Chas B Sing

West Point April 26th 1836

(Boston)

OGS

ELEMENTS

OF

ALGEBRA,

TRANSLATED

FROM THE FRENCH OF M. BOURDON,

FOR THE USE OF THE CADETS

OF THE

U. S. MILITARY ACADEMY.

BY LIEUT. EDWARD C. ROSS,
Assistant Professor of Mathematics.

NEW-YORK:
E. B. CLAYTON—9 CHAMBERS-STREET.
PRINTED BY CLAYTON & VAN NORDEN,
No. 42 William-street.
1831.

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The whole of the *fourth*, *ninth*, and *tenth* chapters of the original, with the exception of the article upon Figurate Numbers, and the Series upon which they depend, have been omitted in the translation; also, that part of the *seventh* chapter which relates to the Theory of Symmetrical Functions. Some of the articles of the 8th chapter have also been omitted. In consequence of these omissions, the numbers which designate the different articles do not succeed each other in regular order.

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Thus, let 12 and 9 be the two numbers, their sum is 21 and their difference 3.

We will find that the product 21 × 3, or 63, is equal to 144, the square of 12, diminished by 81, which is the square of 9.

But to prove this property for any two numbers whatever, we will represent these two numbers by a and b.

The sum will be expressed by $a + b$, and the difference by $a - b$. In order to form the product of these two expressions, we will first suppose that $a + b$ is to be multiplied by a; the product will be $a \times a + b \times a$, or more simply $a^2 + ab$; for it is necessary to take each of the two parts composing $a + b$, as many times as there are units in a, and add together the two products. But it is not by the whole of a that this quantity is to be multiplied, but by a diminished by b; therefore the product $a^2 + ab$ is too great by the product of $a + b$ by b, that is, by $ab + b^2$. Hence $ab + b^2$ must be subtracted from the preceding product $a^2 + ab$; this indicated algebraically gives, $a^2 + ab - ab - b^2$. Now the two products $+ab$, $-ab$, mutually destroy each other, and the required product becomes $a^2 - b^2$.

This result, $a^2 - b^2$, having been obtained, independently of any particular value attributed to a and b, it follows that the theorem is true for any two numbers whatever.

Third Question. THEOREM.

6. If the same number is added to the two terms of a proper fraction, the new fraction resulting from this addition will be greater than the first.

Let the proposed fraction be $\frac{5}{12}$, if 3 be added to its two terms, it becomes $\frac{8}{15}$. These two fractions reduced to the same denominator, become $\frac{75}{180}$, $\frac{96}{180}$, and the second fraction is evidently greater than the first.

To ascertain whether this theorem is true for any fraction whatever, denote this fraction by $\frac{a}{b}$ supposing $a < b$.

Let m represent the number to be added to the two terms, it becomes $\frac{a+m}{b+m}$

In order to compare the two fractions, they should be reduced to the same denominator, which can be done by multiplying

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$4c^3+2c^2 d$, are homogeneous polynomials. $8a^3-4ab+c$ is not homogeneous.

12. Terms composed of the same letters, and affected with the same exponents, are called *similar terms*.

Thus, $7ab$ and $3ab$, $4a^3b^2$, and $5a^3b^2$, are similar terms. $8a^2b$ and $7ab^2$ are not similar terms, for although they are composed of the same letters, yet the same letters are not affected with the same exponents.

It often happens that a polynomial contains several similar terms; it may then be simplified.

Take the polynomial $4a^2b-3a^2c+9ac^2-2a^2b+7a^2c-6b^3$; it may (9) be written thus: $4a^2b-2a^2b+7a^2c-3a^2c+9ac^2-6b^3$; now, $4a^2b-2a^2b$ reduces to $2a^2b$. $7a^2c-3a^2c$ reduces to $4a^2c$; hence the polynomial becomes $2a^2b+4a^2c+9ac^2-6b^3$.

When we have, in any polynomial, the terms $+2a^3bc^2$, $-4a^3bc^2$, $+6a^3bc^2$, $-8a^3bc^2$, $+11a^3bc^2$; the sum of the additive terms may be reduced to $+19a^3bc^2$; and the sum of the subtractive terms to $-12a^3bc^2$; hence the five proposed terms reduce to $19a^3bc^2-12a^3bc^2$, or $7a^3bc^2$.

It may happen that the sum of the subtractive terms exceeds the sum of the additive terms. In this case, subtract the positive coefficient from the negative, and prefix the sign — to the result. Thus, when $+5a^2b$ is the sum of the positive, and $-8a^2b$ the sum of the negative terms, as $-8a^2b$ is the same as $-5a^2b-3a^2b$, it follows that $+5a^2b-8a^2b$ is equivalent to $+5a^2b-5a^2b-3a^2b$, or $-3a^2b$. Hence the following rule for the reduction of similar terms.

Form a single additive term of all the terms preceded by the sign +; which is done by adding together the coefficients of these terms, and annexing to this sum the common literal part. In the same manner form a single subtractive term with all the terms preceded by the sign —; then subtract the least sum from the greatest, and give the result the sign of the greater.

(It should be observed that the reduction affects only the coefficients, and not the exponents.)

From this rule we find that

$4a^2b-8a^2b-9a^2b+11a^2b$ reduces to $-2a^2b$;

$7abc^2-abc^2-7abc^2-8abc^2+6abc^2$ reduces to $-3abc^2$.

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tions, lead to such expressions,) *the product of these two polynomials is also homogeneous.* This is an evident consequence of the rules relative to the letters and exponents in the multiplication of monomials. Moreover, the degree of each term of the product should be equal to the sum of the degrees of any two terms of the multiplier and multiplicand. Thus, in the first of the two preceding examples, all the terms of the multiplicand being of the second degree, as well as those of the multiplier, all the terms of the product are of the fourth degree. In the second, the multiplicand being of the fifth degree, and the multiplier of the third, the product is of the eighth degree. This remark serves to discover any errors in the calculations with respect to the exponents. For example, if it is found that in one of the terms of a product that should be homogeneous, the sum of the exponents is equal to 6, while in all the others their sum is 7, there is a manifest error in the addition of the exponents, and the multiplication of the two terms which have formed this product must be revised.

Second. When, in the multiplication of two polynomials, the product does not present any similar terms for reduction, the total number of terms in the product is equal to the product of the number of terms in the multiplicand, multiplied by the number of terms in the multiplier. This is a consequence of the rule. (No. 17.) Thus, when there are five terms in the multiplicand, and four in the multiplier, there are 5×4, or 20, in the product. In general, when the multiplicand is composed of m terms, and the multiplier of n terms, the product contains $m \times n$.

Third. When some of the terms are similar, the total number of terms in the product, when reduced, may be much less. But we will remark, that among the different terms of the product, there are some that cannot be reduced with any others. These are, 1st. The term produced by the multiplication of the term of the multiplicand, affected with the highest exponent of a certain letter, by the term of the multiplier, affected with the highest exponent of the same letter. 2d. The term produced by the multiplication of the terms, affected with the lowest exponents of the same letter. For these two partial products should contain this letter, affected with a higher or lower exponent than either of the other partial products, and consequently cannot be similar to any of them. This remark, the truth of which is

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The manner in which an algebraic product is formed from its two factors, is called the *law* of this product; and this law remains always the same, whatever values may be attributed to the letters which enter in the two factors.

21. Lastly, a polynomial being given, it may sometimes be decomposed into factors merely by inspection.

Take the polynomial $25\,a^4 - 30\,a^3b + 15\,a^2b^2$, it is evident that 5 and a^2 are factors of each of the terms. We may, therefore, put the polynomial under the form $5\,a^2(5\,a^2 - 6\,a\,b + 3\,b^2)$. In the same way $64\,a^4\,b^6 - 25\,a^2\,b^8$ is transformed into $(8\,a^2\,b^3 + 5\,a\,b^4.)\,(8\,a^2\,b^3 - 5\,a\,b^4)$.

For, $64\,a^4\,b^6$ and $25\,a^2\,b^8$ being the squares of $8\,a^2\,b^3$ and $5\,a\,b^4$, it follows that the proposed expression is the difference of two squares, and that it can be decomposed (No. 19) into the sum of the root multiplied by their difference.

Division.

22. Algebraic division has the same object as arithmetical, viz.: having given a product and one of its factors, to find the other. We will first consider the case of two monomials.

The division of $72\,a^5$ by $8\,a^3$ is indicated thus: $\frac{72\,a^5}{8\,a^3}$.

It is required to find a third monomial, which, multiplied by the second, will produce the first. Now, by the rules for the multiplication of monomials, the required quantity must be such that its coefficient multiplied by 8 should give 72 for a product, and that the exponent of the letter a in this quantity, added to 3, the exponent of the letter a in the divisor, should give 5, the exponent of the dividend. This quantity may therefore, be obtained by dividing 72 by 8, and subtracting the exponent 3 from the exponent 5, which gives $\frac{72\,a^5}{8\,a^3} = 9\,a^2$.

$$\frac{35\,a^3b^2c}{7\,a\,b} = 5\,a^{3-1}b^{2-1}c = 5\,a^2b\,c\,;$$

$$\text{for } 7\,a\,b \times 5\,a^2b\,c = 35\,a^3b^2c.$$

Hence, we see that in order to divide one monomial by another, it is necessary, 1. *to divide the two coefficients one by the other.* 2. *For those letters which are common to the dividend and divisor, write each of them after the coefficient, and*

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As in multiplication the product of two terms having the same sign is affected with the sign +, and the product of two terms having contrary signs is affected with the sign —, we may conclude, 1st. That when the term of the dividend has the sign +, and that of the divisor the sign +, the term of the quotient must have the sign +. 2d. When the term of the dividend has the sign +, and that of the divisor the sign —, the term of the quotient must have the sign —, because it is only the sign —, which, combined with the sign —, can produce the sign + of the dividend. 3d. When the term of the dividend has the sign —, and that of the divisor the sign +, the quotient must have the sign —.

That is, when the two terms of the dividend and divisor have the same sign, the quotient will be affected with the sign +, and when they are affected with contrary signs, the quotient will be affected with the sign —; again, for the sake of brevity, we say that

+ divided by +, and — divided by —, give +;
— divided by +, and + divided by —, give —.

In the proposed example, $10a^4$ and $-5a^2$ being affected with contrary signs, their quotient will have the sign —; moreover, $10a^4$, divided by $5a^2$, gives $2a^2$, (No. 22;) hence, $-2a^2$ is a term of the required quotient. After having written it underneath the divisor, multiply each term of the divisor by it, and subtract the product, $-8a^3b+10a^4-6a^2b^2$, from the dividend, which is done by writing it below the dividend with contrary signs, and performing the reduction. Thus, the result of the first partial operation is

$$57a^2b^2-40a^3b-15b^4+4ab^3.$$

This result is composed of the partial products of each term of the divisor, by each term of the quotient which remains to be determined. We may then consider it as a new dividend, and reason upon it as upon the proposed dividend. We will therefore take in this result, the term $-40a^3b$, affected with the highest exponent of a, and divide it by the term $-5a^2$ of the divisor. Now, from the preceding principles, $-40a^3b$, divided by $-5a^2$ gives $+8ab$ for a new term of the quotient, which is written on the right of the first. Multiplying each term of the divisor by this term, and writing the products with

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sequently must be equal to the part Aa^4 of the dividend affected with the highest power of a. Hence, reciprocally, if we divide Aa^4 by $A'a^2$, we will have the part $A''a^2$ of the quotient; this reduces to dividing A by A', since a^4, divided by a^2, gives a^2. If A and A' are themselves polynomials, composed of one or more letters, we proceed with them as before stated, which requires that the two polynomials should be arranged with reference to one of the letters which enter them. This is the reason why we stated above, that in writing the terms affected with the same power in a column, care should be taken to arrange them with reference to a second letter; they should even be arranged with reference to a third letter, where two or more terms of a column contain the same power of the second letter.

The part $A''a^2$ being obtained, we multiply each part of the divisor by $A''a^2$, and subtract the partial products as soon as obtained; this gives a first remainder, upon which we operate as upon the proposed dividend.

The following are examples of this case:

1st.	$12b^2$ $-29bc$ $+15c^2$	a^3+23b^3 $-31b^2c$ $-9bc^2$ $+15c^3$	a^2+10b^4 $-6b^2c^2$	a	Divisor:	$3b$ $-5c$	$a+2b^2$
					Quotient:	$4b$ $-3c$	a^2+5b^2 $-3c^2$ | a
1st. Rem.		$+15b^3$ $-25b^2c$ $-9bc^2$ $+15bc^3$	a^2+10b^4 $-6b^2c^2$	a			
2d. Rem.			0				

First partial division.

$$\begin{array}{r|l} 12b^2-29bc+15c^2 & 3b-5c \\ \hline -9bc+15c^2 & 4b-3c \\ \hline 0 & \end{array}$$

Second partial division.

$$\begin{array}{r|l} 15b^3-25b^2c-9bc^2+15c^3 & 3b-5c \\ \hline -9bc^2+15c^3 & 5b^2-3c^2 \\ \hline 0 & \end{array}$$

2d. $6b$	a^4 $-7b^2$	a^3 $-3b^3$	a^2 $+4b^3$	$a+b^2-2b$	}	$3b$	$a+b^2-2b$	
-10	$+23b$	$+22b^2$	$-9b^2$			-5		
	-20	$+31b$	$+5b$			$2a^3-3b$	a^2+4b	$a+1$
		$+5$	-5			$+4$	-1	

1st. Rem.	$-9b^2$ $+27b$ -20	a^3
2d. Rem.	$+12b^2$ $-23b$ $+5$	a^2
3d. Rem.	$+3b$ -5	$a+b^2-2b$
4 Rem.	0	

First partial division.

$$\begin{array}{c|c} 6b-10 & 3b-5 \\ \hline 0 & 2 \end{array}$$

Second partial division.

$$\begin{array}{r|l} -9b^2+27b-20 & 3b-5 \\ \hline +12b-20 & -3b+4 \\ \hline 0 & \end{array}$$

Third partial division.

$$\begin{array}{r|l} 12b^2-23b+5 & 3b-5 \\ \hline -3b+5 & 4b-1 \\ \hline 0 & \end{array}$$

Fourth partial division.

$$\begin{array}{r|l} 3b-5 & 3b-5 \\ \hline & 1 \end{array}$$

30. There is another important case of algebraic division, viz.: *when the polynomial dividend contains one or more letters that are not contained in the divisor.* The polynomials might be arranged with reference to a common letter, and the division performed in the common way. But there is a much more simple method for obtaining the quotient.

Suppose, for example, that the dividend contained different powers of the letter a, and that this letter is not contained in the divisor, (it is then said to be *independent* of this letter.) Arranging the dividend with reference to a, it can be put under the form of $A\,a^4+B\,a^3+C\,a^2+D\,a+E$, 4 being supposed the highest exponent of a. Let M be the polynomial divisor, independent of a. Since the divisor multiplied by the quotient must reproduce the dividend, and as the divisor M does not contain a, it is plain that this quotient ought to be a polynomial involving the same powers of the letter a, as those which are found in the dividend. Hence, this quotient is necessarily of the form .... $A'a^4+B'a^3+C'a^2+D'a+E'$. Now, if we conceive that this quotient has been found, and that we have multiplied successively the entire divisor by each of the parts $A'a^4$, $B'a^3$, $C'a^2$, ... the products will be $A'Ma^4$, $B'Ma^3$, $C'Ma^2$ .... and as no reductions can take place between them, since the letter a is affected with different exponents, they must be respectively equal to the terms Aa^4, Ba^3, Ca^2, ... of the dividend.

Therefore, we have $A'M = A$ whence $A' = A : M$

$$B'M = B \quad .. \quad B' = B : M$$
$$C'M = C \quad .. \quad C' = C : M$$

and so with the others; hence the following general proposition. *In order that a polynomial arranged with reference to the powers of a certain letter, may be exactly divisible by a polynomial independent of this letter, it is necessary that each of the coefficients of the different powers of the first polynomial may be exactly divisible by the second. The coefficients of the different powers of the letter in the quotient, are the successive quotients arising from the division of the coefficients of the polynomial dividend by the polynomial divisor.*

Divide the polynomial $3a^2b^3 - 3abc^3 - 2b^3c^2 + b^5 - 3a^2bc^2 + 3ab^3c - a^2c^3 + bc^4 + a^2b^2c$, by $b^2 - c^2$.

The dividend arranged with reference to a, may be put under the form $(3b^3 + b^2c - 3bc^2 - c^3)a^2 + (3b^3c - 3bc^3)a + b^5 - 2b^3c^2 + bc^4$; effecting the three partial divisions denoted by

$$\frac{3b^3 + b^2c - 3bc^2 - c^3}{b^2 - c^2}, \qquad \frac{3b^3c - 3bc^3}{b^2 - c^2},$$
$$\frac{b^5 - 2b^3c^2 + bc^4}{b^2 - c^2}$$

we find the quotients to be: $3b + c$, $3bc$, $b^3 - bc^2$; therefore, the total quotient is, $(3b + c)a^2 + 3bca + b^3 - bc^2$.

The two last quotients can be obtained more easily than by the ordinary process, by observing, 1st, that $3b^3c - 3bc^3$ is equivalent (21) to $3bc(b^2 - c^2)$; 2d, that $b^5 - 2b^3c^2 + bc^4$ is equivalent to $b(b^4 - 2b^2c^2 + c^4)$, or $b(b^2 - c^2)^2$, (19).

It should be observed that, where there exist general rules for performing all the operations required, these rules may frequently be simplified, and we should never omit employing these simplifications, when the occasion presents itself. By so doing, we enter more fully into the spirit of the algebraic language.

