-strength*** to indicate the number of the conditions imposed on the $p$ arbitrary coefficients of the general $\varphi$-function by the requisition that it possess a certain set of coincidences $c_1, c_2, \ldots$. The $\varphi$-strength of a complete set of $2p-2\rho$ $\varphi$-coincidences is then $p-\rho$. Also the $\varphi$-strength of any set of coincidences can never be greater than $p$, for the $\varphi$-strength $p$ already makes the $\varphi$-function vanish identically. For the moment we can replace the expression $\varphi$-strength by the single word strength without fear of ambiguity.

The strength of the set of $Q$ coincidences $c_1, c_2, \ldots c_Q$ above is evidently $p-N_{\varphi_c}$, while that of the set of $Q'$ coincidences $c_1', c_2', \ldots c_{Q'}'$ is $p-N_{\varphi_{c'}}$. On employing for brevity the letters $q$ and $q'$ to indicate the strengths of the sets of $Q$ and $Q'$ coincidences respectively, the reciprocity theorem of Brill and Nöther may evidently also be represented in the form

(2) $$Q-Q'=2(q-q').$$

If we have $Q=Q'=p-\rho$ it follows from the reciprocity theorem that we must also have $q=q'$. Now since $p-\rho$ is the strength of a complete set of $2p-2\rho$ $\varphi$-coincidences, this also will be the strength of some set of $p-\rho$ out of the complete set of $2p-2\rho$ $\varphi$-coincidences. We can therefore evidently subdivide the $2p-2\rho$ coincidences of a complete set of $\varphi$-coincidences into two sets of $p-\rho$ coincidences each having the common strength $p-\rho$. Neither of these two sets can contain more than $q$ coincidences of a set of $\varphi$-coincidences of strength $q$, since in a set whose strength is equal to the number of its coincidences every coincidence must make its contribution to the strength. It follows that the two sets combined, that is the complete system of $2p-2\rho$ $\varphi$-coincidences, cannot contain

* Math. Annal. VII, p. 283.

** The use of the word strength in this connection is borrowed from H. F. Baker's treatise on the Abelian functions.

more than $2q$ coincidences belonging to a set of strength $q$. We conclude therefore that the number $Q$ of a set of $\varphi$-coincidences of strength $q$ cannot be greater than $2q$*. That we may actually have $Q = 2q$ we know, for a complete set of $2p - 2\rho$ $\varphi$-coincidences has $p - \rho$ as its strength.

Let us now consider an arbitrarily chosen set of coincidences $c_1, c_2, \ldots c_Q$, in which the same coincidence may appear any number of times and where $Q$ may be any positive integer. Corresponding to these coincidences we shall have a set of negative coincidences or infinities which we shall indicate by the notation $c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1}$. The repetition of a negative coincidence indicates the order to which the corresponding infinity is to be considered, while in the set of positive coincidences the repetition of a coincidence indicates the order to which the corresponding zero is to be considered.

We shall employ the notation $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$ to indicate the most general rational function of $(z, u)$ whose infinities are all included in the set of $Q$ infinities here in question. The function $H$ is then the most general rational function which can be built on a basis $(\tau)$ in which all the numbers are negative, and the complementary formula as given in (XIII, 9) will, for the case here in question, evidently take the form

$$(3) \qquad N_{\bar{H}} + \rho = N_H + p - Q,$$

where $\bar{H}$ may be supposed to be the most general rational function built on a basis $(\bar{\tau})$ which is complementary to the basis $(\tau)$ with regard to the level furnished by the function $f'_u$. The basis $(\bar{\tau})$ is then made up of the coincidences requisite to the definition of the general $\varphi$-function together with the $Q$ coincidences $c_1, c_2, \ldots c_Q$. As a consequence the function $\bar{H}$ built on the basis $(\bar{\tau})$ is a $\varphi$-function, and in fact the most general $\varphi$-function which possesses the set of coincidences $c_1, c_2, \ldots c_Q$. On employing the letter $q$ to indicate the $\varphi$-strength of this set of $Q$ coincidences we have $N_{\bar{H}} = p - q$, and formula (3) may evidently be written in the form

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* We may also say that the number $Q$ of any arbitrary set of coincidences of strength $q$ cannot be greater than $2q$ so long as we have $Q \leqq 2p$. For if the $Q$ coincidences in question do not constitute a set of $\varphi$-coincidences their strength must evidently be $p$.

(4) $$N_H = Q - q + \rho.$$



We see then that the most general rational function of $(z, u)$, whose infinities are included under a certain set of $Q$ infinities $c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1}$, depends upon $Q - q + \rho$ arbitrary constants where $q$ is the strength of the set of $Q$ coincidences $c_1, c_2, \ldots c_Q$. This is the Riemann-Roch Theorem stated for an equation reducible or irreducible.

In regard to the infinities which actually present themselves in the general function $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$, we may be a little more precise. Namely it is readily seen that a given infinity $c^{-1}$ will or will not present itself according as the omission of the corresponding coincidence $c$ from the set of coincidences $c_1, c_2, \ldots c_Q$ does not or does diminish the strength $q$ of the set. For example the infinity $c_Q^{-1}$ will or will not actually present itself in the general function $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$, according as this function does or does not involve one more arbitrary constant than the general function $H(c_1^{-1}, c_2^{-1}, \ldots c_{Q-1}^{-1})$ — also the numbers of arbitrary constants involved in the two general functions here in question are not or are the same according as $q$ or $q-1$ is the strength of the set of coincidences $c_1, c_2, \ldots c_{Q-1}$, as we see on referring to the formula (4). The infinity $c_Q^{-1}$ then does or does not actually present itself in the general function $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$ according as $q$ or $q-1$ is the strength of the set of coincidences $c_1, c_2, \ldots c_{Q-1}$, that is according as the omission of the coincidence $c_Q$ from the set $c_1, c_2, \ldots c_Q$ does not or does diminish the strength of the set.

The Riemann-Roch Theorem may then be stated in the following form: — The general function $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$ depends on $Q - q + \rho$ arbitrary constants where $q$ is the strength of the set of $Q$ coincidences $c_1, c_2, \ldots c_Q$, and the general function actually becomes infinite for a given one of the indicated infinities when and only when the omission of the corresponding coincidence from the set $c_1, c_2, \ldots c_Q$ does not diminish the strength of the set.