31. Among the different examples of algebraic division, there is one remarkable for its applications. It is so often met with in the resolution of questions, that algebraists have made a kind of *theorem* of it. We have seen (5 and 19) that $(a+b)\times(a-b)$ gives a^2-b^2 for a product; hence, reciprocally a^2-b^2 divided by $a-b$ gives $a+b$ for a quotient. Dividing a^3-b^3 by $a-b$, the quotient is a^2+ab+b^2. In like manner a^4-b^4 divided by $a-b$ gives $a^3+a^2b+ab^2+b^3$ for the quotient. These are results that may be obtained by the ordinary process of division. Analogy would lead to the conclusion that whatever may be the exponents of the letters a and b, the division could be performed exactly; but analogy does not always lead to certainty. To be certain on this point, denote the exponent by m; and proceed to divide a^m-b^m by $a-b$.

$$a^m-b^m \;\big\}\; \frac{a-b}{a^{m-1}}$$

1st Rem. $\quad +a^{m-1}b-b^m$

or, $\quad b\,(a^{m-1}-b^{m-1})$

Dividing a^m by a the quotient is a^{m-1}, by the rule for the exponents (22). The product of $a-b$ by a^{m-1} being subtracted from the dividend, the first remainder is $a^{m-1}b-b^m$, which can be put under the form $b\,(a^{m-1}-b^{m-1})$. Whence, if we suppose $a^{m-1}-b^{m-1}$ divisible by $a-b$, then will a^m-b^m also be divisible by it; that is, if the difference of the similar powers of two quantities of *a certain degree*, is exactly divisible by the difference of these quantities, the difference of the powers of a degree greater by unity, is also divisible by it. Now $\frac{a^2-b^2}{a-b}$ gives an exact quotient, equal to $a+b$; hence, $\frac{a^3-b^3}{a-b}$ gives an exact quotient, and is equal to $a^2+\frac{b(a^2-b^2)}{a-b}$, or $a^2+b(a+b)$, or a^2+ab+b^2. Again $\frac{a^4-b^4}{a-b}$ gives an exact quotient, and is equal to $a^3+b.\frac{(a^3-b^3)}{a-b}$ or $a^3+b\,(a^2+ab+b^2)$ or $a^3+a^2b+ab^2+b^3$.

In general, $\frac{a^m-b^m}{a-b}$ gives an exact quotient, and is equal to $a^{m-1}+a^{m-2}b+a^{m-3}b^2+\ldots\ldots+ab^{m-2}+b^{m-1}$.

The accuracy of this proposition is verified *à posteriori*, by performing the multiplication,

$$(a^{m-1} + a^{m-2}b + \ldots . a b^{m-2} + b^{m-1})(a - b).$$

It will be perceived that the partial products a^m and $-b^m$ are the only ones that do not destroy each other in the reduction.

For example, multiplying $a^{m-2}b$ by a, the product is $a^{m-1}b$; but by multiplying a^{m-1} by $-b$, the product is $-a^{m-1}b$, and this term destroys the preceding. The other terms cancel in the same way. The beginner should reflect upon the first method of demonstrating the proposition, as it is frequently employed in algebra.

32. We have given (23, 26) the principal circumstances by which it may be discovered that the division of monomial or polynomial quantities is not exact; that is, the case in which there does not exist a third entire algebraic quantity, which, multiplied by the second, will produce the first.

We will add, as to polynomials, that it may often be discovered by mere inspection that they cannot be divided by each other. When these polynomials contain two or more letters, before arranging them with reference to a particular letter, observe the two terms of the dividend and divisor, which are affected with the highest exponent of each of the letters. If for one of these letters, the terms with the highest exponents are not divisible by each other, we may conclude that the total division is impossible. This remark applies to each of the operations required by the process for finding the quotient.

Take, for example, $12a^3 - 5a^2b + 7ab^2 - 11b^3$, to be divided by $4a^2 - 8ab + 3b^2$.

By considering only the letter a, the division would appear possible; but regarding the letter b, the division is impossible, since $-11b^3$ is not divisible by $3b^2$.

One polynomial cannot be divided by another containing a letter which is not found in the dividend; for it is impossible that a third quantity, multiplied by a second, depending upon a certain letter, should give a product independent of this letter.

A monomial is never divisible by a polynomial, because (18) every polynomial multiplied by another, gives a product containing at least two terms which are not susceptible of reduction.

OF ALGEBRAIC FRACTIONS.

Greatest Common Divisor.

33. Algebraic fractions should be considered in the same point of view as arithmetical fractions, such as $\frac{3}{4}$, $\frac{1}{1}\frac{1}{2}$, that is, we must conceive that the unit has been divided into as many equal parts as there are units in the denominator, (which may be either a monomial or polynomial,) and that one of these parts is taken as many times as there are units in the numerator. Hence, addition, subtraction, multiplication, and division, are performed according to the rules established for arithmetical fractions. We should, however, in the application of the rules, follow the procedures indicated for the calculus of entire algebraic quantities. It will not, therefore, be necessary to give any examples of these rules; we will hereafter have frequent opportunities of becoming familiar with them.

The reduction of fractions to their most simple terms should have some particular developements.

When a division of monomial or polynomial quantities cannot be performed exactly, it is indicated by means of the known sign, and in this case, the quotient is presented under the form of a fraction, which we have already learned how to simplify. (23.) With respect to polynomial fractions, the following are cases which are easily reduced.

Take, for example, the expression $\frac{a^2-b^2}{a^2-2ab+b^2}$

This fraction can (19) be put under the form $\frac{(a+b)(a-b)}{(a-b)^2}$

Suppressing the factor $a-b$, which is common to the two terms, we obtain $\frac{a+b}{a-b}$

Again, take the expression $\frac{5a^3-10a^2b+5ab^2}{8a^3-8a^2b}$

This expression can be decomposed thus: $\frac{5a(a^2-2ab+b^2)}{8a^2(a-b)}$

or $\frac{5a(a-b)^2}{8a^2(a-b)}$.

Suppressing the common factor, $a(a-b,)$ the result is $\frac{5(a-b)}{8a}$

The particular cases examined above, are those in which the two terms of the fraction can be decomposed into the product of the sum, by the difference of two quantities, and into the square of the sum or difference of two quantities. Practice teaches the manner of performing these decompositions, when they are possible.

But the two terms of the fraction may be more complicated polynomials, and then, their decomposition into factors not being so easy, we have recourse to the process for finding *the greatest common divisor.*

This theory, which is intimately connected with equations, presents some difficulties; it is therefore our intention to give only a part of it here, and resume it again when we shall have acquired the materials necessary to establish it in a complete and rigourous manner.

Of the Greatest Algebraic Common Divisor.

34. Definition. *The greatest common divisor of two polynomials, is the greatest polynomial (with reference to the exponents and coefficients) that will exactly divide the proposed polynomials.*

The characteristic property of the greatest common divisor is, that after having divided the two proposed polynomials by their greatest common divisor, the quotients are *prime with respect to each other;* that is to say, they no longer contain a common factor.

This proposition is evident; for let A and B be the given polynomials, D their common divisor, A' and B' the quotients, we have

$$A = A' \times D \text{ and } B = B' \times D.$$

Now, if A' and B' had a common factor d, it would follow that $d \times D$ would be a divisor, common to the two polynomials, and greater than D, either with respect to the exponents or the coefficients, which would be contrary to the definition.

35. We have seen in arithmetic, 1st. *That the greatest common divisor of two entire numbers contains, as factors, all of the particular divisors common to the two numbers, and does not contain any other factors.* 2d. *That the greatest common divisor of two entire numbers, is the same as that which exists*

between the smallest number and the remainder, after the division.

The theory of the greatest algebraic common divisor also depends upon these two principles, for the demonstration of which see the sixth chapter.

Admitting these principles, let us suppose that it is required to find the greatest common divisor between the two polynomials

$$a^3 - a^2 b + 3 a b^2 - 3 b^3, \text{ and } a^2 - 5 ab + 4 b^2$$

First Operation.

$$\begin{array}{ll} & a^3 - a^2 b + 3 a b^2 - 3 b^3 \left.\right\} \; \dfrac{a^2 - 5 a b + 4 b^2}{a + 4 b} \\ & \underline{+ 4 a^2 b - a b^2 - 3 b^3} \\ \text{1st Rem.} & 19\, a\, b^2 - 19\, b^3 \\ \text{or} & 19\, b^2\, (a - b.) \end{array}$$

Second Operation.

$$\begin{array}{l} a^2 - 5 a b + 4 b^2 \left.\right\} \; \dfrac{a - b}{a - 4 b} \\ \underline{- 4 a b + 4 b^2} \\ \quad 0 \end{array}$$

Hence $a - b$ is the greatest common divisor.

We begin by dividing the polynomial of the highest degree by that of the lowest degree; the quotient is, as we see in the above table, $a + 4 b$, and the remainder is $19 a b^2 - 19 b^3$.

By the second principle, the required common divisor is the same as that which exists between this remainder and the polynomial divisor.

But $19 ab^2 - 19 b^3$ can be put under the form $19 b^2 (a - b.)$ Now the factor, $19 b^2$, will divide this remainder without dividing $a^2 - 5 ab + 4b^2$, hence, by the first principle, this factor cannot enter into the greatest common divisor; we may therefore suppress it, and the question is reduced to finding the greatest common divisor between $a^2 - 5 a b + b^2$ and $a - b$.

Dividing the first of these two polynomials by the second, there is an exact quotient, $a - 4 b$; hence $a - b$ is their greatest common divisor, and it is consequently the greatest common divisor of the two proposed polynomials.

Again, take the same example, and arrange the polynomials with reference to b.

$$-3 b^3 + 3 ab^2 - a^2 b + a^3, \text{ and } 4 b^2 - 5 a b + a^2$$

First Operation.

$$-12b^3+12ab^2-4a^2b+4a^3 \;\big|\; 4b^2-5ab+a^2$$

1st Rem. $\quad -3ab^2-a^2b+4a^3 \;\big|\; -3b, -3a$,

$\quad -12ab^2-4a^2b+16a^3$

2d Rem. $\quad -19a^2b+19a^3$

or $\quad 19a^2(-b+a.)$

Second Operation.

$$4b^2-5ab+a^2 \;\big|\; -b+a$$

$$-ab+a^2 \;\big|\; -4b+a$$

$$0$$

Hence $-b+a$, or $a-b$, is the greatest common divisor.

Here we meet with a difficulty in dividing the two polynomials, because the first term of the dividend is not exactly divisible by the first term of the divisor. But if we observe that the coefficient 4 of this last is not a factor of all the terms of the polynomial $4b^2-5ab+a^2$, and that therefore, by the first principle, 4 cannot form a part of the greatest common divisor, we can, without affecting this common divisor, introduce this factor into the dividend. This gives .... $-12b^3+12ab^2-4a^2b+4a^3$, and then the division of the two first terms is possible.

Effecting this division, the quotient is $-3b$, and the remainder is $-3ab^2-a^2b+4a^3$.

As the exponent of b in this remainder is still equal to that of the divisor, the division may be continued, by multiplying this remainder by 4, in order to render the division of the two first terms possible.

This done, the remainder becomes $-12ab^2-4a^2b+16a^3$, which divided by ..... $4b^2-5ab+a^2$, gives the quotient $-3a$, (which should be separated from the first by a comma, as having no connexion with it,) and the remainder is ..... $-19a^2b+19a^3$.

Placing this last remainder under the form $19a^2(-b+a,)$ and suppressing the factor $19a^2$, as forming no part of the common divisor, the question is reduced to finding the greatest common divisor between $4b^2-5ab+a^2$, and $-b+a$.

Dividing the first of these polynomials by the second, we obtain an exact quotient, $-4b+a$; hence $-b+a$, or $a-b$, is the greatest common divisor required.

36. In the above example, as in all those in which the expo-

nent of the principal letter is greater by unity in the dividend than in the divisor, we can abridge the operation by multiplying every term of the dividend by the square of the coefficient of the first term of the divisor. We may easily conceive that, by this means, the first partial quotient obtained will contain the first power of this coefficient. Multiplying the divisor by the quotient, and making the reductions with the dividend thus prepared, the result will still contain the coefficient as a factor, and the division can be continued until a remainder is obtained of a lower degree than the divisor, with reference to the principal letter.

See the following table.

First Operation.

$$\begin{array}{ll} -48b^3+48ab^2-16a^2b+16a^3 & \left\} \begin{array}{l} 4b^2-5ab+a^2 \\ \hline -12b-3a \end{array} \right. \\ \quad -12ab^2-4a^2b+16a^3 & \\ \hline \text{1st Rem.} \quad -19a^2b+19a^3 & \\ \text{or,} \quad -19a^2(-b+a) & \end{array}$$

Second Operation.

$$\begin{array}{ll} 4b^2-5ab+a^2 & \left\} \begin{array}{l} -b+a \\ \hline -4b+a \end{array} \right. \\ \quad -ab+a^2 & \\ \hline \quad 0 & \end{array}$$

N. B. When the exponent of the principal letter in the dividend exceeds that of the same letter in the divisor by two, three, ... units, multiply the dividend by the third, fourth, ... power of the coefficient of the first term of the divisor. It is easy to see the reason of this.

37. For another example, take the two polynomials

$$15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4$$

and $$12a^3b^2+38a^2b^3+16ab^4-10b^5.$$

Before proceeding to the division of these polynomials, it should be observed, that the first contains a, as a factor common to all its terms; and since this factor does not enter into the second, it can be suppressed, as not forming a part of the common divisor.

For the same reason, the factor $2b^2$, common to all the terms of the second polynomial, and not entering into the first, can be suppressed. Therefore, the question is reduced to finding the common divisor between the polynomials

$$15a^4+10a^3b+4a^2b^2+6ab^3-3b^4$$

and $$6a^3+19a^2b+8ab^2-5b^3.$$

First Operation.

$$\begin{array}{l|l} 30a^4+20a^3b+8a^2b^2+12ab^3-6b^4 & 6a^3+19a^2b+8ab^2-5b^2 \\ -75a^3b-32a^2b^2+37ab^3-6b^4 & \overline{5a,\ -25b} \\ -150a^3b-64a^2b^2+74ab^3-12b^4 & \\ \hline \text{1st Rem.} +411a^2b^2+274ab^3-137b^4 & \\ \text{or}\quad 137b^2\ (3a^2+2ab-b^2.) & \end{array}$$

Second Operation.

$$\begin{array}{l|l} 6a^3+19a^2b+8ab^2-5b^3 & 3a^2+2ab-b^2 \\ +15a^2b+10ab^2-5b^3 & \overline{2a+5b} \\ \hline 0 & \end{array}$$

Hence $3a^2+2ab-b^2$ is the greatest common divisor.

By following the same method as in the preceding example, in would be necessary to multiply the whole dividend by the coefficient 6 of the first term of the divisor, or rather by the square of 6; but as 15 and 6 have a common factor, 3, it is evidently sufficient to multiply the dividend by 2, which is a factor of 6, but not of 15.

After this preparation the division is performed, and we obtain a remainder, the first term of which is $-75a^3b$. As 75 contains the factor 3, which is also a factor of 6, it is only necessary to multiply this remainder by 2, in order to continue the division, which being effected, gives for the first *principal remainder*, $411a^2b^2+274ab^3-137b^4$. Now it is easy to perceive that $137b^2$ is a factor of this remainder, and since it is not a factor of the second polynomial, it may be suppressed, as not forming a part of the common divisor; and the question is reduced to finding the greatest common divisor between the polynomials

$$6a^3+19a^2b+8ab^2-5b^3$$

and $$3a^2+2ab-b^2.$$

Dividing the first of these polynomials by the second, we obtain an exact quotient, $2a+5b$; therefore, the remainder $3a^2+2ab-b^2$ is the greatest common divisor required.

38. *Remark.* It might be asked if the suppression of the factors, common to all the terms of one of the remainders, is *absolutely necessary,* or whether the object is merely to render the

operations more simple. Now, it will easily be perceived that the suppression of these factors is necessary; for, if the factor $137\,b^2$ was not suppressed in the preceding example, it would be necessary to multiply the whole dividend by this factor, in order to render the first term of the dividend divisible by the first term of the divisor; but then, a factor would be introduced into the dividend which was also contained in the divisor; and consequently the required greatest common divisor would be combined with the factor $137\,b^2$, which should not form a part of it.

The following example will serve to illustrate what has just been said.

39. Find the greatest common divisor between the two polynomials,

$$a\,b+2\,a^2-3\,b^2-4\,b\,c-a\,c-c^2$$

and $$9\,a\,c+2\,a^2-5\,a\,b+4\,c^2+8\,b\,c-12\,b^2$$

First operation.

$$\left.\begin{array}{r|l} 2\,a^2+b & a-3\,b^2 \\ -c & -4\,b\,c \\ & -c^2 \end{array}\right\}\begin{array}{r|l} 2\,a^2-5\,b & a-12\,b^2 \\ +9\,c & +8\,b\,c \\ & +4\,c^2 \\ \hline & 1 \end{array}$$

$$\text{1st Rem.}\quad \begin{array}{r|l} 6\,b & a+9\,b^2 \\ -10\,c & -12\,b\,c \\ & -5\,c^2 \end{array}$$

or ..... $$(3\,b-5\,c)\,(2\,a+3\,b+c).$$

Second operation.

$$\left.\begin{array}{r|l} 2\,a^2-5\,b & a-12\,b^2 \\ +9\,c & +8\,b\,c \\ & +4\,c^2 \end{array}\right\}\begin{array}{l} 2\,a+3\,b+c \\ \hline a-4\,b \\ \quad +4\,c \end{array}$$

$$\begin{array}{r|l} -8\,b & a-12\,b^2 \\ +8\,c & +8\,b\,c \\ & +4\,c^2 \\ \hline & 0. \end{array}$$

Hence, $2\,a+3\,b+c$ is the greatest common divisor. After arranging the two polynomials, the division may be performed without any preparation, and the first remainder will be

$$\begin{array}{r|l} 6\,b & a+9\,b^2 \\ -10\,c & -12\,b\,c \\ & -5\,c^2. \end{array}$$

To continue the operation, it is necessary to take the second polynomial for a dividend, and this remainder for a divisor, and multiply the new dividend by $6b-10c$, or simply by ..... $3b-5c$, because 2 is a factor of the first term of the dividend; but before effecting this multiplication, see if this factor ..... $3b-5c$ will not divide the second term of the remainder, viz.: $9b^2-12bc-5c^2$. It does, and gives an exact quotient, $3b+c$; whence it follows, that the remainder can be put under the form $(3b-5c.)\ (2a+3b+c)$.

As $3b-5c$ is a factor of this remainder, and not a factor of the new dividend, (since this factor being independent of the letter a, it should (30) divide each of the coefficients of the different powers of this letter, which it does not,) we may suppress it without affecting the greatest common divisor.

This suppression is indispensable, for, otherwise, a new factor would be introduced into the dividend, and then, the two polynomials containing a factor that they had not before, the greatest common divisor would be changed; it would be combined with the factor $3b-5c$, which should not form a part of it.

Suppressing this factor, and effecting the new division, we obtain an exact quotient; hence, $2a+3b+c$, is the greatest common divisor.

40. For another example, it is proposed to find the greatest common divisor between the two polynomials,
$a^4+3a^3b+4a^2b^2-6ab^3+2b^4$ and $4a^2b+2ab^2-2b^3$, or simply, $2a^2+ab-b^2$, since the factor $2b$ can be suppressed in the second polynomial,

First operation.

$$\begin{array}{l|l} 8a^4+24a^3b+32a^2b^2-48ab^3+16b^4 & 2a^2+ab-b^2 \\ \quad +20a^3b+36a^2b^2-48ab^3+16b^4 & 4a^2+10ab+13b^2 \\ \qquad\qquad +26a^2b^2-38ab^3+16b^4 & \end{array}$$

1st Rem. ..... $-51ab^3+29b^4$

or ..... $-b^3(51a-29b)$.

Second operation.

Multiply by 2601, the square of 51.

$$\begin{array}{l|l} 5202a^2+2601ab-2601b^2 & 51a-29b \\ -5202a^2+2958ab & 102a+109b \\ \qquad +5559ab-2601b^2 & \\ \qquad -5559ab+3161b^2 & \end{array}$$

2d Rem. .... $+560b^2$

The exponent of the letter a in the dividend, exceeding that of the same letter in the divisor by *two* units, we multiply the whole dividend by the cube of 2, or 8. This done, we perform three consecutive divisions, and obtain for the first principal remainder, $-51\,ab^3+29\,b^4$. Suppressing b^3 in this remainder, it becomes $-51\,a+29\,b$ for a new divisor, or, changing the signs, which is permitted, $51\,a-29\,b$: the new dividend is $2a^2+ab-b^2$.

Multiplying this dividend by the square of 51, or 2601, then effecting the division, we obtain for the second principal remainder, $+560\,b^2$, which proves that the two proposed polynomials are *prime with respect to each other*, that is, they have not a common factor. In fact it results from the second principle (35), that the greatest common divisor must be a factor of the remainder of each operation; therefore it should divide the remainder $560\,b^2$; but this remainder is *independent* of the principal letter a; hence, if the two polynomials [illegible] divide the preceding remainder; this last remainder will be the greatest common divisor; but if a remainder is obtained which is *independent* of the principal letter, it indicates that the proposed polynomials are *prime with respect to each other*, or that they have not a common factor, which might not have been discovered at the commencement of the operation.

The above rule applies to the following examples.

1st. $qnp^3+3np^2q^2-2npq^3-2nq^4,$
and $2mp^2q^2-4mp^4-mp^3q+3mpq^3.$
The g. c. d. is $p-q$;

2d. $36a^6-18a^5-27a^4+9a^3,$
and $27a^5b^2-18a^4b^2-9a^3b^2.$
The g. c. d. is $9a^3(a-1)$.

CHAPTER II.

Of Problems of the First Degree.

42. In algebra we commonly consider those problems only, the enunciation of which, when written algebraically, form *equations*. By reflecting upon the resolution of the problem, (No. 3,) we perceive that it is composed of two distinct parts. In the first, the relations established between the known and unknown quantities by the enunciation of the question, are written algebraically. The expression for two equal quantities is thus obtained. This expression is called an *equation*. Such is (No. 3) the expression $2x+b=a$. In the second part, from the enunciation of the problem a series of equations is deduced, the last of which gives the value of the unknown in terms of the known quantities. Such is the result $x=\frac{a-b}{2}$.

As the rules for the first part are rather vague, we will consider at present the second part, which is subjected to fixed and invariable rules.

By the definition, every *equation* is composed of two parts, separated from each other by the sign =.

The part on the left is called the *first member*, and the part on the right the *second member*.

There are several kinds of equalities to be considered.

1st. The equality existing between numbers which are known and given *à priori*, but represented by letters; such are the equalities $a-b=c-d$, $\frac{a}{b}=\frac{c}{d}$ which would be verified immediately, if in place of a, b, c, d, the particular numbers were substituted for which these equalities are supposed to exist.

They are called *equalities*.

2d. The equality which is evident of itself, that which is verified in its actual condition; such are the equalities

$$25=12+13;\ 3a-5b=a-b+2a-4b.$$

These are called *verified equalities.*

3d. The equality which is not verified until after having substituted in the place of one or more of the letters denoting the unknown quantities, certain numbers, of which the values depend upon the known and given numbers in the equality.

To distinguish this equality from others, it is called an *equation.*

There is yet another kind of equality; it is called the *identical equation.* We will speak of it hereafter.

Equations are divided into different classes. Those which contain only the first power of the unknown quantity, are called equations of the *first degree.* Such are the equations $3x+5=17-5x$, $ax+b=cx+d$.

The equation $2x^2-3x=5-2x^2$, is of the 2d degree.

The equation $4x^3-5x^2+x=2x^2+11$, is of the 3d degree.

In general, the *degree* of an equation is denoted by the greatest of the exponents with which the unknown quantity is affected.

Equations are also distinguished as *numerical equations* and *literal equations.* The first are those which contain particular numbers only, with the exception of the unknown quantity, which is always denoted by a letter. Thus, $4x-3=2x+5$, $3x^2-x=8$, are numerical equations. They are the algebraical translation of problems, in which the known quantities are particular numbers.

The equations $ax-b=cx+d$, $ax^2+bx=c$, are literal equations. The given quantities of the problem are represented by letters. It is customary, in order to distinguish the known from the unknown quantities in an equation, to denote the latter by the last letters of the alphabet, x, y, z, &c.

We will now see how (an equation of the first degree with a single unknown quantity being given) we arrive at the *solution; that is, find for the unknown quantity a number, which, substituted in its place in the given equation, will satisfy it; i. e. render the first member identically equal to the second.*

§ 1. *Equations of the First Degree, containing but one unknown quantity.*

43. It may be considered as an axiom, that we can, without

destroying the equality, 1st. *Add* the same number to, or *subtract* it from its two members. 2d. *Multiply* or *divide* its two members by the same number; that is to say, that if the two members were equal before, they will still be equal when changed, as above stated.

The following transformations are of continual use in the resolution of equations.

First Transformation. When the two members of an equation are entire polynomials, it is often necessary to transpose certain terms from one member to the other.

Take the equation $5x-6=8+2x$.

If, in the first place, we subtract $2x$ from both members, the equality is not destroyed, and we have $5x-6-2x=8$. Whence we see that the term $2x$, which was additive in the second member, becomes subtractive in the first. In the second place, if we add 6 to both members, the equality still exists, and we have

$$5x-6-2x+6=8+6.$$

Or, as the terms -6, $+6$, destroy each other,

$$5x-2x=8+6.$$

Hence the term which was subtractive in the first member, passes into the second member with the sign of addition.

Again, take the equation $ax+b=d-cx$. If we add cx to both members, and subtract b from them, the equation becomes

$$ax+b+cx-b=d-cx+cx-b.$$

Or, reducing $$ax+cx=d-b.$$

Therefore, *when a term is transposed from one member to the other, its sign must be changed.*

44. *Second Transformation.*—Again, it often happens that some of the terms of an equation are fractions, and it is necessary to reduce it to another, in which all the terms are entire. Take the equation, ...

$$\frac{2x}{3}-\frac{3}{4}=11+\frac{x}{5}.$$

First reduce all the fractions to the same denominator, by the known rule; the equation becomes,

$$\frac{40x}{60}-\frac{45}{60}=11+\frac{12x}{60};$$

and, since we can (No. 43) multiply both members by the same number, we will multiply them by 60, which amounts to suppressing the denominator 60 in the fractional terms, and multiplying the entire term by 60; the equation then becomes

$$40x-45=660+12x.$$

Here we will remark, that we can pass immediately from the proposed equation, to that we wish to obtain, by omitting the common denominator, and multiplying each of the entire terms by this denominator.

Take for a second example, $\frac{5x}{12}-\frac{4x}{3}-13=\frac{7}{8}-\frac{13x}{6}$.

Some of the denominators evidently have common factors, and the smallest number which is a *multiple of these denominators*, is 24. (See Arithmetic, No. 152.) Hence all the fractions must be reduced to this denominator. Performing this operation, and omitting the common denominator 24, the equation becomes $10x-32x-312=21-52x$.

(The entire term—13, we perceive, has been multiplied by 24.)

This equation is exact, since after having reduced the fractions to the same denominator, we have multiplied both members by the same number 24.

From what has been said, we can deduce the following rule: *To make the denominators disappear from an equation, form the most simple multiple of all the denominators, (this number is the product of all the denominators, when there are no common factors;) then multiply each of the entire terms by this multiple, and each of the fractional terms by the quotient of this multiple, divided by the denominator of the term thus multiplied, and omit the denominator of this term.*

To apply this rule, take the equation

$$\frac{ax}{b}-\frac{2c^2x}{ab}+4a=\frac{4bc^2x}{a^3}-\frac{5a^3}{b^2}+\frac{2c^2}{a}-3b.$$

The most simple multiple of all the denominators is evidently a^3b^2; therefore, multiply each entire term by a^3b^2, and each fractional term by the quotient of a^3b^2, divided by the denominator of this term. After performing these operations, the equation becomes,

$$a^4bx-2a^2bc^2x+4a^4b^3=4b^3c^2x-5a^6+2a^2b^2c^2-3a^3b^3.$$

45. We will now apply the preceding principles to the resolution of the equation,

$$4x-3=2x+5.$$

By transposing the terms — 3 and $2x$, it becomes

$$4x - 2x = 5 + 3.$$

Or reducing, $2x = 8$.

Dividing both numbers by 2, we find, $x = \frac{8}{2} = 4$

Now, if 4 be substituted in place of x in the equation, it becomes $4 \times 4 - 3 = 2 \times 4 + 5$, or $13 = 13$.

Again, take for an example the equation (No. 44).

$$\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$$

By making the denominator disappear,

$$10x - 32x - 312 = 21 - 52x.$$

By transposing the term which contains x into the first member, and the known terms into the second, we have

$10x - 32x + 52x = 21 + 312$, or reducing $30x = 333$.

Dividing both members by 30, we obtain $x = \frac{333}{30} = \frac{111}{10}$; a result which may be verified by substituting it for x in the proposed equation.

Take the equation $(3a - x)(a - b) + 2ax = 4b(x + a)$. It is first necessary to perform the multiplications indicated, in order to reduce the two members to two polynomials, and thus be able to disengage the unknown quantity x, from the known quantities. Now if we apply the rule (No. 17) for the multiplication of polynomials, the equation becomes . . . . $3a^2 - ax - 3ab + bx + 2ax = 4bx + 4ab$; transposing and reducing $ax - 3bx = 7ab - 3a^2$. Observe now that $ax - 3bx$ is the same thing as $(a - 3b)x$, hence . . . . $(a - 3b)x = 7ab - 3a^2$. Dividing both members by $a - 3b$, we find

$$x = \frac{7ab - 3a^2}{a - 3b}.$$

In general, in order to resolve an equation of the first degree, however complicated it may be, it is necessary, 1st. *To commence by making the denominators disappear, if there are any, and perform all the algebraic operations indicated in both members; we thus obtain an equation, the two members of which are entire polynomials.* 2d. *Transpose into the same member (commonly the first) all the terms affected with the unknown quantity, and into the other all of the known terms.* 3d. Reduce *to a single term all the terms involving x, if the equation is numerical; and if it is algebraical, of all these terms form a single product, composed of two factors, one of which shall be x, and the others all the multipliers of x, connected with their respective signs.* 4th. *Divide both members by the number or polynomial by which the unknown quantity is multiplied.*

The following example is sufficiently complicated to enable us to illustrate every part of the preceding rule. Resolve the equation

$$\frac{(a+b)(x-b)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b};$$

By first making the denominators disappear, it becomes

$$b(a+b)^2(x-b)-3ab(a^2-b^2)=b(a-b)(4ab-b^2)-2b(a^2-b^2)x+(a^2-b^2)(a^2-bx);$$

performing the multiplications indicated,

$$a^2bx+2ab^2x+b^3x-a^2b^2-2ab^3-b^4-3a^3b+3ab^3=4a^2b^2-ab^3-4ab^3+b^4-2a^2bx+2b^3x+a^4-a^2b^2-a^2bx+b^3x;$$

transposing and reducing

$$4a^2bx+2ab^2x-2b^3x=4a^2b^2-6ab^3+2b^4+3a^3b+a^4;$$

forming a single term of all those involving x, we have

$$b(4a^2+2ab-2b^2)x=4a^2b^2-6ab^3+2b^4+3a^3b+a^4.$$

Hence, $$x=\frac{a^4+3a^3b+4a^2b^2-6ab^3+2b^4}{b(4a^2+2ab-2b^2)}.$$

An expression which cannot be reduced to an entire polynomial. (No. 40).

46. If it was required to resolve the equation $3x-2=4x-7$, by transposing the term involving x into the first member, and the known terms into the second, it would become ...... $3x-4x=2-7$, or reducing $-x=-5$. To interpret this result, it is only necessary to observe, that we may invert the order of the transposition, that is, pass the terms involving x into the second member; this gives $7-2=4x-3x$, whence $5=x$, or $x=5$; consequently, when we arrive at such a result as $-x=-5$, we have only to change the signs of both members.

This evidently amounts to transposing the terms affected with the unknown quantity into the second member, and the known terms into the first; then write the second member first, and reciprocally.

We may now proceed to the resolution of problems.

47. We have already said, that the first part of the algebraic resolution of a problem is not subjected to any well defined rule. Sometimes the enunciation of the problem furnishes the equation immediately; sometimes it is necessary to discover from the enunciation conditions from which an equation may be formed; sometimes the conditions enunciated are not exactly those which are written algebraically, but rather conditions which may be regarded as consequences of those enunciated. The conditions enunciated are called *explicit conditions*, and those which are deduced from them *implicit conditions*. The application of the following rule will almost always enable us to form the equation. It is enunciated thus: *Consider the problem solved, and indicate, by means of the algebraic signs, upon the known quantities, represented either by numbers or letters, and the unknown quantity, which is always represented by a letter, the same course of reasoning and operations which it would be necessary to perform, in order to verify the unknown quantity, if it had been given.*

Prob. 1st. Find a number such that the sum of one half, one third, and one fourth of it, augmented by 45, shall be equal to 448.

Let x be the required number, then $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$, will represent the half, third, and fourth of this number. Now, from the enunciation, it is necessary that the sum of the three parts augmented by 45 should give 448. Hence the equation of the problem is

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+45=448.$$

Or subtracting 45 from both members,

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=403;$$

making the denominators disappear, $6x+4x+3x=4836$, and reducing $13x=4836$. Hence $x=\frac{4836}{13}=372$,

and $\frac{372}{2}+\frac{372}{3}+\frac{372}{4}+45=186+124+93+45=448.$

This is one of those questions which, in arithmetic, is solved by the rule of false position, and we see with what facility algebra makes known the answer to it.

Problem 2d. *A person engages a workman for 48 days. For each day that he laboured he received 24 sols, and for each day that he was idle, he paid 12 sols for his board. At the end of 48 days, the account was settled, when the labourer received 504 sols. Required, the number of working days, and the number of days he was idle.*

If these two numbers were known, by multiplying them respectively by 24 and 12, then subtracting the last product from the first, the result would be 504. Let us indicate these operations by means of algebraic signs. Let x be the number of working days; $48-x$ will represent the number of idle days; $24\times x$, or $24x$, will denote the amount received by the workman, and $12(48-x)$ the sum which he pays for his board. The equation of the problem, therefore, is

$$24x-12(48-x)=504;$$

performing the operation indicated, $24x-576+12x=504$;

hence $\qquad 36x=504+576=1080,$

and $\qquad x=\frac{1080}{36}=30;$

whence $\qquad 48-x=48-30=18.$

Consequently, the workman laboured 30 days, and was idle 18. In fact, for 30 days' labour he should receive 24×30, or 720 sols, but he being idle 18 days, his board during this time amounts to 12×18, or 216 sols; now we have

$$720-216=504.$$

This problem may be made general, by denoting the whole number of working and idle days by n, the sum he received for each day he worked by a, the sum paid back for each idle day by b, and balance due the labourer, or the result of the account, by c. As before, let x represent the number of working days; $n-x$ will express the number of idle days. Hence ax will represent what the labourer earns, and $b(n-x)$ the sum to be deducted from it. The equation of the problem therefore is,

$$ax-b(n-x)=c;$$

whence $\qquad ax-bn+bx=c,$

$$(a+b)x=c+bn,$$

and $\qquad x=\frac{c+bn}{a+b};$

and consequently, $n-x=n-\frac{c+bn}{a+b}=\frac{an+bn-c-bn}{a+b},$

or $\qquad n-x=\frac{an-c}{a+b}.$

Problem 3d. *A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6; but three leaps of the greyhound are equivalent to 7 of the fox. How many leaps must the greyhound make to overtake the fox?*

From the enunciation, it is evident that the space passed over by the greyhound is composed of the 60 leaps which the fox is in advance, plus the space that the fox passes over from the moment when the greyhound starts in pursuit of him. Hence, if we can find the expressions for these two spaces, it will be easy to form the equation of the problem.