The strength of a set of coincidences can evidently not be greater than the number of coincidences in the set. If the strength of a set of coincidences $c_1, c_2, \ldots c_Q$ be just equal to the number of coincidences in the set, we see from the Riemann-Roch theorem that the general function $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$ can possess no infinities and must therefore reduce to an arbitrary constant*. If the strength of the set be less than the number of its coincidences the general function in question will actually present a number of infinities.



It is evident that the strength of a set of any number of coincidences can never be greater than $p$, for the strength $p$ is already sufficient to make the $p$ arbitrary coefficients in the general $\varphi$-function vanish. It follows that it is always possible to construct a function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_{p+1}^{-1})$ other than a constant, since the number of the coincidences $c_1, c_2, \ldots c_{p+1}$ is certainly greater than their strength. Also if $p$ be the strength of this set of $p+1$ coincidences and if the strength remain the same after the omission of any arbitrary one of the coincidences from the set, the function must actually possess the $p+1$ infinities in question, for its infinities cannot be included in any set of $p$ out of the $p+1$ infinities, since the strength of the corresponding set of $p$ coincidences is equal to their number. This remark has a significance only in connection with an irreducible algebraic equation, for in the case of a reducible equation it is readily seen that the omission of a coincidence from a set of $p+1$ coincidences of strength $p$ diminishes the strength of the set excepting where the omitted coincidence corresponds to a certain definite one of the constituent irreducible equations. — For other than special sets of infinities then, in the case of an irreducible algebraic equation, a function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_{p+1}^{-1})$ which is not a constant must actually possess each one of the $p+1$ indicated infinities.

In the case where $p=0$ we may construct a rational function possessing a single infinity or any arbitrary combination of infinities. Here we have no $\varphi$-functions, but on regarding 0 as the strength of any coincidence or set of coincidences this case may evidently also be included in the reasoning above.

In the statement of the Riemann-Roch theorem above, the number $\rho$ of the irreducible equations into which the equation $f(z,u)=0$ resolves

* In the case of a reducible algebraic equation the constant may of course have a different value for each of the constituent irreducible equations.

itself appears explicitly. We might also state the theorem without explicit reference to the reducibility or irreducibility of the equation $f(z, u) = 0$, on introducing* what one may call the *adjoint strength* of a set of coincidences. By the adjoint strength of a set of $\bar{Q}$ coincidences namely, we shall mean the number $\bar{q}$ of conditions which we impose upon the coefficients of the general adjoint function in attempting to add the $\bar{Q}$ coincidences in question to those already implied in the definition of the adjoint function.

The general $\varphi$-function corresponding to the equation $f(z, u) = 0$ is derived from the general adjoint function on adding 2 to the order of coincidence of the latter function with each of the $n$ branches at $\infty$. This implies the addition of $2n$ coincidences to the coincidences at $\infty$ possessed by the general adjoint function. These $2n$ coincidences in what follows we shall simply refer to as the $2n$ coincidences at $\infty$. In Chapter XIII we have seen that in order to obtain the general $\varphi$-function corresponding to the equation $f(z, u) = 0$ from the general adjoint function, we must subject the coefficients of the latter to $2n - \rho$ independent conditions. The adjoint strength of the set of $2n$ coincidences at $\infty$ is therefore $2n - \rho$.

If $q$ be the $\varphi$-strength of the set of $Q$ coincidences $c_1, c_2, \ldots c_Q$ and $\bar{q}$ the adjoint strength of the set of $\bar{Q} = Q + 2n$ coincidences made up of these $Q$ coincidences and of the $2n$ coincidences at $\infty$, we shall evidently have $\bar{q} = q + 2n - \rho$ and consequently

$$Q - q + \rho = \bar{Q} - \bar{q}.$$

To the Riemann-Roch theorem we may then give the following formulation. — The general function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1})$ depends on $\bar{Q} - \bar{q}$ arbitrary constants, where $\bar{q}$ is the adjoint strength of the set of $\bar{Q} = Q + 2n$ coincidences made up of the $Q$ coincidences $c_1, c_2, \ldots c_Q$ and of the $2n$ coincidences at $\infty$. Furthermore the general function in question does or does not actually possess a given one of the $Q$ indicated infinities, according as the omission of the corresponding coincidence from the set of $\bar{Q}$ coincidences does not or does diminish the adjoint strength of the set.

Consider a set of $Q$ infinities arranged in any arbitrary order $c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1}$. The $\varphi$-strength of the set of $k$ coincidences $c_1, c_2, \ldots c_k$ we shall indicate by $q_k$, so that $q_Q = q$ is the strength of the complete set $c_1, c_2, \ldots c_Q$. From what we have seen in the foregoing it will or will not be possible to construct a function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_k^{-1})$ which actually possesses the infinity $c_k^{-1}$, according as we have $q_k = q_{k-1}$ or $q_k = q_{k-1} + 1$. The latter case however will present itself for $q$ values of $k$, for we evidently have $\sum_{k=1}^{Q}(q_k - q_{k-1}) = q$. Corresponding to the arrangement $c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1}$ of our $Q$ infinities then there are just $q$ values of $k$ for which it is impossible to construct a function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_k^{-1})$ which actually possesses the infinity $c_k^{-1}$. The number of such impossible functions corresponding to a given arrangement of a set of any number of infinities can never be greater than $p$, for this is the greatest value which the strength of a set of coincidences can have.



Suppose the fundamental algebraic equation to be irreducible. The strength of a complete set of $2p-2$ coincidences is $p-1$, for these coincidences determine the $\varphi$-function to a constant factor. As a consequence the strength of any set whatever of $2p-1$ or more coincidences must be $p$. Corresponding to any arrangement $c_1^{-1}, c_2^{-1}, \ldots c_Q^{-1}$ of a set of $Q$ infinities, where $Q$ is $\geqq 2p-1$, then it follows that there are just $p$ values of $k$ for which it is impossible to construct a function of the type $H(c_1^{-1}, c_2^{-1}, \ldots c_k^{-1})$ which actually possesses the infinity $c_k^{-1}$. This is in effect Nöther's generalization* of the Weierstrassian Gap Theorem.

To derive the latter theorem from the former it is only necessary to suppose the $c$'s to be all the same, and we see that there are just $p$ values of $k$ for which it is impossible to construct a rational function of $(z, u)$ which possesses a given infinity $c^{-1}$ to the $k$th order and which has no other infinities.

In this connection we might prove a theorem of Hurwitz relating to the gaps of the gap theorem. The $p$ gaps namely in the series of possible functions $H(c_1^{-1}, c_2^{-1}, \ldots c_k^{-1})$ occur, as we have seen, for the values $k$ for which $q_k = q_{k-1} + 1$. Here we may evidently assume $k \leqq 2p-1$ and we shall consequently also have $k-1 \leqq 2q_{k-1}$ for, as has already been pointed

* Crelle, Bd. 97, P. 228.

out, if $q$ is the strength of a set of $Q$ coincidences we have $Q \leqq 2q$ so long as we have $Q \leqq 2p$. For the sum of the numbers indicating the places of the gaps we therefore have



$$(5) \qquad \sum k \leqq \sum_{q_{k-1}=0}^{p-1} (2q_{k-1}+1) = 1 + 3 + \cdots + (2p-1) = p^2.$$

In particular on supposing the $c$'s to be all the same, we conclude that the sum of the values $k$ which represent the orders of the infinity $c^{-1}$ for which it is impossible to construct a rational function possessing no other infinities, is $\leqq p^2$. This theorem is due to Hurwitz*.

In the generalized gap theorem we see that the gaps are transformed into gaps by a birational transformation, so that the genus of our fundamental algebraic equation is not altered by such a transformation. The same thing is also evident from the Riemann-Roch theorem. — If, for example, $c_1, c_2, \ldots c_p$ constitute a set of $p$ coincidences of strength $p$ corresponding to the equation $f(z,u)=0$, we know that a function of the type $H(c_1^{-1}, \ldots c_p^{-1})$ other than a constant does not exist, whereas it is always possible to construct a function of the type $H(c_1^{-1}, \ldots c_p^{-1}, c_{p+1}^{-1})$ which actually possesses a number of the indicated infinities. Supposing the equation $f(z,u)=0$ to be transformed by a birational transformation into the equation $g(\xi,\eta)=0$, and supposing at the same time that the $p$ coincidences $c_1, \ldots c_p$ go over into $p$ coincidences $\gamma_1, \ldots \gamma_p$ associated with the transformed equation, we see that a rational function of $(\xi,\eta)$ of the type $H(\gamma_1^{-1}, \ldots \gamma_p^{-1})$ cannot become infinite, as otherwise it would transform into a function of the type $H(c_1^{-1}, \ldots c_p^{-1})$ which is not a constant. It follows that the genus of the equation $g(\xi,\eta)=0$ cannot be less than $p$, the genus of the equation $f(z,u)=0$. In like manner it may be shewn conversely that the genus of the equation $f(z,u)=0$ cannot be less than the genus of the equation $g(\xi,\eta)=0$. The genera of the two equations must therefore be the same.

By a birational transformation the quotient of two $\varphi$-functions transforms into the quotient of two $\varphi$-functions. To prove this we first prove

* Cf. Hurwitz, Math. Annal. Bd. XLI, p. 410 and H. F. Baker, Abelian Functions, p. 42.

that any system of $2p-2$ $\varphi$-coincidences transforms into a system of $2p-2$ $\varphi$-coincidences by a birational transformation. — Let $c_1, \dots c_{2p-2}$ be a system of $2p-2$ $\varphi$-coincidences corresponding to the equation $f(z, u) = 0$. These $2p-2$ $\varphi$-coincidences will be completely determined by some $p-1$ among them, — say by $c_1, \dots c_{p-1}$. By a birational transformation the equation $f(z, u) = 0$ transforms into an equation $g(\xi, \eta) = 0$ and the set of $\varphi$-coincidences $c_1, \dots c_{2p-2}$ goes over into a set of $2p-2$ coincidences $\gamma_1, \dots \gamma_{2p-2}$. This latter set of $2p-2$ coincidences we shall prove to constitute a complete system of $\varphi$-coincidences for the equation $g(\xi, \eta) = 0$.

Since $p-1$ is the strength of the complete system of $2p-2$ coincidences $c_1, \dots c_{2p-2}$ as also of the set of $p-1$ coincidences $c_1, \dots c_{p-1}$, it is possible to construct functions other than constants of the types

$$H(c_1^{-1}, \dots c_{p-1}^{-1}, c_p^{-1}),\; H(c_1^{-1}, \dots c_{p-1}^{-1}, c_{p+1}^{-1}), \dots H(c_1^{-1}, \dots c_{p-1}^{-1}, c_{2p-2}^{-1})$$

whereas a function of the type $H(c_1^{-1}, \dots c_{p-1}^{-1})$ must be a constant. It follows therefore that it is possible to construct functions of $(\xi, \eta)$ other than constants of the types

$$H(\gamma_1^{-1}, \dots \gamma_{p-1}^{-1}, \gamma_p^{-1}),\; H(\gamma_1^{-1}, \dots \gamma_{p-1}^{-1}, \gamma_{p+1}^{-1}), \dots H(\gamma_1^{-1}, \dots \gamma_{p-1}^{-1}, \gamma_{2p-2}^{-1})$$

whereas a function of the type $H(\gamma_1^{-1}, \dots \gamma_{p-1}^{-1})$ must be a constant. We derive therefrom that the $2p-2$ coincidences $\gamma_1, \dots \gamma_{2p-2}$ constitute the complete system of coincidences of a $\varphi$-function corresponding to the equation $g(\xi, \eta) = 0$, this system of coincidences being determined by the $p-1$ coincidences $\gamma_1, \dots \gamma_{p-1}$. A complete system of $\varphi$-coincidences $c_1, \dots c_{2p-2}$ corresponding to the equation $f(z, u) = 0$ is therefore transformed by a birational transformation into a complete system of $\varphi$-coincidences corresponding to the transformed equation $g(\xi, \eta) = 0$*.