Let x be the number of leaps made by the greyhound. Since the fox makes 9 leaps whilst the greyhound makes 6, it follows that the fox makes $\frac{9}{6}$ or $\frac{3}{2}$ leaps, whilst the greyhound makes 1; and consequently, that he makes a number expressed by $\frac{3x}{2}$, whilst the greyhound makes a number represented by x. It might be supposed that in order to obtain the equation, it would be sufficient to place x equal to $60 + \frac{3}{2}x$; but in doing so, a manifest error would be committed; for the leaps of the greyhound are greater than those of the fox, and we would then equate heterogeneous numbers, that is, numbers referred to different units. Hence it is necessary to express the leaps of the fox by means of those of the greyhound, or reciprocally. Now, according to the enunciation, 3 leaps of the greyhound are equivalent to 7 leaps of the fox, then 1 leap of the greyhound is equivalent to $\frac{7}{3}$ leaps of the fox, and consequently x leaps of the greyhound are equivalent to $7\frac{x}{3}$ of the fox.

Hence we have the equation $\qquad \frac{7x}{3} = 60 + \frac{3}{2}x;$

making the denominators disappear . . $14x = 360 + 9x,$

Whence $\qquad 5x = 360$ and $x = 72.$

Therefore, the greyhound will make 72 leaps to overtake the fox, and during this time the fox will make $72 \times \frac{3}{2}$ or 108.

Verification.

The 72 leaps of the greyhound being equivalent to $\frac{72 \times 7}{3}$, or 168 leaps of the fox, we evidently have $168 = 60 + 108$.

48. Problem 4. *A father who had three children, ordered in his will, that his property should be divided amongst them in the following manner: the first to have a sum a, plus the nth part of what remained; the second a sum 2a, plus the nth part of what remained after subtracting from it the first part and 2a; the third to have a sum 3a, plus the nth part of what remained after subtracting from it the two first parts and 3a. In this manner his property was entirely divided; required the amount of it.*

Let x denote the property of the father. If by means of this quantity, algebraic expressions could be formed for the three parts, we might subtract their sum from the whole property x, and the remainder placed equal to zero, would give the equation of the problem. We will then endeavour to determine successively these three parts.

Since x denotes the property of the father, $x - a$ is what remains after having subtracted a from it; therefore the part which the first child is to have, is $a + \frac{x-a}{n}$, or reducing to a common denominator

$$\frac{an+x-a}{n} \ldots\ldots\ \text{1st part.}$$

In order to form the 2d part, this first part and $2a$ must be subtracted from x, this gives $x - 2a - \frac{(an+x-a)}{n}$, or reducing to a common denominator and subtracting,

$$\frac{nx-3an-x+a}{n} \ldots\ldots\ \text{1st remainder.}$$

Now, the second part is composed of $2a$, plus the nth part of this remainder; therefore, it is $2a + \frac{nx-3an-x+a}{n^2}$, or reducing to a common denominator,

$$\frac{2an^2+nx-3an-x+a}{n^2} \ldots\ldots\ \text{2d part.}$$

Subtracting the two first parts plus $3a$, from x, we have

$$x-3a-\frac{(an+x-a)}{n}-\frac{(2an^2+nx-3an-x+a)}{n^2}$$

Or, reducing to a common denominator, and performing the operations indicated,

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2} \text{ . . . . 2d remainder.}$$

Hence the 3d part is $3a+\frac{n^2x-6an^2-2nx+4an+x-a}{n3}$.

Or, reducing to a common denominator,

$$\frac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3} \text{ . . . . 3d part.}$$

But from the enunciation the estate of the father is found to be entirely divided. Hence the difference between x, and the sum of the three parts should be equal to zero. This gives the equation

$$\left.\begin{matrix} x-\frac{an+x-a}{n}-\frac{2an^2+nx-3an-x+a}{n^2} \\ -\frac{3an^3+n^2x-6an^2-2nx+4an+x-a}{n^3} \end{matrix}\right\}=0.$$

by making the denominators disappear, and performing the operations indicated, we have

$$n^3x-6an^3-3n^2x+10an^2+3nx-5an-x+a=0.$$

Whence

$$x=\frac{6an^3-10an^2+5an-a}{n^3-3n^2+3n-1}=\frac{a(6n^3-10n^2+5n-1)}{n^3-3n^2+3n-1}.$$

A more simple equation and result may be obtained, by observing, that the part which goes to the third child is composed of $3a$, plus the nth part of what remains, and that the estate is then entirely divided, that is, the third child has only the sum $3a$, and the remainder just mentioned is nothing. Now the expression for this remainder has been found to be

$$\frac{n^2x-6an^2-2nx+4an+x-a}{n^2}.$$

Placing this equal to zero, and making the denominator disappear, we have

$$n^2x-6an^2-2nx+4an+x-a=0.$$

Whence $$x=\frac{6an^2-4an+a}{n^2-2n+1}=\frac{a(6n^2-4n+1)}{n^2-2n+1}.$$

To prove the *numerical identity* of this expression with the preceding, it is only necessary to show that the second can be deduced from the first, by suppressing a factor common to its numerator and denominator. Now if we apply the rule for finding the greatest common divisor (No. 41.) to the two polynomials .... $a(6n^3-10n^2+5n-1)$ and n^3-3n^2+3n-1, it will be seen that $n-1$ is a common factor, and by dividing the numerator and denominator of the first expression by this factor, the result will be the second.

This problem shows the beginner how important it is to seize upon every circumstance in the enunciation of a question, which may facilitate the formation of the equation, otherwise he runs the risk of arriving at results more complicated than the nature of the question requires.

The conditions which have served to form successively the expressions for the three parts, are the *explicit conditions* of the problem; and the condition which has served to determine the most simple equation of the problem, is an *implicit condition*, which a little attention has sufficed to show, was comprehended in the enunciation.

To obtain the values of the three parts, it is only necessary to substitute for x its value in the three expressions obtained for these parts.

Apply the formula $x=\dfrac{a\,(6n^2-4\,n+1)}{n^2-2\,n+1}$ to a particular example.

Let $a=10000,\ n=5.$

We have

$$x=\frac{10000(6\times25-4\times5+1)}{25-10+1}=\frac{10000\times131}{16}=\frac{1310000}{16}=81875.$$

To verify the enunciation in this case:

The first child should have $10000+\dfrac{81875-10000}{5}$, or 24375.

There remains then $81875-24375$, or 57500, to divide between the other two children.

The second should have $20000+\dfrac{57500-20000}{5}$, or 27500.

Then there remains $57500-27500$, or 30000, for the third child. Now 30000 is triple of 10000; hence the problem is verified.

We can give a more simple and elegant solution to this problem, but it is less direct. It also depends upon the remark, that after having subtracted $3a$ from the two first parts, nothing remains.

Denote the three remainders mentioned in the enunciation by r, r', r''. The algebraic expressions for the three parts will be

$$a+\frac{r}{n},\quad 2a+\frac{r'}{n},\quad 3a+\frac{r''}{n}.$$

Now, 1st. From the enunciation, it is evident that $r''=o$. Therefore the third part is $3a$.

2d. What remains after giving to the second child $2a+\frac{r'}{n}$ can be represented by $r'-\frac{r'}{n}$, or $\frac{(n-1)r'}{n}$.

Moreover, this remainder also forms the third part. Therefore we have

$$\frac{(n-1)r'}{n}=3a;\ \text{whence } r'=\frac{3an}{n-1}.$$

Then the second part is $2a+\frac{3an}{n-1}:n=2a+\frac{3a}{n-1}$, or converting the whole number into a fraction, and reducing, $\frac{2an+a}{n-1}$.

3d. The remainder, after giving to the first $a+\frac{r}{n}$, can be expressed by $r-\frac{r}{n}$ or $\frac{(n-1)r}{n}$. Now this remainder should form the two other parts, or $3a+\frac{2an+a}{n-1}$.

Therefore, $\frac{(n-1)r}{n}=3a+\frac{2an+a}{n-1}=\frac{5an-2a}{n-1}$.

Hence, $r=\frac{5an-2a}{n-1}\times\frac{n}{(n-1)}=\frac{5an^2-2an}{(n-1)^2}$.

And consequently the first part is

$$a+\frac{5an^2-2an}{(n-1)^2}:n=a+\frac{5an-2a}{(n-1)^2}.$$

$$=a+\frac{5an-2a}{n^2-2n+1}=\frac{an^2+3an-a}{n^2-2n+1}.$$

Then the whole estate is $3a+\frac{2an+a}{n-1}+\frac{an^2+3an-a}{n^2-2n+1}$.

Or, by reducing the whole number and fractions to a common denominator,

$$\frac{3a(n^2-2n+1)+(2an+a)(n-1)+an^2+3an-a}{n^2-2n+1};$$

performing the operations indicated, and reducing

$$\frac{6an^2-4an+a}{n^2-2n+1}=\frac{a(6n^2-4n+1}{(n-1)^2},$$

which agrees with the preceding result.

This solution is more complete than the preceding, since we obtain from it the estate of the father, and the expressions for the three parts.

49. Problem 5. *A father ordered in his will, that the eldest of his children should have a sum a, out of his estate, plus the n^{th} part of the remainder; that the second should have a sum 2 a, plus the n^{th} part of what remained after having subtracted from it the first part and 2 a; that the third should have a sum 3 a, plus the n^{th} part of the new remainder—and so on. It is moreover supposed that the children share equally. Required, the value of the father's estate, the share of each child, and the number of children.*

This problem is remarkable, because the number of conditions contained in the enunciation is greater than the number of unknown values required to be found.

Let the estate of the father be represented by x, $x-a$ will express what remains after having taken from it the sum a. Therefore the share of the eldest is

$$a+\frac{x-a}{n} \text{ or } \frac{an+x-a}{n} \quad \ldots\ldots \text{ 1st part.}$$

Subtracting the first part, and $2a$, from x, we have

$$x-2a-\frac{(an+x-a)}{n}, \quad \text{or } \frac{nx-3an-x+a}{n},$$

the n^{th} part of which is, $\dfrac{nx-3an-x+a}{n^2}$.

Hence, the share of the second child is

$$2a+\frac{nx-3an-x+a}{n^2}, \text{ or } \frac{2an^2+nx-3an-x+a}{n^2} \quad \text{2d part.}$$

In like manner, the other parts might be formed, but as all the parts should be equal, to form the equation of the problem, it suffices to equate the two first parts, which gives

$$\frac{a'n+x-a}{n}=\frac{2\,a\,n^2+n\,x-3an-x+a}{n^2},$$

whence $$x=a\,n^2-2\,a\,n+a.$$

Substituting this value of x in the expression for the first part, we find $$\frac{a\,n+a\,n^2-2\,a\,n+a-a}{n};$$

or reducing, $\frac{a\,n^2-a\,n}{n}=a\,n-a=(n-1)\,a$;

and as the parts are equal, by dividing the whole estate by the first part, we will obtain a quotient which will show the number of children; therefore $\frac{a\,n^2-2\,a\,n+a}{a\,n-a}$, or $n-1$, denotes the number of children.

The father's estate, $a\,n^2-2\,a\,n+a$,

The share of each child, $a\,(n-1)$,

Whole number of children, $(n-1)$.

It now remains to be known, whether the other conditions of the problem are satisfied; that is, whether by giving to the second child $2\,a$, plus the n^{th} part of what remains; to the third, $3\,a$, plus the n^{th} part of what remains . . . . . ., the share of each child was in fact $(n-1)\,a$.

The difference between the estate of the father and the first part being $a\,(n-1)^2-a\,(n-1,)$ the share of the second child will be

$$2\,a+\frac{a(n-1)^2-a(n-1)-2a}{n} \text{ or } \frac{2a(n-1)+a(n-1)^2-a(n-1)}{n},$$

and reducing

$$\frac{a\,(n-1)+a\,(n-1)^2}{n} \text{ or } \frac{a\,(n-1)\,(1+n-1)}{n},$$

or $$a(n-1).$$

In like manner the difference between $a(n-1)^2$ and the two first parts being . . . $a(n-1)^2-2a(n-1)$, the third part will be $3a+\frac{a(n-1)^2-2a(n-1)-3a}{n}$, which being reduced, evidently becomes $\frac{a(n-1)+a(n-1)^2}{n}$, or $a(n-1)$.

In the same way we would obtain for the fourth part

$4a+\frac{a(n-1)^2-3a(n-1)-4a}{n}$, or $\frac{a(n-1)+a(n-1)^2}{n}$, and so on. Hence all the conditions of the enunciation are satisfied.

§ II. *Of Problems and equations of the first degree, involving two or more unknown quantities.*

50. Although several of the questions hitherto resolved, contained in their enunciation more than one unknown quantity, we have resolved them by employing but one symbol. The reason for this is, that we have been able, from the conditions of the enunciation, to express easily the other unknown quantities by means of this symbol; but this is not the case in all problems containing more than one unknown quantity.

To ascertain how problems of this kind are resolved: first, take some of those which have been resolved by means of one unknown quantity.

Given the sum a, of two numbers, and their difference b, it is required to find these numbers.

Axiom. If to two equal numbers A and B, two equal numbers C and D be respectively added, the results $A+C$ and $B+D$ will be equal; that is to say, if we have the equations $A=B$ and $C=D$, there results from them $A+C=B+D$. In like manner, if from two equal numbers, two other equal numbers be subtracted, the remainders will be equal; that is, from the two equations $A=B$ and $C=D$ we deduce $A-C=B-D$.

Applying this axiom to the two equations of the proposed problem,

We have by addition, $2x=a+b$,

and by subtraction $2y=a-b$.

Each of these equations containing but one unknown quantity, we deduce from the first, $x=\frac{a+b}{2}$;

and from the second $y=\frac{a-b}{2}$.

Indeed, we have

$$\frac{a+b}{2}+\frac{a-b}{2}=\frac{2a}{2}=a, \text{ and } \frac{a+b}{2}-\frac{a-b}{2}=\frac{2b}{2}=b.$$

Again, take the problem of the workman (No. 39).

Let x represent the number of working days, and y the number of idle days, ax will express the sum which the labourer receives for the working days, and by that which is charged to him for the idle days.

We will therefore have the two equations $\begin{cases} x+y=n \\ ax-by=c. \end{cases}$

It has already been shown that the two members of an equation can be multiplied by the same number, without destroying the equality; therefore the two members of the first equation may be multiplied by b, the coefficient of y in the second, and it becomes $bx+by=bn$, an equation which added to the second $ax-by=c$ gives, $bx+ax=bn+c$, whence $x=\frac{bn+c}{a+b}$. In like manner multiplying the two members of the first equation by a, the coefficient of x in the second, it becomes $ax+ay=an$, an equation from which if the second $ax-by=c$, be subtracted, gives

$$ay+by=an-c, \text{ whence } y=\frac{an-c}{a+b}.$$

By introducing a symbol to represent each of the unknown quantities in the preceding problems, the solution which has just been given has the advantage of making known the two required numbers, independently of each other.

Elimination.

51. Take the two equations . . . . . $\begin{cases} 5x+7y=43, \\ 11x+9y=69. \end{cases}$ which may be regarded as the algebraic enunciation of a problem containing two unknown quantities. If, in these equations, one of the unknown quantities was affected with the same coefficient, we might, by a simple subtraction, form a new equation which would contain but one unknown quantity, and from which the value of this unknown quantity could be deduced.

Now, if both members of the first equation be multiplied by 9, the coefficient of y in the second, and the two members of the second by 7, the coefficient of y in the first, we will obtain

$$\begin{cases} 45x+63y=387, \\ 77x+63y=483, \end{cases}$$

equations which may be substituted for the two first, and in which y is affected with the same coefficient.

Subtracting then the first of these equations from the second, there results $32x=96$, whence $x=3$.

Again, if we multiply both members of the first equation by 11, the coefficient of x in the second, and both members of the second by 5, the coefficient of x in the first, we will form the two equations $\begin{cases} 55x+77y=473, \\ 55x+45y=345, \end{cases}$ which may be substituted for the two proposed equations, and in which the coefficients of x are the same.

Subtracting then the second of these two equations from the first, there results $32y=128$, whence $y=4$.

Therefore $x=3$ and $y=4$ are the values of x and y, which should verify the enunciation of the question. Indeed we have,

1st. $5\times3+7\times4=15+28=43$;

2d. $11\times3+9\times4=33+36=69$.

The above operation, by means of which we obtain the values for the unknown quantities which satisfy the given equations, is called *elimination*.

The preceding method is analogous to the reduction of fractions to the same denominator, and is also, like that operation, susceptible of some simplifications.

Take for another example, the two equations,

$$8x-21y=33, \ldots 6x+35y=177.$$

In order to render the coefficients of y equal to each other, we will remark that 21 and 35 have the common factor 7; hence it is only necessary to multiply the first equation by 5, and the second by 3; this gives the two equations

$$40x-105y=165, \ldots . 18x+105y=531,$$

which added together give

$$58x=696, \text{ whence } x=12.$$

In like manner the coefficients of x contain the common factor 2; therefore in order to render these coefficients equal to each other, it is only necessary to multiply the first by 3 and the second by 4, which gives

$$24x-63y=99, \ldots . 24x+140y=708.$$

Subtracting the first from the second, we have

$$203y=609, \text{ whence } y=3.$$

N. B. It is important to ascertain whether the coefficients have any common factor, as the calculations are then much more simple.

Take for the third example the equations

$$\frac{2x}{3}-4+\frac{y}{2}+x=8-\frac{3y}{4}+\frac{1}{12},$$

$$\frac{y}{6}-\frac{x}{2}+2=\frac{1}{6}-2x+6.$$

In the first place it is necessary to make the denominators disappear (No. 44), and we thus obtain the two equations,

$8x-48+6y+12x=96+1-9y, \; . \; . \; y-3x+12=1-12x+36,$

$$\text{reducing, } \begin{cases} 20x+15y=145, \\ 9x+y=25, \end{cases} \text{ or, } \begin{cases} 4x+3y=29, \\ 9x+y=25. \end{cases}$$

Multiplying the second equation by 3, and subtracting the first from it, we find $23x=46$, whence $x=2$; but we have moreover $y=25-9x$; therefore $y=25-9\times 2=7$.

52. Let us now consider the case of three equations involving three unknown quantities.

$$\text{Take the equations, } \begin{cases} 5x-6y+4z=15. \\ 7x+4y-3z=19. \\ 2x+\;y+6z=46. \end{cases}$$

To eliminate z by means of the two first equations, multiply the first by 3 and the second by 4, then add the two results together, (since the coefficients of z have contrary signs), this gives a new equation . . . . . . $43x-2y=121$

Multiplying the second equation by 2, a factor of the coefficient of z in the third equation, and adding them together, we have . . $16x+9y=84$

The question is then reduced to finding the values of x and y, which will satisfy these new equations.

Now, if the first be multiplied by 9, the second by 2, and the results be added together, we find $-419x=1257$, whence $x=3$.

We might, by means of the two equations involving x and y, determine y in the same way we have determined x; but the value of y may be determined more simply, by observing that the last of thes etwo equations becomes, by substituting for x its value found above,

$$48+9y=84 \qquad \text{whence } y=\frac{84-48}{9}=4.$$

In the same manner the first of the three proposed equations, becomes, by substituting the values of x and y,

$$15-24+4z=15, \text{ whence } z=\frac{24}{4}=6.$$

In general, let there be a number m of equations involving a like number of unknown quantities. In order to find the values of the unknown quantities, *combine successively any one of the equations with each of the $m-1$ others, to eliminate the same unknown quantity ; $m-1$ new equations, containing $m-1$ unknown quantities will thus be obtained, upon which operate in the same manner as upon the proposed equations, that is, elimiminate another unknown quantity by combining one of these new equations with the $m-2$ others, this will give $m-2$ equations containing $m-2$ unknown quantities. Continue this series of operations until a single equation containing but one unknown quantity is obtained, from which the value of this unknown quantity is easily found.* Then by going back through the series of equations which have been obtained, the value of the other unknown quantities may be successively determined.

53. The method of elimination just exposed is called the *method by addition and subtraction*, because the unknown quantities disappear by additions and subtractions, after having prepared the equations in such a manner that one unknown quantity shall have the same coefficient in two of them.

There are two other principal methods of eliminations. The first, called the *method by substitution*, consists in finding the value of one of the unknown quantities in one of the equations, as if the other unknown quantities were already determined, and in substituting this value in the other equations; in this way new equations are formed, which contain one unknown quantity less than the other, and upon which we operate as upon the proposed equations.

The second, called the *method by comparison*, consists in finding the value of the same unknown quantity in all the equations, placing them (two and two) equal to each other, which necessarily gives a new sect of equations, containing one unknown quantity less than the other, upon which we operate as upon the proposed equations.

But there is an inconvenience in these two methods, which the *method by addition and subtraction* is not subject to, viz.: they produce new equations, containing denominators, which it is afterwards necessary to make disappear. The *method by sub-*

stitution is, however, advantageously employed whenever the coefficient of one of the unknown quantities is equal to unity in one of the equations, because then the inconvenience of which we have just spoken does not occur. We will sometimes have occasion to employ it, but generally the method of *addition and subtraction* is preferable. It moreover presents this advantage, viz.: when the coefficients are not too great, we can perform the addition or subtraction at the same time with the multiplication, which is necessary to render the coefficients equal to each other.

54. It often happens that each of the proposed equations does not contain all the unknown quantities. In this case, with a little address, the elimination is very quickly performed.

Take the four equations involving four unknown quantities:

$$\left.\begin{array}{r} 2x-3y+2z=13 \\ 4u-2x=30 \end{array}\right\} \begin{array}{l} \text{- - - (1)} \\ \text{- - - (2)} \end{array} \quad \begin{array}{r} 4y+2z=14 \\ 5y+3u=32 \end{array} \begin{array}{l} \text{- - - (3)} \\ \text{- - - (4)} \end{array}$$

By inspecting these equations, we see that the elimination of z in the two equations, (1) and (3,) will give an equation involving x and y; and if we eliminate u in the equatio ns (2) and (4,) we will obtain a second equation, involving x and y. These two last unknown quantities may therefore be easily determined. In the first place, the elimination of z in (1) and (3) gives $7y-2x=1$

That of u in (2) and (4,) gives - - - - - $20y+6x=38$

Multiplying the first of these equations by 3, and adding - - - - - - - - - - - - - - - - $41y=41$

Whence - - - - - - - - - - - - - - - - - - - $y=1$

Substituting this value in $7y-2x=1$, we find $x=3$

Substituting for x its value in equation, (2,) it becomes $4u-6=30$, whence - - - - - - - $u=9$

And substituting for 9 its value in equation, (3,) there results - - - - - - - - - - - - - - - $z=5$

The student may exercise himself with the following equations.

$$\left.\begin{array}{r} 7x-2z+3u=17 \\ 4y-2z+t=11 \\ 5y-3x-2u=8 \\ 4y-3u+2t=9 \\ 3z+8u=33 \end{array}\right\} \qquad x=2,\ y=4,\ z=3,\ u=3,\ t=1.$$

55. In all the preceding reasoning, we have supposed the

number of equations equal to the number of symbols employed to denote the unknown quantities. This must be the case in every problem involving two or more unknown quantities, in order that it may be *determinate;* that is, in order that it shall not admit of an infinite number of solutions.

Suppose, for example, that a problem involving two unknown quantities, x and y, leads to the single equation, $5x - 3y = 12$; we deduce from it $x = \frac{12+3y}{5}$. Now, by making successively

$$y = 1,\ 2,\ 3,\ 4,\ 5,\ 6,$$

there results, $x = 3,\ \frac{18}{5},\ \frac{21}{5},\ \frac{24}{5},\ \frac{27}{5},\ 6,$

and every system of values,

$$(x = 3, y = 1),\ (x = \frac{18}{5},\ y = 2),\ (x = \frac{21}{5},\ y = 3,$$

substituted for x and y in the equation, will satisfy it equally well.

If we had two equations involving three unknown quantities, we could in the first place eliminate one of the unknown quantities by means of the proposed equations, and thus obtain an equation, which, containing two unknown quantities, would be satisfied by an infinite number of systems of values taken for these unknown quantities. Therefore, in order that a problem may be determined, its enunciation must contain at least as many different conditions as there are unknown quantities, and these conditions must be such that each of them may be expressed by an equation.

56. We will now proceed to the resolution of problems involving two or more unknown quantities.

Problem 6th. *A person possessed a capital of* 30,000 *francs, for which he drew a certain interest; but he owed a sum of* 20,000 *francs, for which he paid a certain interest. The interest that he received exceeded that which he paid by* 800 *francs. Another person possessed* 35,000 *francs, for which he received interest at the second of the above rates,* (y)*; but he owed* 20,000 *francs, for which he paid interest at the first of the above rates. The interest that he received exceeded that which he paid by* 310 *francs. Required, the two rates of interest.*

Solution. Let x and y denote the two rates of interest for 100 francs. To obtain the interest of 30,000 francs at the rate denoted by x, we will form the proportion . . . . $100:x::30,000:\frac{30000\,x}{100}$, or $300\,x$. In the same way we will obtain $\frac{20000\,y}{100}$, or $200\,y$, for the interest of 20,000 francs, at the rate denoted by y. But from the enunciation, the difference between these two interests is equal to 800 francs.

We have, then, for the first equation of the problem,

$$300\,x-200\,y=800.$$

By writing algebraically the second condition of the problem, we obtain the other equation,

$$350\,y-240\,x=310.$$

Both members of the first equation being divisible by 100, and those of the second by 10, we may put the following in place of them:

$$3x-2\,y=8, \quad 35\,y-24\,x=31.$$

To eliminate x, multiply the first equation by 8, and then add it to the second; there results $19\,y=95$, whence $y=5$.

Substituting for y, in the first equation, its value, this equation becomes

$$3\,x-10=8, \text{ whence } x=6.$$

Therefore, the first rate is 6 per cent., and the second 5.

In fact,

30,000 francs, placed at 6 pr ct. gives 300×6, or 1800 francs.
20,000 do. " 5 " " 200×5, or 1000 do.

And we have $1800-1000=800$.

The second condition can be verified in the same manner.

Problem 7th. *There are three ingots composed of different metals mixed together. A pound of the first contains* 7 *ounces of silver,* 3 *ounces of copper, and* 6 *of pewter. A pound of the second contains* 12 *ounces of silver,* 3 *ounces of copper, and* 1 *of pewter. A pound of the third contains* 4 *ounces of silver,* 7 *ounces of copper, and* 5 *of pewter. It is required to find how much it will take of each of the three ingots to form a fourth, which shall contain in a pound,* 8 *ounces of silver,* $3\frac{3}{4}$ *of copper, and* $4\frac{1}{4}$ *of pewter.*

Solution. Let x, y and z represent the number of ounces which it is necessary to take from the three ingots respectively, in order to form a pound of the required ingot. Since there are 7 ounces of silver in a pound, or 16 ounces, of the first ingot, it follows that one ounce of it contains $\frac{7}{16}$ of an ounce of silver, and consequently in a number of ounces denoted by x, there is $\frac{7x}{16}$ ounces of silver. In the same manner we would find that $\frac{12y}{16}$, $\frac{4z}{16}$, expresses the number of ounces of silver taken from the second and third, when the fourth is formed; but from the enunciation, this fourth ingot contains 8 ounces of silver. We have, then, for the first equation

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8,$$

or, making the denominators disappear $7x+12y+4z=128$
as respects the copper we would find - - - - - $3x+3y+7z=60$
and with reference to the pewter - - - - - - - $6x+y+5z=68$

As the coefficients of y in these three equations, are the most simple, it is most convenient to eliminate this unknown quantity first.

Multiplying the second equation by 4, and subtracting the first equation from the product, we have - - $5x+24z=112$

Multiplying the third equation by 3, and subtracting the second from the product - - $15x+8z=144$

Multiplying this last equation by 3, and subtracting the preceding one from the product, we obtain $40x=320$, whence $x=8$.

Substitute this value for x in the equation $15x+8z=144$; it becomes

$$120+8z=144, \text{ whence } z=3.$$

Lastly, the two values $x=8$, $z=3$, being substituted in the equation $6x+y+5z=68$, give $48+y+15=68$, whence $y=5$.

Therefore in order to form a pound of the fourth ingot, we must take 8 ounces of the first, 5 ounces of the second, and 3 of the third. Indeed, if there are 7 ounces of silver in 16 ounces of the first ingot, in 8 ounces of it there should be a number of ounces of silver expressed by $\frac{7\times 8}{16}$.

In like manner $\frac{12\times 5}{16}$ and $\frac{4\times 3}{16}$ represents the quantity of silver contained in 5 ounces of the second ingot and 3 ounces of the third. Now we have $\frac{7\times8}{16}+\frac{12\times5}{16}+\frac{4\times3}{16}=\frac{128}{16}=8$; Therefore the fourth ingot contains 8 ounces of silver, as required by the enunciation. The same conditions may be verified relative to the copper and pewter.

57. The following are problems for the exercise of the student.

Problem 8. *A labourer can do a certain work expressed by* a, *in a time expressed by* b; *a second labourer, the work* c *in a time* d; *a third, the work* e, *in a time* f. *It is required to find the time it would take the three labourers, working together, to perform the work* g.

$$\text{Answer } x=\frac{bdfg}{adf+bcf+bde}.$$

Application.

$a=27$; $b=4$ | $c=35$; $d=6$ | $e=40$; $f=12$ | $g=19$;

x will be found equal to 12.

Problem 9. *If sea water contains* 1 *pound of salt in* 32 *pounds of this water, how much fresh water must be added to these* 32 *pounds, in order that the quantity of salt contained in* 32 *pounds of the new mixture shall be reduced to* 2 *ounces, or* $\frac{1}{8}$ *of a pound?* (Ans. 224lb.)

Problem 10. *How many times will the hour and minute hand of a watch be in conjunction, between noon and midnight, and at what time will each conjunction take place?*

(Ans. *Number of conjunctions* 11; 1*st conjunction at* $1^h 5' \frac{5}{11}$; 2*nd at* $2^h 10' \frac{10}{11}$; 3*rd at* $3^h 16' \frac{4}{11}$ . . .)

Problem 11. *A number is composed of three figures; the sum of these figures is* 11; *the figure in the place of units is double that in the place of hundreds; and when* 297 *is added to this number, the sum obtained is this number reversed. What is the number?* (Ans. 326.)

Problem 12. *A person who possesses* 100,000 *francs, placed a part of it out at* 5 *per cent. interest, and the other part at* 4 *per cent. The interest which he received for the whole amounted to* 4640 *francs. Required, the two parts.*

(Ans. 64.000 *and* 36.000.)

Problem 13. *A person possessed a certain capital, which he placed out at a certain interest. Another person who possessed*

10,000 francs more than the first, and who put out his capital 1 per cent. more advantageously than he did, had an income greater by 800 francs. A third person who possessed 15,000 francs more than the first, and who put out his capital 2 per cent. more advantageously than he did, had an income greater by 1500 francs. Required, the capitals of the three persons, and the three rates of interest.

(*Sums placed at interest* 30,000, 40,000 45,000
Rates of interest, 4, 5, 6.)

§ III. *Problems which give rise to negative results. Theory of negative quantities.*

58. The use of algebraic signs in the resolution of problems, often gives rise to some singular circumstances, which at the first view are embarrassing; but by reflecting upon them, we arrive at their explication, and even take advantage of them to render *the algebraic language* still more general.

Let the following question be proposed: *Find a number which, added to the number* b, *gives for a sum the number* a.

Solution. Let x be the number sought, we evidently have the equation

$$b+x=a, \text{ whence } x=a-b.$$

This expression or *formula* will give the value of x, in all the particular cases of this proposed problem.

For example, let $a=47$, $b=29$, then $x=47-29=18$.

Again, let $a=24$, $b=31$; then will $x=24-31$.

As 31 is equal to $24+7$, this value of x can be put under the form . . . $x=24-24-7$, or, reducing, $x=-7$. This value obtained for x is called a *negative solution.* How is it to be interpreted?

By returning to the enunciation of the problem, we see that it is impossible that 31 augmented by a number should give 24 for the sum, since 24 is less than 31. Therefore no number can verify the enunciation in this particular case. Nevertheless, if in the equation of the problem . . . $31+x=24$, we put in place of the term $+x$, its negative value -7, there results $31-7=24$, an exact equation, which means that the number 31 diminished by 7 gives 24 for the difference.

Hence the negative solution $x=-7$, indicates the impossibility of satisfying the enunciation of the problem in the sense in which it has been stated; but by considering this solution independently of its sign, that is to say, $x=7$,—it satisfies the enunciation when modified, thus:

Find a number, which, subtracted from* 31, *gives* 24 *for the difference, which enunciation only differs from the first, viz.: ***Find a number, which, added to* 31, *gives* 24 *for the sum,*** in this, that the words, *added to*, are replaced by the words, subtracted from, and the word, *sum*, by the word, *difference*.

The new question, when solved directly, gives the equation $31-x=24$; whence, $31-24=x$, or $x=7$.

Again, let the following problem be proposed: *A father has lived a number* a *of years, his son a number* b. *Find in how many years the age of the son will be one fourth the age of the father.*

Solution. Let x denote the required number of years, $a+x$ and $b+x$ will represent the ages of the father and son, at the end of this number of years; therefore we have the equation, $b+x=\frac{a+x}{4}$; whence $x=\frac{a-4b}{3}$.

Suppose $a=54$, $b=9$, we have $x=\frac{54-36}{3}=\frac{18}{3}=6$.

The father having lived 54 years, and the son 9, in 6 years the father will have lived 60 years, and his son 15; now 15 is the fourth of 60; hence, $x=6$ satisfies the enunciation.

Let us suppose that $a=45$, $b=15$; there results, $x=\frac{54-60}{3}$.

This expression may be reduced to $x=-5$, by performing the operations indicated. Now how is the negative solution $x=-5$ to be interpreted?

Let us return to the equation of the problem, which in the particular case we are considering, is $15+x=\frac{45+x}{4}$. It contains a manifest contradiction, because the second member becomes $\frac{45}{4}+\frac{x}{4}$; and each of these two parts is less than each of the two parts of the first member. But if we substitute -5 for $+x$ in the equation, it becomes $15-5=\frac{45-5}{4}$, or $10=\frac{40}{4}$, an exact equation; which means, that if, instead of adding a certain nnmber of years to the 2 years, we subtract 5 years from them, the age of the son will be the fourth of that of the father. Therefore, the solution that we have just found, being considered independently of its sign, satisfies this new enunciation, viz.: *A father has lived* 45 *years, his son* 15. Find when the age of the son was one fourth that of the father.

The equation resulting from this new enunciation will be $15 - x = \frac{45 - x}{4}$, whence we deduce $60 - 4x = 45 - x$, and $x = 5$.

In fact, we see from the enunciation of the problem, that the ratio of the age of the son to that of the father being $\frac{15}{45}$, or $\frac{1}{3}$, cannot become the fourth of that of the father; but it has been heretofore, because we have proved (No. 6) in a general manner, that by adding the same number to the two terms of a fraction, the value of the fraction is augmented. On the contrary, the value of the fraction is diminished when we subtract the same number from both of its terms.

59. Reasoning from analogy, we can establish this general principle. 1st. Every negative value found for the unknown quantity in a problem of the first degree, indicates a fault in the construction of the enunciation, or at least in the equation, which is the algebraic translation of it. (See the remark at the end of this No.) 2d. This value, considered without reference to its sign, may be considered as the answer to a problem, of which the enunciation only differs from that of the proposed problem in this, that certain quantities, which were additive, have become subtractive, and reciprocally.

Demonstration. The first part of this principle is easily demonstrated. Indeed, when we find a negative value for x, it necessarily arises from this, that from the nature of the equation, we have been led to subtract a greater number from a less, an operation which cannot be performed.

It is in this way that the values $x = -7$, $x = -5$, are obtained (No. 58) from the equations $x = 24 - 31$, $x = \frac{45 - 60}{3}$.