We can now prove that the quotient of two $\varphi$-functions corresponding to the equation $f(z, u) = 0$ transforms into the quotient of two $\varphi$-functions

* By a transformation birational for a reducible equation then a complete system of $2p-2p$ $\varphi$-coincidences corresponding to this equation will be transformed into a complete system of $\varphi$-coincidences corresponding to the transformed equation. Also the quotient of two $\varphi$-functions corresponding to a reducible equation will be transformed into the quotient of two $\varphi$-functions corresponding to the transformed equation.

corresponding to the equation $g(\xi, \eta) = 0$. A $\varphi$-function is determined to a constant factor by the system of its $2p-2$ $\varphi$-coincidences. Employing then the notation $\varphi(c_1, \dots c_{2p-2})$ to indicate a $\varphi$-function possessing the system of $\varphi$-coincidences $c_1, \dots c_{2p-2}$, the function is designated to a constant factor by this notation. — Suppose $c_1, \dots c_{2p-2}$ and $c'_1, \dots c'_{2p-2}$ to be two complete systems of $\varphi$-coincidences corresponding to the equation $f(z, u) = 0$, which by the birational transformation go over into the two systems of $\varphi$-coincidences $\gamma_1, \dots \gamma_{2p-2}$ and $\gamma'_1, \dots \gamma'_{2p-2}$ corresponding to the equation $g(\xi, \eta) = 0$. The quotient

$$\frac{\varphi(c_1, \dots c_{2p-2})}{\varphi(c'_1, \dots c'_{2p-2})}$$

will then transform into a quotient

$$\frac{\varphi(\gamma_1, \dots \gamma_{2p-2})}{\varphi(\gamma'_1, \dots \gamma'_{2p-2})}$$

for the former quotient will evidently transform into a function of $(\xi, \eta)$ whose zeros and infinities are those of the latter quotient, and since a rational function is determined to a constant factor by its zeros and infinities we may write

$$\frac{\varphi(c_1, \dots c_{2p-2})}{\varphi(c'_1, \dots c'_{2p-2})} = \frac{\varphi(\gamma_1, \dots \gamma_{2p-2})}{\varphi(\gamma'_1, \dots \gamma'_{2p-2})}$$

on taking account of the constant factor in the $\varphi$-functions $\varphi(\gamma_1, \dots \gamma_{2p-2})$ and $\varphi(\gamma'_1, \dots \gamma'_{2p-2})$.

In particular a transformation to tangential coordinates is a birational transformation. The genus $p$ of the curve $f(z, u) = 0$ is therefore also the genus of the curve referred to tangential coordinates.

Representing the equation of our curve transformed to tangential coordinates by $g(\xi, \eta) = 0$ and regarding $\eta$ as the dependent variable, we shall indicate the degree of the transformed curve in this variable by $\bar{n}$. The $\bar{n}$ branches of the curve corresponding to a value $\xi = \alpha_k$ we shall suppose to consist of $\bar{r}_k$ cycles of orders $\bar{\nu}_1^{(k)}, \bar{\nu}_2^{(k)}, \dots \bar{\nu}_{\bar{r}_k}^{(k)}$ respectively, and the orders of coincidence of the branches of these cycles each with the product of the remaining $\bar{n}-1$ branches of the curve we shall indicate by $\bar{\mu}_1^{(k)}, \ldots \bar{\nu}_{\bar{r}_k}^{(k)}$ respectively. Furthermore by $m_1^{(k)}, m_2^{(k)}, \ldots n_{r_k}^{(k)}$ we shall indicate the orders of coincidence of an arbitrarily chosen rational function $H(z, u)$ with the branches of the several cycles of the equation $f(z, u)=0$ corresponding to a value $z=c_k$, and by $\bar{m}_1^{(k)}, \bar{m}_2^{(k)}, \ldots \bar{m}_{\bar{r}_k}^{(k)}$ the orders of coincidence of any arbitrary rational function $\bar{H}(\xi, \eta)$ with the branches of the several cycles of the equation $g(\xi, \eta)=0$ corresponding to the value $\xi=\alpha_k$.



Now the aggregate sum of the orders of coincidence of a rational function of $(z, u)$ with all the branches of the curve $f(z, u)=0$ corresponding to the different values of $z$, is 0. We therefore have

$$\sum_k \sum_{s=1}^{r_k} m_s^{(k)} \nu_s^{(k)} = 0.$$

Also from (XIII, 6) we have the formula

$$p = -n + 1 + \frac{1}{2} \sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1).$$

Similarly corresponding to the equation $g(\xi, \eta)=0$ we have the formulae

$$\sum_k \sum_{s=1}^{\bar{r}_k} \bar{m}_s^{(k)} \bar{\nu}_s^{(k)} = 0$$

and

$$p = -\bar{n} + 1 + \frac{1}{2} \sum_k \sum_{s=1}^{\bar{r}_k} (\bar{\nu}_s^{(k)} - 1).$$

From these four formulae eliminating $p$ we obtain the three formulae

$$(6) \qquad \begin{aligned} &\sum_k \sum_{s=1}^{r_k} m_s^{(k)} \nu_s^{(k)} = 0 \\ &\sum_k \sum_{s=1}^{\bar{r}_k} \bar{m}_s^{(k)} \bar{\nu}_s^{(k)} = 0 \\ &\sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1) - \sum_k \sum_{s=1}^{\bar{r}_k} (\bar{\nu}_s^{(k)} - 1) = 2(n - \bar{n}). \end{aligned}$$

These three formulae from the point of view of the theory of the algebraic functions may be regarded as the generalization of Plücker's formulae.



In particular selecting for $H(z,u)$ and $\bar{H}(\xi,\eta)$ respectively the functions $f'_u(z,u)$ and $g'_\eta(\xi,\eta)$, the formulae (6) become

$$\sum_k \sum_{s=1}^{r_k} \mu_s^{(k)} \nu_s^{(k)} = 0$$

$$(7) \qquad \sum_k \sum_{s=1}^{\bar{r}_k} \bar{\mu}_s^{(k)} \bar{\nu}_s^{(k)} = 0$$

$$\sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1) - \sum_k \sum_{s=1}^{\bar{r}_k} (\bar{\nu}_s^{(k)} - 1) = 2(n - \bar{n}).$$

Suppose now that the equation $f(z,u)=0$ is integral of degree $n$ and that it possesses only the simplest point and line singularities, namely double points and double tangents and ordinary cusps and stationary tangents. Furthermore assume that neither the equation $f(z,u)=0$ nor its transformed equation $g(\xi,\eta)=0$ presents any singularity at $\infty$. We shall evidently then have

$$\nu_s^{(\infty)} = 1, \quad \mu_s^{(\infty)} = -(n-1), \qquad (s = 1, 2, \dots r_\infty)$$
$$\bar{\nu}_s^{(\infty)} = 1, \quad \bar{\mu}_s^{(\infty)} = -(\bar{n}-1), \qquad (s = 1, 2, \dots \bar{r}_\infty)$$

and the formulae (7) take the forms

$$-n(n-1) + \sum_k{}' \sum_{s=1}^{r_k} \mu_s^{(k)} \nu_s^{(k)} = 0$$

$$(8) \qquad -\bar{n}(\bar{n}-1) + \sum_k{}' \sum_{s=1}^{\bar{r}_k} \bar{\mu}_s^{(k)} \bar{\nu}_s^{(k)} = 0$$

$$\sum_k{}' \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1) - \sum_k{}' \sum_{s=1}^{\bar{r}_k} (\bar{\nu}_s^{(k)} - 1) = 2(n - \bar{n}).$$

Considering the first of these three formulae we see that any term in the double summation which has a value other than 0 must correspond to a double point, a cusp, or the point of contact of a tangent parallel to the axis of $u$. To a double point evidently correspond two terms in each of which we have $\mu_s^{(k)} = 1$, $\nu_s^{(k)} = 1$. To a cusp corresponds a term in which we have $\mu_s^{(k)} = \frac{3}{2}$, $\nu_s^{(k)} = 2$, and to the point of contact of a tangent parallel to the axis of $u$ corresponds a term in which we have $\mu_s^{(k)} = \frac{1}{2}$, $\nu_s^{(k)} = 2$. The increment in the double summation due to a double point is then 2, that furnished by a cusp is 3, while every tangent parallel to the axis of $u$ gives an increment 1 to the summation. Now the number of the tangents parallel to the axis of $u$ is equal to $\bar{n}$ the class of the curve. On indicating then by $\delta$ and $\varkappa$ the number of the double points and the number of the cusps respectively, the first of the three formulae in (3) assumes the form



$$-n(n-1) + 2\delta + 3\varkappa + \bar{n} = 0.$$

On indicating by $\tau$ and $\iota$ respectively the number of the double tangents and the number of the stationary tangents, the second formula in (8) will in like manner assume the form

$$-\bar{n}(\bar{n}-1) + 2\tau + 3\iota + n = 0,$$

while the third formula evidently takes the form

$$(\bar{n} + \varkappa) - (n + \iota) = 2(n - \bar{n}).$$

For the three formulae (8) then we have

$$\begin{aligned} \bar{n} &= n(n-1) - 2\delta - 3\varkappa \\ n &= \bar{n}(\bar{n}-1) - 2\tau - 3\iota \qquad (9) \\ \iota - \varkappa &= 3(\bar{n} - n) \end{aligned}$$

and these three formulae we know to be equivalent to Plücker's formulae.

We might here make reference to the theory of the coresidual sets of points on an algebraic curve* — not so much with the purpose of introducing the modified notation which we shall find it convenient to employ, as with the object of recalling the theory in association with the increased precision which we have given to the conception of adjointness.

* See Brill and Nöther, Math. Annal. Bd. VII, p. 272; H. F. Baker, Abelian Functions, p. 136; Salmon's Higher Plane Curves (3rd. ed.) p. 137.

— Instead of speaking of »sets of points» we shall employ the expression *sets of coincidences* in that which follows.



Suppose $(\tau)$ to be a basis of coincidences whose numbers for cycles corresponding to finite values of the variable $z$ are all equal to or greater than the corresponding adjoint numbers — such a basis we shall say is adjoint for finite values of the variable $z$. Furthermore assume the basis $(\tau)$ to be a possible basis for each of the irreducible equations included in our fundamental algebraic equation, and let $M(z, u)$ and $N(z, u)$ be rational functions existent for each of the irreducible equations in question, and built on the basis $(\tau)$ by the addition of coincidences all of which correspond to finite values of the variable $z$. The sets of coincidences possessed by the functions $M(z, u)$ and $N(z, u)$ over and above those required by the basis $(\tau)$, we shall indicate by $P$ and $Q$ respectively. Each of these sets of coincidences we call a residual of the basis $(\tau)$ and the two sets we say are coresiduals relative to this basis.

If the sets of coincidences $P$ and $Q$ are coresiduals relative to the basis $(\tau)$, and if either of these sets is a residual of a second basis $(t)$ which is adjoint for finite values of the variable $z$, then are the two sets also coresiduals relative to this second basis. For supposing the rational function $\overline{M}(z, u)$ to be built on the basis $(t)$ by the addition of the coincidences of the set $P$ to those already required by the basis and constructing the function

$$\overline{N}(z, u) = \frac{N(z, u)}{M(z, u)} \cdot \overline{M}(z, u), \tag{10}$$

we see that the function $\overline{N}(z, u)$ is built on the basis $(t)$ by the addition of the coincidences of the set $Q$, and that as a consequence the sets $P$ and $Q$ are coresiduals relative to the basis $(t)$. If then two sets of coincidences corresponding to finite values of the variable $z$ are coresiduals relative to a basis of coincidences which is adjoint for finite values of the variable, and if either of the sets be a residual of another such basis, then are the two sets also coresiduals of this second basis. We may then simply refer to the two sets of coincidences as coresiduals.

If the fundamental algebraic equation $f(z, u) = 0$ is integral the functions $M(z,u)$, $N(z,u)$, $\overline{M}(z,u)$ and $\overline{N}(z,u)$ which appear in the preceding are all polynomials in $(z,u)$, for rational functions which are adjoint relatively to an integral algebraic equation for finite values of the variable $z$ are, as we have seen, necessarily integral.