Now, if no absolute number, substituted for x, can verify the equation we have obtained by executing upon that of the problem the transformations indicated, (No. 43 . . . 45), this first *equation itself* cannot be verified in the sense in which it has been formed; for the accuracy of these transformations is well established for every equation susceptible of being verified by any *absolute* number, substituted in the equation in place of the unknown quantity.

The impossibility of resolving the question or equation in the sense in which it has been stated, is often evident by merely inspecting the enunciation or equation. The two preceding problems are examples of it. At other times it is difficult to discover this impossibility at first sight, but the operations of the calculus always end by making it appear quite evident.

Let us proceed to the second part of the principle.

We will observe in the first place, that if in the equation we replace x by $-x$, all the additive terms containing x become subtractive, and reciprocally; for if we have, for example, the term $+ax$, by putting $-x$ in place of x, it will become $+a\times-x$, or $-ax$. In like manner, if we have the term $-bx$, it will become $-b\times-x$, or $+bx$.

Therefore, by translating the new equation into ordinary language, we will necessarily have a new enunciation, which only differs from the first in this, that there are certain quantities that were additive, which will become subtractive, and reciprocally.

It remains now to show, that the substitution of $-x$ in place of x in the equation, gives rise to the result $x=p$, when we have in the first place obtained $x=-p$. (p is here considered an absolute number.)

Now whatever the primitive equation of the problem may be, we can always by known transformations suppose it to be reduced to the form $ax=-b$, (a and b being absolute numbers.)

From this equation we deduce $x=\frac{-b}{a}$, or $x=-\frac{b}{a}$, or, lastly, $x=-p$; p expressing the absolute number $\frac{b}{a}$. But if we put $-x$ in place of x in the primitive equation, by operating upon the new equation as upon the first, we will arrive at the equation $-ax=-b$.

$$\text{whence } x=\frac{b}{a}, \text{ or } x=p.$$

From the preceding reasoning we see the manner in which negative solutions should be interpreted. They are to be regarded, (when considered without reference to their signs,) as answers, not to the questions as they have been stated, but to questions of the same nature, of which the conditions have been modified; and the surest way of obtaining the new enunciation, is to return to the equation of the problem, change x into $-x$, then translate the new equation into ordinary language. (See No 70. for the demonstration of the same principle, in equations of the first degree involving two or more unknown quantities.)

60. *Remark.* The principle just established is only rigorously true for the equations, and is not always true for the enunciation of the problem: that is to say, the problem may have an exact enunciation, whenever by resolving the equation deduced from it, we arrive at a negative value. The reason for this is, the algebraist, in the application of his method to the resolution of a problem, often takes certain conditions in a sense contrary to that in which they should be taken ; and in this case, the negative solution that he obtains, corrects the view in which he considered the conditions. Thus, the equation is faulty, although the problem is susceptible of being resolved; and it is only when the equation is the faithful translation of the enunciation and of the meaning of all its conditions, that the principle is applicable to the enunciation. In the sequel we will see some examples of this kind, but it is principally in the application of algebra to geometrical questions, that the principle is less applicable to the enunciation than to the equation. By a little reflection upon the demonstration which has been given, it will be seen that the reasoning refers more to the equations than to the enunciations, of which these equations are considered as the algebraic translations.

61. In the preceding demonstration we have had occasion to multiply $+a$ by $-x$, to divide $-b$ by $+a$, $-b$ by $-a$; and the results were obtained by applying to monomials the rule for the signs established for the multiplication and division of polynomials. It may at first view appear necessary to demonstrate these rules with respect to monomials, and in fact almost every author has attempted it. But the demonstrations which they have given for them have only the appearance of rigour, and leave much uncertainty in the mind.

We will therefore say that we have extended to monomials the rules for the signs demonstrated for polynomials, in order to interpret the peculiar results furnished by algebra. By not admitting this extension we would deprive ourselves of one of the principal advantages of the algebraic language, which consists in embracing in one and the same formula, the solutions of several questions of the same nature, but the enunciations of which differ as to the meaning of certain conditions, viz. certain quantities are additive in the one and subtractive in the others, and reciprocally.

Again, the following considerations may be urged as a reason for extending to monomial quantities the rules laid down for polynomials.

The demonstration given (No. 17.) for the multiplication of a binomial $a-b$ by a binomial $c-d$, evidently supposes $a>b$, and $c>d$. If the contrary was the case, the reasoning in that demonstration would not present any precise meaning to the mind: and yet, the rule for the signs once established, we never think of questioning it, whatever may be the relative magnitudes of a, b, c, d.

This being the case, the product of $a-b$ by c being $ac-bc$, it follows that the product of a negative expression $a-b$, (a being $<b$), by a positive quantity, c is negative.

In like manner, the product of b by $c-d$ being $bc-bd$, it follows that the product of a positive quantity b by a negative expression $c-d$ (c being $<d$) is negative.

Lastly, the product of $a-b$ by $c-d$ being, as we have seen, $ac-bc-ad+bd$, an expression which may be put under the form $bd-ad-bc+ac$, or $d(b-a)-c(b-a)$; if we suppose $d>c$ and $b>a$ or $b-a$ positive, there necessarily results $d(b-a)>c(b-a)$, and consequently $d(b-a)-c(b-a)$, positive. Hence the product of a negative expression $(a-b)$ by a negative expression $c-d$ (a being $<b$ and $c<d$) is positive.

This constitutes one of the characteristics of algebra. In arithmetic and in geometry, the reasoning always has reference to something real that the mind can grasp; while in algebra, we very often reason and operate upon *imaginary* expressions, or upon symbols presenting operations impossible to execute; but the accuracy of the results obtained by these means, and which we would arrive at by more rigorous, but much longer procedures, sufficiently justify the steps that we have followed.

62. As the rules for the signs relative to monomials are of constant use in algebra, it will not be amiss to recapitulate them. We will moreover derive from them some new expressions peculiar to the algebraic language.

First, addition and subtraction.

To add $+b$ or $-b$ to a quantity expressed by a, it is necessary to write $a+b$ or $a-b$, that is, write the two monomials one after the other with their respective signs. (See No. 13.)

To subtract $+b$ or $-b$ from a, write $a-b$ or $a+b$, that is, change the sign of the monomial to be subtracted, and write it with its new sign, after that from which the subtraction has been made. (See No. 14.)

For multiplication and division,

$+a\times+b$ or $-a\times-b$ gives for a product $+ab$.
$-a\times+b$ or $+a\times-b$ gives for a product $-ab$. } (No. 17.)

$+a:+b$ or $-a:-b$ gives for a quotient $+\frac{a}{b}$.
$-a:+b$ or $+a:-b$ gives for a quotient $-\frac{a}{b}$. } (No. 25.)

These rules give rise to the following important remarks.

1st. In algebra, the words *add* and *sum* do not always (as in arithmetic) convey the idea of augmentation; for $a-b$, which results from the addition of $-b$ to a, is, properly speaking, a difference between the number of units expressed by a, and the number of units expressed by b. Consequently, this result is less than a. To distinguish this sum from an arithmetical sum, it is called the algebraic sum. Thus, the polynomial . . . . . . $2a^3-3a^2b+3b^2c-2a^2c$ is an algebraic sum, so long as it is considered as the result of the union of the monomials . . . . . $2a^3, -3a^2b, +3b^2c, -2a^2c$, with their respective signs; and, in its *proper acceptation*, it is the arithmetical difference between the sum of the units contained in the additive terms, and the sum of the units contained in the subtractive terms.

It follows from this, that an algebraic sum may, in the numerical applications, be reduced to a *negative* number, or a number affected with the sign $-$.

2d. The words *subtraction* and *difference* do not always convey the idea of diminution, for the difference between $+a$ and $-b$ being $a+b$, exceeds a. This result is an *algebraic difference*, because it can be put under the form of $a-(-b)$.

By means of these denominations, negative values may be considered as answers to questions. For example, in the equation $31+x=24$, the result, $x=-7$, indicates that it is necessary to add -7 to 31, in order to obtain 24, and in fact . . . . $31+(-7)$, or $31-7$, is equal to 24.

Likewise, in the equation $15+x=\frac{45+x}{4}$, the result, $x=-5$, indicates that -5 must be added to the two ages, that the age of the son may be one fourth of that of the father. For,

$$15+(-5), \text{ or } 15-5=10,$$
$$45+(-5), \text{ or } 45-5=40.$$

63. The necessity of employing negative expressions in algebraic calculations, and of working upon them as upon absolute quantities, has led algebraists to two other propositions, which will hereafter be of frequent occurrence. These are: *That every negative quantity* — a, *is less than nothing; and of two negative quantities, the least is that of which the numerical or absolute value is the greatest.*

Thus we have $-a<0$, and $-a<-b$, if a is numerically greater than b.

Demonstration. To explain these two propositions, we observe, that if from the same number we subtract a series of successively increasing numbers, the remainders will proportionally be diminished. For instance, take any whole number, say 6, and subtract from it successively, 1, 2, 3, 4, 5, 6, 7, 8, 9; we find, in stating the differences on the same line,

$6-1, 6-2, 6-3, 6-4, 6-5, 6-6, 6-7, 6-8, 6-9$;

Which, on being reduced, become

$$5,\ 4,\ 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3.$$

Hence, we see that -1 must be regarded as less than 0, because the latter expresses the difference between 6 and itself, while -1 expresses the difference between 6 and a greater number.

For the same reason, -1 is greater than -2, and -3 is greater than -4, although the numerical values of the former expressions are less than those of the latter.

Another Demonstration. When, in order to interpret the singular results obtained by the algebraic solution of a question, we agree to consider the *negative expressions* as quantities, it is necessary that we should obtain correct results, by subjecting them to the same operation as the absolute numbers. Now, it may be considered as an *axiom*, that if a number a is greater than a number b, by adding to each of them the number d, the result, $a+d$, will be greater than $b+d$.

This being the case, admitting the inequalities $0>-a$, and $-a>-(a+m)$, (a and m being absolute numbers), if $a+m$ be added to both members of each of them, the result will be $a+m>m$ and $m>0$, which is evidently correct. On the contrary, if we suppose $0<-a$, and $-a<-(a+m)$, the result would be $a+m<m$, and $m<0$, which is absurd.

In general, the two preceding propositions must be admitted, if we wish to operate upon *negative expressions* as upon absolute quantities. These propositions are merely algebraic figures of speech, analogous to those frequently used in common language. Thus, we often say of a person, he is worth less than nothing, when we wish to express that he owes more than he has the means of paying; and of two persons possessing an equal amount of property, that he is the richest who owes the least.

Discussion of Problems involving two or more unknown quantities.

64. When a problem has been resolved generally, that is, by representing the given quantities by letters, it may be required to determine what the values of the unknown quantities become, when particular suppositions are made upon the given quantities. The determination of these values, and the interpretation of the peculiar results obtained, form what is called the *discussion of the problem.*

The discussion of the following question presents nearly all the circumstances which are met with in problems of the first degree.

Problem 4th. *Two couriers start at the same time from two points, A and B, upon the same line A R, and travel in the same direction A B. The courier starting from A travels a number of miles per hour denoted by* m, *the courier from B, a number* n. *It is required, to find at what distances from the points A and B the couriers will be together.*

Solution. Let R be the point at which A overtakes B; call x and y the unknown distances $A\,R$ and $B\,R$, expressed in miles, and a the distance between the points of departure A and B. It is plain that the first equation will be

$$x-y=a \quad - \; - \quad (1).$$

But as m and n express the number of miles travelled per hour, (or the respective velocities of the two couriers,) it follows that the time employed in passing over the spaces x and y will be indicated by $\frac{x}{m}$ and $\frac{y}{n}$; moreover, these times are equal, therefore the second equation of the problem will be, $\frac{x}{m}=\frac{y}{n}$, or . . . . .
$nx-my=0 \quad - \; - \quad (2).$

Combining the equations (1) and (2) by the known methods of elimination, we have

$$x=\frac{a\,m}{m-n}, \quad y=\frac{a\,n}{m-n},$$

values which may be easily verified.

Discussion. So long as we suppose $m>n$, whence $m-n>0$, or positive, the values of x and y will be positive, and the problem will be resolved in the true sense of the enunciation. In fact, if the courier from A is supposed to travel faster than the one from B, he must be continually gaining upon this last; the interval which separates them diminishes more and more, until it vanishes, and then the couriers will be found upon the same point of the line.

But if we suppose $m<n$, whence $m-n<0$, or negative, the two values become negative at the same time, and we have

$$x=-\frac{a\,m}{n-m}, \quad y=-\frac{a\,n}{n-m}.$$

To interpret these results, we will observe, that it is impossible that the two couriers should come together in the direction $A\,B$; for as B goes faster than A, the distance between them would be increasing continually. But if, instead of supposing that they travel in the direction $A\,B$, we suppose they go in the direction $B\,A$, the circumstances become the same as in the case where $m>n$; it is therefore evident that the couriers would come together at a point R', upon the prolongation of $B\,A$.

The principle established in 59 would indicate this; for if we change the signs in the two equations, we have

$$\left.\begin{array}{l} -x+y=a \\ -\frac{x}{m}=-\frac{y}{n} \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} y-x=a, \\ \frac{x}{m}=\frac{y}{n}, \end{array}\right.$$

from which we derive

$$x=\frac{am}{n-m}, \quad y=\frac{an}{n-m}.$$

These values verify the new enunciation, in which the couriers are supposed to travel in the direction BA.

When $m=n$, then $m-n=0$, and the values of x and y reduce to

$$x=\frac{am}{0}, \quad y=\frac{an}{0}.$$

In order to interpret these new results, we will go back to the enunciation, and it will be perceived that it is absolutely impossible to satisfy it; for in whatever direction of the line AB the two couriers travel, they can never come together, since they were in the first place separated by an interval a, and as they go equally fast, they ought always to preserve the same distance between each other. Hence the result, $\frac{am}{0}$, may be regarded as a new sign of impossibility. In fact, resuming the equations of the problem, they become in the case when $m=n$,

$$\left.\begin{array}{l} x-y=a \\ \frac{x}{m}=\frac{y}{m} \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} x-y=a, \\ x-y=0, \end{array}\right.$$

which are evidently incompatible.

Nevertheless, algebraists consider the results, $x=\frac{am}{0}$, $y=\frac{an}{0}$, as forming a species of value, to which they have given the name of *infinite value*, for this reason:

When the difference $m-n$, without being absolutely nothing, is supposed to be very small, the results, $\frac{am}{m-n}$, $\frac{an}{m-n}$, are very great.

Take, for example, $m-n=0,01$, $m=3$; then .... $n=3-0,01=2,99$, and

$$\frac{am}{m-n}=\frac{3a}{0,01}=300a, \quad \frac{an}{m-n}=299a.$$

Again, take $m-n=0,0001$, $m=3$, then $n=2,9999$, and

$$\frac{a\,m}{m-n}=30000\,a,\quad \frac{a\,n}{m-n}=29999\,a.$$

In short, if the difference between the two velocities is not zero, the couriers will come together at some point of the line, but the distances from this point to the points of departure become greater and greater as this difference diminishes.

Hence, *if the difference between the velocities is less than any given magnitude, the distances* $\frac{a\,m}{m-n}$, $\frac{a\,n}{m-n}$, *will be greater than any given quantity*, or infinite. Therefore, for brevity, we say, let $m-n=0$; the results become $x=\frac{a\,m}{0}$, $y=\frac{a\,n}{0}$, infinite values.

As 0 is less than any absolute quantity, this character may be taken to denote the *last* (or evanescent) *value* of a quantity which may become as small as we please. Again, as the value of a fraction increases as its numerator becomes greater with reference to its denominator, the expression $\frac{A}{0}$ (A being any absolute number) is a proper one to represent an *infinite* quantity; that is, a quantity greater than any assignable quantity.

Infinity is also expressed thus ∞; and consequently, a quantity less than any given quantity may be expressed by $\frac{A}{\infty}$; for a fraction diminishes as its denominator becomes greater with reference to its numerator. Hence, 0 and $\frac{A}{\infty}$ are synonymous symbols, and so are $\frac{A}{0}$ and ∞.

We have been thus particular in explaining these ideas of infinity, because there are some questions of such a nature, that infinity may be considered as the true answer to the enunciation.

In the case where $m=n$ it will be perceived that there is not, properly speaking, any solution in *finite and determinate numbers;* but the values of the unknown quantities are found to be infinite.

If, in addition to the hypothesis $m=n$, we suppose that $a=0$, the two values become $x=\frac{0}{0}$, $y=\frac{0}{0}$.

To interpret these results, reconsider the enunciation, and it will be perceived, that if the two couriers travel equally fast, and start from the same point, they ought always to be together, and consequently the required point is any point whatever of the line travelled over. In fact, in the hypothesis $m=n$, $a=0$, the equations become

$$\left.\begin{array}{l} x-y=0 \\ \frac{x}{m}-\frac{y}{m}=0 \end{array}\right\} \text{ or, } \left\{\begin{array}{l} x-y=0 \\ x-y=0 \end{array}\right.$$

which are identically the same. Hence the question is indeterminate, since we have, in fact, but one equation involving two unknown quantities. Therefore the expression $\frac{0}{0}$ is in this case, the symbol of an *indeterminate quantity.*

If the couriers do not travel equally fast, that is, if $m>$, or $m<n$, and $a=0$, then will $x=0$, $y=0$.

In fact, it is evident, that if the couriers start from the same point, and travel with different velocities, they can only be together at the point of departure.

The preceding suppositions are the only ones that lead to remarkable results; and they are sufficient to show beginners the manner in which algebra answers to all the circumstances of the enunciation of a problem.

Before we generalize the preceding results, we will make a remark of the utmost importance in the applications of algebra.

65. When a problem has been resolved in a general manner, we can, by means of the formulas, or values found for the unknown quantities, *obtain by a simple change of signs*, those which agree to new general problems, the enunciations of which only differ from that of the proposed problem, by certain quantities, which, having been considered as additive, have become subtractive, and reciprocally.

Take for example the problem of the labourer (47). Supposing that the labourer receives a sum c, we have the equations

$$\left.\begin{array}{l} x+y=n \\ ax-by=c \end{array}\right\}, \text{ whence } x=\frac{bn+c}{a+b},\ y=\frac{an-c}{a+b}.$$

But if we suppose that the labourer, instead of receiving, owes a sum c, the equations will then be

$$\left.\begin{array}{l} x+y=n \\ by-ax=c \end{array}\right\} \text{ or, } \left\{\begin{array}{l} x+y=n, \\ ax-by=-c. \end{array}\right.$$

(By changing the signs of the second equation.)

Now it is visible that we can obtain immediately the values of x and y which correspond to the preceding values, by merely changing the sign of c in each of those values; this gives

$$x=\frac{bn-c}{a+b},\quad y=\frac{an+c}{a+b}.$$

To prove this rigorously, denote for the present $-c$ by d;

The equations then become $\begin{cases} x+\ y=n \\ ax-by=d \end{cases}$ and they only differ from those of the first enunciation by having d in the place of c. We would, therefore, necessarily find

$$x=\frac{bn+d}{a+b},\quad y=\frac{an-d}{a+b}.$$

And by substituting $-c$ for d, we have

$$x=\frac{bn+(-c)}{a+b};\quad y=\frac{an-(-c)}{a+b};$$

or by applying the rules of 62,

$$x=\frac{bn-c}{a+b};\quad y=\frac{an+c}{a+b}.$$

The results which agree to both enunciations may be comprehended in the same formula, by writing

$$x=\frac{bn\pm c}{a+b};\quad y=\frac{an\mp c}{a+b}.$$

The double sign $\pm$ is read *plus or minus*, the superior signs correspond to the case in which the labourer received, and the inferior signs to the case in which he owed a sum c. These formulas comprehend the case in which, in a settlement between the labourer and his employer, their accounts balance. This supposes $c=0$, which gives

$$n=\frac{bn}{a+b};\quad y=\frac{an}{a+b}.$$

Again, take the two general equations $\begin{matrix} ax+by=c, \\ dx+fy=g, \end{matrix}$ arising from the algebraic translation of any problem whatever. Multiplying the first equation by f, and the second by b, and subtracting the second from the first we have,

$$(af-bd)x=cf-bg,\ \text{whence}\ x=\frac{cf-bg}{af-bd}.$$

In like manner,

$$y=\frac{ag-cd}{af-bd}.$$

Now, in order to pass from these formulas,

1st. To those which agree to the equations $\begin{cases} ax-by=c, \\ dx+fy=g, \end{cases}$

it is only necessary to change b into $-b$, which gives

$$x=\frac{cf+bg}{af+bd};\quad y=\frac{ag-cd}{af+bd};$$

2d. To the formulas relative to the equations $\begin{cases} ax-by=c, \\ dx-fy=g, \end{cases}$

it will be sufficient to change $+b$ to $-b$, and $+f$ to $-f$, which gives the formulas,

$$x=\frac{-cf+bg}{-af+bd}=\frac{bg-cf}{bd-af};\quad y=\frac{ag-cd}{bd-af}.$$

The demonstration being precisely the same as in the preceding example, we will not repeat it.

§ IV. *General discussion of Problems and Equations of the First Degree.*

66. In order to generalize the discussion of problems of the first degree, involving one or more unknown quantities, we will proceed to find formulas which can represent the values of the unknown quantities, for any system of equations whatever, containing the same number of them.

In the first place, every equation of the first degree, involving but one unknown quantity, can be reduced to the form $ax=b$; a denoting *the algebraic sum* of the quantities by which the unknown quantity is multiplied, and b *the algebraic sum* of all the known terms.

From this equation we deduce, $x=\frac{b}{a}$.

Secondly. Every equation of the first degree, involving two unknown quantities, may be put under the form $ax+by=c$. For if the proposed equation contained denominators, we would make them disappear. (44). Afterwards, collect all the terms involving x, and those involving y, in the first member, then transpose all of the known terms into the second ; the *algebraic sum* of the first can be designated by ax, that of the second by by, and the third by c. Hence, a, b, c are entire.

Take the two equations $\begin{cases} ax+by=c, \\ a'x+b'y=c'. \end{cases}$

Multiplying the first by b', and the second by b, and subtracting one from the other, we have

$$(ab'-ba')x=cb'-bc', \text{ whence } x=\frac{cb'-bc'}{ab'-ba'}.$$

In a similar manner, we find $\quad y=\dfrac{ac'-ca'}{ab'-ba'}$.

Again, take the three equations

$$ax+by+cz=d \text{ - - - - - - } (1),$$
$$a'x+b'y+c'z=d' \text{ - - - - - - } (2),$$
$$a''x+b''y+c''z=d'' \text{ - - - - - - } (3).$$

To eliminate z, multiply the first equation by c', and the second by c, and subtract the second from the first; there results

$$(ac'-ca')x+(bc'-cb')y=dc'-cd' \text{ . . . . } (4).$$

Combining the second equation with the third in the same manner, we find

$$(a'c''-c'a'')x+(b'c''-c'b'')y=d'c''-c'd'' \text{ - - - - } (5).$$

To eliminate y, multiply equation (4) by $b'c''-c'b''$, and equation (5) by $bc'-cb'$, then subtract one from the other, and we have

$$[(ac'-ca')(b'c''-c'b'')-(a'c''-c'a'')(bc'-cb')]x=$$
$$(dc'-cd')(b'c''-c'b'')-(c'd''-d'c'')(bc'-cb');$$

or, by performing the operations indicated, reducing and dividing by c',

$$(ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a'')x=$$
$$db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''.$$

Hence, $\quad x=\dfrac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$

By performing analogous operations, to eliminate x and z, and y and z, we will find,

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}.$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}.$$

As the beginner will not have had much practice in abbreviating the operations as much as possible, we will indicate a method of deducing the values of y and z, from that of x, without being obliged to go through all of the preceding operations.

It will be observed, that the system of equations (1), (2) and

(3), remain the same when we substitute y, b, b' and b'', for the quantities x, a, a', a'', and *reciprocally;* therefore if in the expression for the value of x, we change x into y; also a, a', a'', which are the coefficients of x, into b, b', b'', which are the coefficients of y, and reciprocally. The result will be, the value of y, viz.:

$$y=\frac{da'c''-dc'a''+cd'a''-ad'c''+ac'd''-ca'd''}{ba'c''-bc'a''+cb'a''-ab'c''+ac'b''-ca'b''}.$$

or, changing the signs of the numerator and denominator, and writing in each the three last terms first, and the three first last,

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+cb'b''-ba'c''+bc'a''-cb'a''}.$$

In a similar manner we might obtain the value of z, by changing x, a, a', a'' into z, c, c' c''. and *reciprocally.*

This is sufficient to indicate the course to be pursued, in the case of four equations involving four unknown quantities, &c.

N. B. By a little reflection upon the manner in which these formulas have been obtained, we are sensible, that for any number of equations whatever, containing a like number of unknown quantities x, y, z .... *there can be, in general, but one system of values*, which will verify the equations.

This proposition is evident for an equation involving but one unknown quantity, $ax=b$. There is but one value $\frac{b}{a}$, which will satisfy it.

In the case of two equations involving two unknown quantities, after having multiplied the first equation by the coefficient of y in the second, and reciprocally, the result obtained by subtracting one from the other, may be substituted in one of the two proposed equations. Now as this equation will then contain only one unknown quantity, it will admit of but one value for this unknown, which, being substituted in one of the equations, will in like manner give but one value for y. The same reasoning will apply to three equations involving three unknown quantities.

67. From the use of accents in the notation of the coefficients, *a law* has been observed, from which the preceding formulas can be deduced, without the aid of the rules for elimination.

In the case of two equations involving two unknown quantities we have found

$$x=\frac{cb'-bc'}{ab'-ba'};\quad y=\frac{ac'-ca'}{ab'-ba'}.$$

1st. To obtain the common denominator of these two values, take the letters, a, and b, which are the coefficients of x, and y, in the first equation, and form the two arrangements ab and ba, and connect them with the sign —, which gives $ab-ba$; then accent the last letter of each term; and we have

$$ab'-ba'.$$

2nd. To obtain the numerator relative to each unknown quantity, take this denominator, and replace the letter which denotes the coefficient of this unknown quantity, by the letter which represents the known term of the equation; taking care to leave the accents as they were. In this way, $ab-ba'$ is changed to $cb'-bc'$, for the value of x, and to $ac'-ca'$, for the value of y. Let us now consider the case of three equations involving three unknown quantities, a, b, c, denoting the coefficients of x, y, z, and d the known term. 1st. To find the common denominator, take the denominator $ab-ba$, which agrees to the case of two unknown quantities (excepting the accents); introduce the letter c into each of the terms ab and ba, in the following places; viz. on the right, in the middle, and on the left, then interpose alternately the positive and negative signs; the result will be $abc-acb+cab-bac+bca-cba$. Now over the second letter of each term place the accent ($'$), and over the third letter the accent ($''$). The denominator will be

$$ab'c''-ac'b'',+ca'b''-ba'c''+bc'a''+cb'a''.$$

2d. To form the numerator for the value of each unknown quantity, take the denominator, and replace the letter denoting the coefficient of this unknown, by the letter which denotes the known term, leaving the accents the same. Thus for x, change a into d; for y, b into d; and for z, c into d.

This *law* which may be regarded as resulting from observation, in the case of two or three equations, can be extended to any number of equations, but the demonstration of it is very complicated. (See the second part of Garnier's Algebra.)

68. To show the use of these formulas in the applications to particular cases, we will take the equations

$$5x-7y=34,\quad 3x-13y=-6.$$

Comparing them with the two general equations $ax+by=c$, $a'x+b'y=c'$, we have $a=5$, $b=-7$, $c=34$, $a'=3$, $b'=-13$, $c'=-6$.

Substitute these values in the place of a, b, c, a', b', c', in the formulas

$$x=\frac{cb'-bc'}{ab'-ba'},\quad y=\frac{ac'-ca'}{ab'-ba'},\text{ we find}$$

1st. $$x=\frac{34\times-13-(-7)\times-6}{5\times13-(-7)\times3}=\frac{-34\times13-7\times6}{-5\times13+7\times3}$$

$$=\frac{-442-42}{-65+21}=\frac{-484}{-44}=11.$$

2d. $$y=\frac{5\times-6-34\times3}{5\times-13-(-7)\times3}=\frac{-30-102}{-65+21}=\frac{-132}{-44}=3.$$

And the values $x=11$, and $y=3$, will satisfy the proposed equations.

We might assure ourselves of this at once, by substituting them in the equations. But, in order that, the demonstration may be independent of any particular example, we will remark that we may pass from the formulas relative to the equations $ax+by=c$, $a'x+b'y=c'$ to those which agree to the equations $ax-by=c$, and $a'x-b'y=-c'$, by (65) changing b to $-b$, b' to $-b'$ and c' to $-c'$ which gives

$$x=\frac{c\times-b'-(-b)\times-c'}{a\times-b'-(-b)\times a'},\quad y=\frac{a\times-c'-c\times a'}{a\times-b'-(-b)\times a'}$$

and to deduce from these new general formulas, the values which agree to the particular equations, it will be necessary to make $a=5$, $b=7$, $c=34$, $a'=3$, $b'=13$, $c'=6$.

Hence, the values relative to the proposed equations may be obtained from the first general formulas, by making $a=5$, $b=-7$, $c=34$, $a'=3$, $b'=-13$, $c'=-6$, then performing the operations indicated, by the rules established for monomials.

In general, the rule is to substitute in place of the coefficients a, b, a', b' - - -, their values considered, with the same signs with which they are affected in the particular equations, and then perform the operations indicated.

Hence we again perceive the necessity for admitting *the rule for the signs* relative to monomials, if we wish to render the general formulas of the first degree, applicable to each particular example.

We will now proceed with the discussion of these formulas.

69. By inspecting the preceding formulas, it will be perceived that, in the particular applications, four different kinds of values may be obtained for solutions of problems of the first degree, viz.; *positive values*, *negative values*, *values of the form* $\frac{A}{0}$, and of the form $\frac{0}{0}$. These four results are obtained in the problem of the couriers, and it is now proposed to interpret them in a general manner.

In the first place, *the positive values* are commonly solutions of questions, in the sense enunciated. Nevertheless we will observe that, for certain problems, all the positive values will not satisfy the enunciation. If, for example, the nature of the problem required that the numbers sought should be entire, and we found fractions, the problem could not be solved. Sometimes, the nature of the problem does not admit the unknown quantities to be greater or less than certain numbers which are known and given *à priori*. Therefore, *the positive values are*, properly speaking, direct answers to the equations; and they are not *solutions of the question*, *unless they are of such a nature as to agree with the conditions of the enunciation.* In order to conceive that a number may verify an equation, without verifying the problem of which it is the algebraic translation, it is only necessary to observe that *the same equation is the algebraic translation of an infinite number of problems*, some of which admit of all possible numbers for solutions, and others admit only of numbers of a certain nature.

70. We already know in what light to view *negative values*, for problems involving but one unknown quantity. That nothing may be wanting on this subject, we will demonstrate the principle of No. 59, for a problem involving two or more unknown quantities.

It is evident that, if negative values are obtained for some of the unknown quantities, the equations of the problem cannot be satisfied in the sense in which they have been stated; for if a system of absolute numbers substituted for x, y, z, . . . could verify them, the equations deduced from them by the method of elimination, would exist for this system. Therefore the equation which contains but one of the unknown quantities, for which a negative result has been obtained, should be verified by an absolute number, which is contrary to the hypothesis. *Hence it is necessary to rectify the enunciation of the problem, or rather, the equations which are the algebraic translation of it.*

If we go back to the equations, and change the signs of the unknown quantities for which negative results have been obtained, the terms affected with these unknown quantities will necessarily change their sign, and the enunciation of the problem will generally be modified in such a manner, that *certain quantities which were additive will become subtractive, and reciprocally.*

After the equations are thus modified, *the new enunciation is verified by the values obtained for these unknown quantities in the first instance.*

To illustrate this:

Take three equations involving three unknown quantities; viz.

$$ax+by+cz=d,\quad a'x+b'y+c'z=d',\quad a''x+b''y+c''z=d'',$$

and suppose that they have given $x=p$, $y=-q$, $z=-r$; now in these equations change y and z into $-y$ and $-z$, or into y' and z' (denoting for the present, $-y$ and $-z$ by y' and z'), they become

$$ax+by'+cz'=d,\quad a'x+b'y'+c'z'=d',\quad a''x+b''y'+c''z'=d''.$$

These equations only differ from the preceding, in having y and z replaced by y' and z'; they will therefore give, $x=p$, $y'=-q$, $z'=-r$; hence, substituting $-y$ and $-z$ for y' and z' . . . . $x=p$, $-y=-q$, $-z=-r$, or $x=p, y=q$, $z=r$.

Therefore the principle of No. 59, is true for problems of the first degree involving two or more unknown quantities.

Sometimes the problem is of such a nature, that its enunciation is not susceptible of any modification; in this case, the *negative values* are only solutions of the modified equations, which may be considered as the algebraic translation of other problems which are susceptible of modification.

71. It now remains to interpret the expressions $\frac{A}{0}$ and $\frac{0}{0}$.

Take the equation $ax=b$, involving one unknown quantity, whence $x=\frac{b}{a}$.

1st. If, for a particular supposition made with reference to the given quantities of the question, we have $a=0$, there results $x=\frac{b}{0}$.

Now in this case the equation becomes $0\times x=b$, and evidently cannot be satisfied by any finite number. We will however remark that, as the equation can be put under the form $\frac{b}{x}=0$, if we substitute for x, numbers greater and greater, $\frac{b}{x}$ will differ less and less from 0, and the equation will become more and more exact; so that we may take a value for x so great that $\frac{b}{x}$ will be less than any assignable quantity.

It is in consequence of this that algebraists say that infinity satisfies the equation in this case; and there are some questions for which this kind of result forms a true solution; at least, it is certain that the equation does not admit of a solution in *finite* numbers, and this is all that we wish to prove.

2d. If we have $a=0$, $b=0$, at the same time, the value of x takes the form $x=\frac{0}{0}$.

In this case, the equation becomes $0\times x=0$, and *every finite number*, positive or negative, will satisfy it. Therefore *the equation (or the problem of which it is the algebraic translation) is indeterminate.*

72. It should be observed, that the expression $\frac{0}{0}$, does not always indicate an *indetermination*, it frequently indicates *the existence of a common factor* to the two terms of the fraction, which factor becomes nothing, in consequence of a particular hypothesis.

For example, suppose that we find for the solution of a problem, $x=\frac{a^3-b^3}{a^2-b^2}$. If, in the formula, a is made equal to b, there results $x=\frac{0}{0}$.

But it will be observed, that a^3-b^3 can (31) be put under the form $(a-b)(a^2+ab+b^2)$, and that a^2-b^2 is equal to..... $(a-b)(a+b)$, therefore the value of x becomes

$$x=\frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}.$$

Now, if we suppress the common factor$(a-b)$, before making the supposition $a=b$, the value of x becomes $x=\frac{a^2+ab+b^2}{a+b}$, which reduces to $x=\frac{3a^2}{2a}$, or $x=\frac{3a}{2}$, when $a=b$.

For another example, take the expression

$$\frac{a^2-b^2}{(a-b)^2}=\frac{(a+b)\,(a-b)}{(a-b)\,(a-b)}.$$

Making $a=b$, we find $x=\frac{0}{0}$, because the factor $(a-b)$ is common to the two terms; but if we first suppress this factor, there results $x=\frac{a+b}{a-b}$, which reduces to $x=\frac{2a}{0}$, when $a=b$.

From this we conclude, that *the symbol* $\frac{0}{0}$ *sometimes indicates the existence of a common factor to the two terms of the fraction which reduces to this form.* Therefore, before pronouncing upon the true value of the fraction, it is necessary to ascertain whether the two terms do not contain a common factor. If they do not, we conclude that the equation is really *indeterminate.* If they do contain one, suppress it, and then make the particular hypothesis; this will give the true value of the fraction, which will assume one of the three forms $\frac{A}{B}$, $\frac{A}{0}$, $\frac{0}{0}$, in which case, the equation is *determinate*, *impossible* in finite numbers, or *indeterminate.*

This observation is very useful in the discussion of problems.