If the zeros and infinities of a rational function $H(z,u)$ correspond to finite values of the variable $z$, it is readily seen that they constitute coresidual sets of coincidences. For on constructing a rational function $M(z,u)$ which is adjoint for all finite values of $z$ and which over and above this includes among its coincidences ones corresponding to the infinities of the function $H(z,u)$, we see that the product

$$H(z,u) \,.\, M(z,u) = N(z,u)$$

is also adjoint for all finite values of the variable $z$ and that its coincidences are the same as those of the function $M(z,u)$, save that among these latter the coincidences corresponding to the infinities of the function $H(z,u)$ are replaced by the zeros of this function. From this it follows that the infinities of the function $H(z,u)$ are coresidual to its zeros. It is furthermore evident that the rational function $H(z,u)$ may be represented in an infinite number of ways in the fractional form

$$H(z,u) = \frac{N(z,u)}{M(z,u)},$$

where numerator and denominator are adjoint relatively to all finite values of the variable $z$, for the function $M(z,u)$ as characterized above may be chosen in an infinite number of ways. — Here of course we do not regard as distinct, a representation obtained on multiplying numerator and denominator in a given fractional form by the same rational function of $z$.

In what precedes, coresidual sets of coincidences correspond only to finite values of the variable $z$. We may readily extend our definition however so that the value $z = \infty$ will not be excluded. — Suppose $(\tau)$ to be any basis which is a possible one for each of the irreducible equations included in the fundamental algebraic equation. Let $M(z,u)$ and $N(z,u)$ be rational functions existent for each of the irreducible equations in question and built on the basis $(\tau)$ by the addition of coincidences which correspond to cycles for each one of which the basis is adjoint. The sets of these coincidences possessed by the functions $M(z, u)$ and $N(z, u)$, over and above those required by the basis $(\tau)$, we shall indicate by $P$ and $Q$ respectively. Each of these sets of coincidences we call a residual of the basis $(\tau)$ and the two sets we say are coresiduals relative to this basis.



If the sets of coincidences $P$ and $Q$ are coresiduals relative to the basis $(\tau)$ and if either of these sets is a residual of a second basis $(t)$, which is at the same time adjoint for all the cycles to which the coincidences of the other set correspond, then is this other set also a residual of the basis $(t)$ and the two sets are therefore coresiduals relative to this basis. For supposing a rational function $\bar{M}(z, u)$ to be built on the basis $(t)$ by the addition of the coincidences of the set $P$ to those already required by the basis, and constructing a function $\bar{N}(z, u)$ as in formula (10), we see that this function is built on the basis $(t)$ by the addition of the coincidences of the set $Q$, and that as a consequence the sets $P$ and $Q$ are coresiduals relative to the basis $(t)$. With regard to any basis $(t)$ then, which is adjoint for all the cycles to which the coincidences in the sets $P$ and $Q$ correspond, and which is possible for each of the irreducible equations included in the fundamental algebraic equation, we may say that the two sets are either coresiduals relative to the basis or that neither set is a residual of the basis, in the case where the sets have already been seen to be coresiduals relative to some one basis $(\tau)$.

With the generalized definition of coresiduals it is readily seen that the zeros and infinities of any rational function $H(z, u)$ constitute coresidual sets of coincidences. For on constructing a rational function $M(z, u)$ which is adjoint for all those cycles to which zeros or infinities of the function $H(z, u)$ correspond, and which over and above this includes among its coincidences ones corresponding to the infinities here in question, we see that the product

$$H(z, u) \,.\, M(z, u) = N(z, u)$$

is also adjoint for the cycles just referred to and that its coincidences are the same as those of the function $M(z, u)$, save that among these latter the coincidences corresponding to the infinities of the function $H(z, u)$ are replaced by the zeros of this function. It immediately follows that the coincidences corresponding to the infinities of the function $H(z, u)$ and those consisting of its zeros are residuals of a basis $(\tau)$ which is adjoint to all those cycles to which these coincidences correspond. The sets of coincidences in question are therefore coresiduals according to our definition.



From the above it evidently follows that any rational function of $(z, u)$ can be represented as the quotient of rational functions which are adjoint for all those cycles to which zeros or infinities of the function correspond. It is however also readily seen that any rational function can be represented as the quotient of rational functions which are adjoint for all cycles save $\rho$, which may be otherwise arbitrarily chosen so long as one cycle corresponds to each of the $\rho$ irreducible equations included in the fundamental algebraic equation.

# CHAPTER XV.

## The Abelian integrals.

Existence of $p$ linearly independent Abelian integrals of the first kind. Existence of the elementary Abelian integrals of the second and third kinds. Reduction of the general Abelian integral. Periodicity of the Abelian integrals.

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The theory of the algebraic functions developed in the preceding chapters furnishes an algebraic basis on which to build up the theory of the Abelian transcendents. The existence of the Abelian integrals of the three kinds immediately follows from our theory of the rational functions of $(z, u)$. — For example, the general Abelian integral of the first kind evidently has as integrand the most general rational function of $(z, u)$ built on a basis $(\bar{\tau})$, in which the numbers corresponding to a finite value $z = a_k$ are

$$-1 + \frac{1}{\nu_1^{(k)}}, \; -1 + \frac{1}{\nu_2^{(k)}}, \; \ldots \; -1 + \frac{1}{\nu_{r_k}^{(k)}},$$

while for the value $z = \infty$ its numbers are

$$1 + \frac{1}{\nu_1^{(\infty)}}, \; 1 + \frac{1}{\nu_2^{(\infty)}}, \; \ldots \; 1 + \frac{1}{\nu_{r_\infty}^{(\infty)}}.$$

These numbers represent the orders of coincidence which must be possessed by the integrand if the integral is to remain finite for all values of the variable $z$.



If the basis $(\tau)$ be complementary to the basis $(\bar{\tau})$ with regard to the level furnished by the function $f'_u(z, u)$, the numbers of the basis $(\tau)$ will coincide with the orders of coincidence of the function $f'_u(z, u)$ corresponding to the various values of $z$, the value $z = \infty$ included. In the complementary formula as stated in (XIII, 9) we shall then have

$$\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)} = \sum_k \sum_{s=1}^{r_k} \mu_s^{(k)} \nu_s^{(k)} = 0.$$

At the same time we shall also evidently have $N_H = \rho$, for $H$ here represents the most general rational function which possesses the same orders of coincidence as the function $f'_u(z, u)$. It follows then from the formula in question that we must have $N_{\bar{H}} = \rho$. The most general function $\bar{H}$ built on the basis $(\bar{\tau})$, that is the integrand of the general Abelian integral of the first kind, will then depend on $p$ arbitrary constants.

The same result is also otherwise apparent on recalling the orders of coincidence which define the general $\varphi$-function and the orders of coincidence possessed by the function $f'_u(z, u)$. For from these orders of coincidence it immediately follows that the integrand of the general Abelian integral of the first kind must be represented by the quotient of the general $\varphi$-function by the function $f'_u(z, u)$, and the general $\varphi$-function as we have already seen involves $p$ arbitrary constants. The general Abelian integral of the first kind we might then represent by the notation

$$\mathrm{I} = \int \frac{\varphi(z, u)}{f'_u(z, u)} dz$$

where $\varphi(z, u)$ is the general $\varphi$-function.

Suppose now that we introduce an extra infinity — an extra negative coincidence as we shall call it — into the basis $(\bar{\tau})$. The modified basis we shall indicate by the notation $(\bar{\tau}_1)$. At the same time a corresponding positive coincidence is introduced into the basis $(\tau)$, giving us a modified basis which we shall indicate by the notation $(\tau_1)$. The numbers of the basis $(\tau_1)$ are the same as the orders of coincidence of the function $f'_u(z, u)$ for the various values of $z$, together with an extra coincidence corresponding to some one cycle. The sum of the numbers of the basis $(\tau_1)$ is therefore 1 and the basis itself is an impossible basis for some one of the irreducible equations whose aggregate constitutes the fundamental algebraic equation. On indicating by $H_1$ and $\bar{H}_1$ the general rational functions built on the bases $(\tau_1)$ and $(\bar{\tau}_1)$ respectively, we shall then have $N_{H_1} = \rho - 1$ and from the formula (XIII, 9) we derive $N_{\bar{H}_1} = p$. The number of arbitrary constants involved in the general rational function $\bar{H}_1$ built on the basis $(\bar{\tau}_1)$, is then the same as the number of arbitrary constants involved in the general rational function $\bar{H}$ built on the basis $(\bar{\tau})$. The functions $\bar{H}_1$ and $\bar{H}$ then are one and the same, and the extra infinity introduced in the basis $(\bar{\tau}_1)$ is not actually possessed by the function $\bar{H}_1$. It follows that a rational function of $(z, u)$ cannot possess a single infinity of order 1, all its other infinities being of lower order. An Abelian integral which possesses a single logarithmic infinity and no other infinities therefore does not exist — what, for the rest, we know from the elementary theory of the residues of a rational function.

We shall now introduce two extra negative coincidences into the basis $(\bar{\tau})$, indicating the modified basis by the notation $(\bar{\tau}_2)$. Introducing at the same time the two corresponding positive coincidences into the basis $(\tau)$ we indicate the modified basis so obtained by the notation $(\tau_2)$. The formula (XIII, 9) then assumes the form

$$N_{\bar{H}_2} + \rho = N_{H_2} + p + 2$$

where $\bar{H}_2$ and $H_2$ represent the most general rational functions built on the respective bases $(\bar{\tau}_2)$ and $(\tau_2)$. If the two extra negative coincidences introduced into the basis $(\bar{\tau})$, and consequently also the two extra positive coincidences introduced into the basis $(\tau)$, correspond to the same irreducible equation, included in our fundamental algebraic equation, we evidently have $N_{H_2} = \rho - 1$ and from the formula above we derive $N_{\bar{H}_2} = p + 1$.

If the two extra coincidences introduced into either of the bases above correspond to different irreducible equations, included in the fundamental algebraic equation, we have $N_{H_2} = \rho - 2$ and therefore $N_{\bar{H}_2} = p = N_{\bar{H}}$.



The most general rational function $\bar{H}_2$ built on the basis $(\bar{\tau}_2)$ will then involve one more arbitrary constant than the most general rational function built on the basis $(\bar{\tau})$ if the two extra infinities involved in the former basis correspond to the same irreducible equation, included in our fundamental algebraic equation. The two extra infinities in question may or may not coincide. In the case where they are distinct the rational function $\bar{H}_2$ possesses two distinct infinities of the order 1 and the integral $\int \bar{H}_2 dz$ therefore possesses two logarithmic infinities and is an elementary Abelian integral of the third kind. In the case where they coincide the rational function $\bar{H}_2$ possesses an order of coincidence $-1 - \frac{1}{\nu_s^{(k)}}$ for the branches of a cycle of order $\nu_s^{(k)}$ corresponding to some finite value $z = a_k$, or an order of coincidence $1 - \frac{1}{\nu_s^{(\infty)}}$ for the branches of a cycle of order $\nu_s^{(\infty)}$ corresponding to the value $z = \infty$. In this case then the integral $\int \bar{H}_2 dz$ possesses a single algebraic infinity and is therefore an elementary Abelian integral of the second kind. That it does not at the same time possess a logarithmic infinity evidently follows from the principle of residues.

Let us now introduce an extra negative coincidence $i + 1$ times over into the basis $(\bar{\tau})$, indicating the modified basis by the notation $(\bar{\tau}_{i+1})$. Introducing at the same time the corresponding positive coincidence $i + 1$ times over into the basis $(\tau)$, we indicate the modified basis so obtained by the notation $(\tau_{i+1})$. The formula (XIII, 9) then assumes the form

$$N_{\bar{H}} + \rho = N_H + p + i + 1.$$

Here we evidently have $N_H = \rho - 1$ and consequently $N_{\bar{H}} = p + i$. The function $\bar{H}$ will have an order of coincidence $-1 - \frac{i}{\nu_s^{(k)}}$ with the branches of some one of the cycles corresponding to a finite value $z = a_k$, or an order of coincidence $1 - \frac{i}{\nu_s^{(\infty)}}$ with the branches of a cycle corresponding to the value $z=\infty$. The integral $\int \bar{H}\,dz$ will then have an infinity of order $\frac{i}{\nu_s^{(k)}}$ for a cycle corresponding to the value $z=a_k$ or an infinity of order $\frac{i}{\nu_s^{(\infty)}}$ for a cycle corresponding to the value $z=\infty$. Also we may evidently so dispose of $i-1$ out of the $p+i$ arbitrary constants involved in the expression of the general function $\bar{H}$ that a term in $(z-a_k)^{-\frac{i}{\nu_s^{(k)}}}$ — or $\left(\frac{1}{z}\right)^{-\frac{i}{\nu_s^{(\infty)}}}$ — is the only term of negative exponent which appears in the development of the integral $\int \bar{H}\,dz$ for a branch of the cycle of order $\nu_s^{(k)}$ — or $\nu_s^{(\infty)}$ — corresponding to the value $z=a_k$ — or $z=\infty$ —. The function $\bar{H}$ so restricted will then depend on $p+1$ arbitrary constants and the corresponding integral $\int \bar{H}\,dz$ may also be called an elementary Abelian integral of the second kind.

In the foregoing we have proved the existence of $p$ linearly independent Abelian integrals of the first kind, as also the existence of the elementary Abelian integrals of the second kind corresponding to any arbitrarily chosen cycle of the fundamental algebraic equation. We have at the same time proved the existence of elementary Abelian integrals of the third kind possessing logarithmic infinities corresponding to two arbitrarily chosen cycles of the fundamental algebraic equation, and we have furthermore seen that the integrand of an elementary Abelian integral in its most general form depends on p + 1 arbitrary constants.

It is evident that any Abelian integral can be represented as the sum of a number of elementary Abelian integrals. Also if we in any manner particularize the elementary Abelian integrals so that they involve no arbitrary constants, the general Abelian integral can evidently be represented as a linear expression with constant coefficients in such elementary Abelian integrals of the second and third kinds, together with an Abelian integral of the first kind. Again if $c_1, c_2, \ldots c_p$ be a set of $p$ coincidences of strength $p$ — that is an unconditioned set of $p$ coincidences — we know from the Riemann-Roch theorem that we can construct a rational function of $(z, u)$ which actually possesses any arbitrarily assigned infinity or combination of infinities, together with some or all of the infinities $c_1^{-1}, c_2^{-1}, \ldots c_p^{-1}$. If then $\int H(z,u)\,dz$

be an Abelian integral presenting any combination of algebraic infinities, it evidently follows that we can construct a rational function $H_1(z, u)$ presenting the same combination of infinities together with some or all of the infinities $c_1^{-1}, c_2^{-1}, \ldots c_p^{-1}$ and furthermore that the constants in this function may be so chosen that the only algebraic infinities presented by the difference



$$\int H(z, u)\, dz - H_1(z, u)$$

are included among the $p$ infinities $c_1^{-1}, c_2^{-1}, \ldots c_p^{-1}$. It follows that any Abelian integral can be represented in the form

$$\int H(z, u)\, dz = H_1(z, u) + \mathrm{I} + d_1 \mathrm{II}(c_1^{-1}) + d_2 \mathrm{II}(c_2^{-1}) + \cdots + d_p \mathrm{II}(c_p^{-1}) + \mathrm{III}$$

where $H_1(z, u)$ is a rational function of $(z, u)$, where I and III are Abelian integrals of the first and third kinds respectively, where the coefficients $d$ are constants, and where $\mathrm{II}(c_1^{-1}), \ldots \mathrm{II}(c_p^{-1})$ represent assigned elementary Abelian integrals of the second kind possessing respectively the infinities $c_1^{-1}, \ldots c_p^{-1}$. The Abelian integral of the third kind III is of course representable as a sum with constant coefficients of a number of elementary Abelian integrals of the third kind.

If we would consider the Abelian integrals with reference to their periodicity we might, from an arbitrarily selected point O in the $z$-plane, construct a number of closed loops about those points for which cycles of order greater than 1 exist, the number of loops corresponding to a cycle of order $\nu$ being $\nu - 1$. In all we should then have

$$\sum_k \sum_{s=1}^{r_k} (\nu_s^{(k)} - 1) = 2p + 2(n-1)$$

loops. Any closed path circuiting a number of the branch values of the variable $z$ may for our purpose be regarded as made up of a number of the loops in question, with repetitions it may be of the same loop and with descriptions of a loop in either sense. If we set out from the point O with a certain value of the variable $u$ and on completing the circuit of a number of the loops return to our point of departure with the value of $u$ with which we set out, we shall call the circuit a complete circuit.

It is evident that an Abelian integral which is not itself a rational function of $(z, u)$ must possess periods. Apart from the periods due to the presence of logarithmic infinities in an Abelian integral of the third kind a period will evidently be the result of effecting the indicated integration about a complete circuit. Now it might be shewn as in the treatise of Clebsch and Gordan*, that all complete circuits can be reduced to terms of some $2p$ complete circuits and we may therefore regard $2p$ as the number of the periods of an Abelian integral, apart from logarithmic periods.

That the $2p$ periods of the general Abelian integral are linearly independent of one another may readily be shewn. For let $\mathrm{I}_1, \mathrm{I}_2, \ldots \mathrm{I}_p$ represent a complete set of linearly independent integrals of the first kind and let $\mathrm{II}(c_1^{-1}), \mathrm{II}(c_2^{-1}), \ldots \mathrm{II}(c_p^{-1})$ be a set of $p$ elementary Abelian integrals of the second kind. In the event of a like linear relation holding between the corresponding periods of these $2p$ integrals we could so choose the coefficients $d$ in the expression

$$d_1 \mathrm{II}(c_1^{-1}) + \cdots + d_p \mathrm{II}(c_p^{-1}) + d_{p+1} \mathrm{I}_1 + \cdots + d_{2p} \mathrm{I}_p$$

that the periods of the Abelian integral represented by the sum would all be equal to 0. In such case the sum in question would have to represent a rational function of $(z, u)$ — what is evidently impossible if the set of $p$ coincidences $c_1, \ldots c_p$ have the strength $p$. If then the corresponding periods of the $2p$ integrals here in question are not connected by a like linear relation it is certainly impossible to assign a specific linear relation which holds for the periods of an arbitrary Abelian integral or of an arbitrary Abelian integral which presents no logarithmic infinities. In particular such a specific linear relation with *integer* coefficients does not exist. It follows therefore that no assigned combination of $2p-1$ periods, can be equivalent to the remaining period or to an assigned multiple of the remaining period for all Abelian integrals simultaneously or for all Abelian integrals which do not present logarithmic infinities.

* Theorie der Abelschen Functionen, p. 85.

A more detailed consideration of the periodic properties of the Abelian integrals and the treatment of their other transcendental properties do not lie within the scope of the present volume. Our object has been simply to develop an algebraic theory of the algebraic functions of a complex variable and there leave it to be utilized as may be found convenient in connection with the theory of the Abelian transcendents or elsewhere.