73. Let us resume our subject, and consider now the two equations

$$ax+by=c,\ a'x+b'y=c',$$

involving two unknown quantities.

We have found (No. 66)

$$x=\frac{cb'-bc'}{ab'-ba'},\ y=\frac{ac'-ca'}{ab'-ba'}.$$

Suppose that $ab'-ba'=0$, the numerators $cb'-bc'$, $ac'-ca'$, being different from 0; the values of x and y reduce to

$$x=\frac{A}{0},\ y=\frac{B}{0}.$$

To interpret these results, it should be observed, that from the equation $ab'-ba'=0$, we find $a'=\frac{ab'}{b}$; and substituting this value in the equation $a'x+b'y=c'$, it becomes $\frac{ab'}{b}.x+b'y=c'$; or clearing the fraction, and dividing by b', $ax+by=\frac{bc'}{b'}$.

Now the first member of this equation is identical with that of the first

$$ax+by=c,$$

while the second number is essentially different; for from the inequality $cb' \gtrless bc'$, we deduce $c \gtrless \frac{bc'}{b'}$.

Hence the two equations cannot be satisfied simultaneously, by any system of finite values for x and y.

If we have $ab'-ba'=0$, and $cb'-bc'=0$, at the same time, the value of x is reduced to $x=\frac{0}{0}$, which it is necessary to interpret.

Since $ab'-ba'=0$, the proposed equations, can be put under the form $ax+by=c$, $ax+by=\frac{bc'}{b'}$; but these two equations are in fact but one and the same, for from the relation $cb'-bc'=0$, we deduce $c=\frac{bc'}{b'}$.

Therefore we have in reality but one equation, involving two unknown quantities, for the resolution of the problem. Hence the question is indeterminate.

As the relation $ab'-ba'=0$, gives $b'=\frac{ba'}{a}$; whence, by substituting in the relation $bc'-b'c=0$, there results $\frac{cba'}{a}-bc'=0$, or reducing $ca'-ac'=0$, we may conclude that *when the value of* x *is of the form* $\frac{0}{0}$, *the value of* y *is* generally *of the same form, and reciprocally.*

I say *generally*, because, if we had at the same time $b=0$, $b'=0$, the two expressions $ab'-ba'$, $cb'-bc'$, would necessarily be nothing, from which there would not result any determinate value for $ac'-ca'$.

In this particular case, the two values of x and y reduce to

$$x=\frac{0}{0}, \text{ and } y=\frac{ac'-ca'}{0}, \text{ or } \frac{A}{0}.$$

Reciprocally, if we had $a=0$, $a'=0$, there would result

$$x=\frac{cb'-bc'}{0}, \text{ or } \frac{A}{0}, \text{ and } y=\frac{0}{0}.$$

But these particular cases are scarcely admissible, since then, the two equations would reduce to two equations involving but *one* unknown quantity, viz.

$$\left.\begin{aligned} ax&=c \\ a'x&=c' \end{aligned}\right\} \text{ by supposing } b=0,\ b'=0;$$

and $$\left.\begin{aligned} by&=c \\ b'y&=c' \end{aligned}\right\} \text{ by supposing } a=0,\ a'=0;$$

whilst we are here treating of the case of two equations involving two unknown quantities.

74. The preceding course of reasoning cannot be so easily applied to the case in which we have more than two equations; but we can supply it by the following:

Let us consider four equations, (1), (2), (3), (4), containing the four unknown quantities, x, y, z, u.

Let $\frac{A}{D}$ denote the value of x obtained by elimination, and suppose that $D=0$ for a certain hypothesis made with respect to the given quantities, A being any quantity or 0. I say that, in the first case, *the proposed equations cannot be satisfied by any finite values;* and that, in the second, *they are indeterminate*, or susceptible of being verified by an infinite number of systems of values for x, y, z, u. In fact, it follows from the method of elimination, that the system of equations, (1), (2), (3), (4), may be replaced by four other equations, of which one is $Dx=A$; the second would be an equation involving x and y; the third an equation involving x, y, z; and the fourth, one of the proposed equations, (1), for example.

This being the case, it follows, 1st. When the value of x assumes the form $\frac{A}{0}$, as the equation involving x then becomes $0\times x=A$, and as it is moreover a necessary consequence of the simultaneous existence of the proposed equations, these equations must be *impossible* in finite numbers, since the equation $0\times x=A$ cannot be satisfied by a finite number.

2d. When the value of x assumes the form $\frac{0}{0}$, (not having a factor common to the numerator and denominator in its expression,) the equation $Dx=A$ becomes $0\times x=0$, and can be satisfied by an infinite number of values. By substituting each of these values in the equation involving x and y, which we have mentioned above, we would obtain an infinite number of corresponding values for y; substituting all of these systems of values of x and y in the equation involving x, y, z, we would obtain an infinite number of values for z. Finally, by substituting all of these systems of values of x, y, and z, in the equation (1), there would result an infinite number of values for u; and all of these systems, thus obtained, would necessarily satisfy the proposed equations.

75. The first part of this proposition is not subject to any restriction; when we find for one of the unknown quantities a result of the form $\frac{A}{0}$, it is a sure sign that the equations are *impossible* in *finite* numbers, at least for this unknown quantity.

As to the second part, it is subject to several modifications; that is to say, we may obtain results of the form $\frac{0}{0}$ for one or more of the unknown quantities, and yet have no right to conclude that the equations are *indeterminate*. Sometimes, even, one of the values being of the form $\frac{0}{0}$, we obtain the others under the form $\frac{A}{0}$.

The following systems of equations furnish the proof of this:

$$\text{1st System} \cdots\cdots \begin{cases} ax+by+cz=d, \\ ax+by+cz=d', \\ ax+by+cz=d''. \end{cases}$$

$$\text{2d System} \cdots\cdots \begin{cases} a\,x+by+cz=d, \\ a'x+by+cz=d', \\ a''x+by+cz=d''. \end{cases}$$

$$\text{3d System} \cdots\cdots \begin{cases} x+y+mz=p, \\ x+y+mz=q, \\ x+y+nz=r. \end{cases}$$

Applying the general formulas of No. 66, to the two first systems, we find

$$x=\frac{0}{0},\quad y=\frac{0}{0},\quad z=\frac{0}{0};$$

and yet, by inspecting the first, it is easily perceived that it cannot exist in *finite* numbers (the first members remaining the same,) unless we have $d=d'=d''$. It is true that when this relation does exist, the system is *indeterminate*, since it then reduces to *one equation involving three unknown quantities*; but it is not less certain that, in the actual condition of the quantities, the equations are incompatible.

The first of the formulas (No. 66.) applied to the second system, gives $a=\frac{0}{0}$, which it is necessary to interpret.

In order that the second system may exist (the first members remaining the same) it is necessary that

$$d-ax=d'-a'x, \text{ and } d-ax=d''-a''x.$$

Now the first of these two relations gives

$$x=\frac{d'-d}{a'-a}; \text{ and the second, } x=\frac{d''-d}{a''-a}.$$

As the two values of x should agree with each other, there will result from them the equality of condition

$$\frac{d'-d}{a'-a}=\frac{d''-d}{a''-a}.$$

So long as this relation between the quantities a, d, a', d', a'', d'', is not satisfied, the second system will be impossible, although we have obtained values of the form $\frac{0}{0}$, for each of the unknown quantities. When this relation is satisfied, the value of x will be *determinate*, and equal to $\frac{d'-d}{a'-a}$ or $\frac{d''-d}{a''-a}$; but the values of y and z will be indeterminate, since we will have but one equation involving two unknown quantities.

Applying the formulas (66) to the third system we find,

$$x=\frac{(m-n)(p-q)}{0}, \quad y=\frac{(n-m)(p-q)}{0}, \quad z=\frac{0}{0};$$

that is, two of the values are of the form $\frac{A}{0}$, and the other of the form $\frac{0}{0}$.

In this example, in order that the two first equations may be *possible* simultaneously, (the first members remaining the same), it is necessary that $p=q$; in which case, the two values of x and y, reduce to the form $\frac{0}{0}$.

This condition admitted, I say that the value of z, for which we have found $\frac{0}{0}$, becomes *determinate*, and that the two others are *indeterminate*. In fact, the system then reduces to the two equations

$$x+y+mz=p, \quad x+y+nz=r,$$

which, by subtracting one from the other, gives

$$(m-n)z=p-r\,; \text{ whence } z=\frac{p-r}{m-n}.$$

Substituting this value in the two equations, there results the single equation

$$x+y=\frac{mr-pn}{m-n}.$$

The cases just examined are sufficient to convince us, 1st, that,

In the applications of general formulas to particular systems, the values of some of the unknown quantities may be presented under the form $\frac{A}{0}$ and others under the form $\frac{0}{0}$;

2d. that the symbol $\frac{0}{0}$, obtained for each of them, does not necessarily indicate the *indetermination* of the equations. The symbol $\frac{A}{0}$ is always a characteristic of *impossibility*, but the symbol $\frac{0}{0}$ is sometimes a characteristic of indetermination, sometimes of impossibility. Sometimes it also indicates the presence of a common factor (72).

In order to ascertain its true signification, the best way is to go back to the equations of the system, and seek directly for the values of the unknown quantities, from these equations.

76. Let there be, for example, the system of equations

$$x+9y+6z=16$$
$$2x+3y+2z=\ 7$$
$$3x+6y+4z=13.$$

By applying the general formulas, we find,

$$x=\frac{0}{0},\quad y=\frac{0}{0},\quad z=\frac{0}{0};$$

but *if we operate directly* upon the equations, by multiplying the second by 3, and subtracting the first from this product, we have

$$5x=21-16=5\,; \quad \text{whence } x=1.$$

Substituting this value in the three equations we obtain

for the first, $9y+6z=15$, or $3y+2z=5$;
. . . . 2d, ... $3y+2z=5$;
. . . . 3d, $6y+4z=10$, or $3y+2z=5$.

Hence the value of x is *determinate* and equal to 1; as to the values of y and z, they are *indeterminate*, since we have but one equation involving two unknown quantities.

The proposed system corresponds with the second of the preceding No., since by dividing the first equation by 3, and the third by 2, the coefficients of y and z become the same.

We will now show how this system of equations has been formed: After having taken arbitrarily the first members of these equations, in such a manner however, that the coefficients of y and z, multiplied together crosswise in the equations taken two and two, form equal products, we have also taken arbitrarily the second members of the first and second equations, which give,

$$\left.\begin{array}{l} x+9y+6z=16 \\ 2x+3y+2z=7 \\ 3x+6y+4z=k \end{array}\right\} \text{ whence } \left\{\begin{array}{l} \frac{1}{3}x+3y+2z=\frac{16}{3}, \\ 2x+3y+2z=7, \\ \frac{3}{2}x+3y+2z=\frac{k}{2}. \end{array}\right.$$

Then $\frac{k}{2}$, or d'' is determined from the following relation:

$$\frac{d''-d}{a''-a}=\frac{d'-d}{a'-a} \text{ of No. 75, by taking}$$

$$a=\frac{1}{3},\ d=\frac{16}{3},\ a'=2,\ d'=7,\ a''=\frac{3}{2},$$

which gives $d''=\frac{13}{2}$, whence $2d''$, or $k=13$.

Let there be another system,

$$\begin{aligned} 11x-\ 8y+\ 6z&=49, \\ 5x-12y+\ 9z&=16, \\ 4x-20y+15z&=15, \end{aligned}$$

which also corresponds with the second in No. 75, but does not satisfy the relation

$$\frac{d''-d}{a''-a}=\frac{d'-d}{a'-a}.$$

By applying the formulas, we would find

$$x=\frac{0}{0},\ y=\frac{A}{0},\ z=\frac{B}{0}.$$

Now let us operate directly upon the equations.

Multiplying the first by 3, and the second by 2, and then subtracting, we have $23x=115$; whence $x=5$. Substituting this value in the three equations, they become

$$\left.\begin{aligned}8y-\ 6z&=6\\12y-\ 9z&=9\\20y-15z&=5\end{aligned}\right\}\quad\text{or}\quad\left\{\begin{aligned}4y-3z&=3,\\4y-3z&=3,\\4y-3z&=1.\end{aligned}\right.$$

The two last equations are evidently *impossible* simultaneously; and if we should apply to them the general formulas relative to two unknown quantities, we would obtain

$$y=\frac{A}{0},\ z=\frac{B}{0}.$$

Therefore, of the three values $\frac{0}{0}$, obtained above, for x, y, z, the first has a determinate signification, $x=5$, and the two others are *infinite*.

77. When we operate directly upon particular examples, there are other characteristics of *impossibility* or *indetermination*.

In the first system, treated of in No. 76, if we take the two equations $\left.\begin{aligned}3y+2z&=5,\\3y+2z&=5,\end{aligned}\right\}$ which have been obtained by the elimination of x, and in order to obtain y or z, we subtract one of the equations from the other, there results $0=0$.

In like manner, in the second system, by taking the two equations $\left.\begin{aligned}4y-3z&=3,\\4y-3z&=1,\end{aligned}\right\}$ and subtracting one from the other, there would result the absurd equality, $0=2$.

The results $0=0$ and $0=A$ are true characteristics of the indetermination, or of the *simultaneous impossibility*, of the equations.

78. We will finish the discussion of equations of the first

degree, by the examination of a particular case. It is that where, in general equations, all of the known quantities which form the second members are supposed to be nothing at the same time. In this case, it evidently follows from the law of the formation of the numerators, in the general values for the unknown quantities, (67), that these numerators all vanish at the same time; that is, we have $A=0$, $B=0$. . . . . Moreover, as there does not exist any particular relation between the coefficients $a, b, c, a', b'\, c'$, ..., of the unknown quantities, D, which results from a certain combination of these coefficients, is generally different from 0. Therefore the values of the unknown quantities will be $x=0$, $y=0$, $z=0$. . . . . . . These values evidently verify the proposed equations.

If, however, besides the hypothesis that the known quantities composing the second members are nothing at the same time, there exists the relation $D=0$ between the coefficients of the unknown quantities, the general values reduce to the form

$$x=\frac{0}{0},\ y=\frac{0}{0},\ \&c.$$

Now I say, that in this case the equations are indeterminate, but the ratios of the unknown quantities are *constant* numbers, which may be obtained from the proposed equations.

Take the three equations

$$ax+by+cz=0,\ a'x+b'y+c'z=0,\ a''x+b''y+c''z=0,$$

in which we suppose that D, or

$$ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''=0.$$

They can be put under the form

$$a\frac{x}{z}+b\frac{y}{z}+c=0,\ a'\frac{x}{z}+b'\frac{y}{z}+c'=0,\ a''\frac{x}{z}+b''\frac{y}{z}+c''=0.$$

Then, by treating $\frac{x}{z}$ and $\frac{y}{z}$ as two unknown quantities, the two first give

$$\frac{x}{z}=\frac{bc'-cb'}{ab'-ba'},\ \frac{y}{z}=\frac{ca'-ac'}{ab'-ba'}.$$

Whence we see, that by giving to z values which are entirely arbitrary, *the values of* x *and* y *could be obtained from these two proportions, of which the second ratios are constant, and equal to known quantities.*

It remains to be shown whether these values satisfy the third equation. By substituting them in this equation it becomes,

$$a''\times\frac{bc'-cb'}{ab'-ba'}+b''\times\frac{ca'-ac'}{ab'-ba'}+c''=0,$$

or, reducing and writing the terms in order,

$$ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''=0,$$

the condition which is satisfied by hypothesis.

79. This naturally brings us to the examination of a circumstance, of which the second problem of the will (49) presents an example: it is that in which the enunciation gives rise to a greater number of different equations than there are unknown quantities to be determined.

Suppose, in general, that the question contains n unknown quantities, and gives rise to m different equations, m being $>n$. *It is necessary in the first place to combine together* n, *of the proposed equations, in order to find the* n *values for the unknown quantities; then to substitute these values in the* m—n *remaining equations, which form as many conditions between the given quantities; and these conditions should be fulfilled, in order that the problem may be possible, in the sense in which it was enunciated.* The $m-n$ equations thus obtained are called *equations of condition.*

80. *Recapitulation of the preceding discussion.* It follows from this discussion, 1st. that a system composed of any number of equations of the first degree, involving the same number of unknown quantities, cannot, in general, be satisfied but in one manner. (66).

2d. That every *positive value*, found for an unknown quantity, answers directly to the equations of this problem, without always answering to its enunciation. (69).

3d. That every *negative value* only answers indirectly to the enunciation, or to the equations which are the algebraic translations of it, but always answers to the equations considered in a sense purely algebraic. (59 and 70).

4th. That *every expression of the form* $\frac{A}{0}$, found for one or more unknown quantities, indicates an incompatibility in the proposed system of equations. (71, 73, and 74).

5th. That every expression of the form $\frac{0}{0}$, indicates either an indetermination, or an incompatibility; (71, 73, 74 and 75,); but that the value of an unknown quantity may be reduced to $\frac{0}{0}$, in consequence of the presence of a common factor to the two terms of the fraction; and it must be attentively examined, in order to see if this is the case. (72).

6th. That if all the second members of the proposed system of equations are nothing, the values of the unknown quantities also become 0; that if to this hypothesis, it is added, that the common denominator of the values of the unknown quantities be 0, the number of systems of values is infinite; but that these values must have a constant ratio to each other. (78).

7th. That when the number of equations exceeds the number of unknown quantities, the problem is only possible so long as the values of the unknown quantities, determined by an equal number of equations, will satisfy the other conditions. (79).

81. The following are the enunciations of some problems which are susceptible of discussion, or present something interesting in their resolution.

Problem 15. *A Banker has two kinds of money; it takes* a *pieces of the first to make crown; and* b *of the second to make the same sum. Some one offers him a crown for* c *pieces. How many of each kind must the banker give him?*

$$\left(\text{Ans.} \ldots \text{1st kind, } \frac{a(c-b)}{a-b}; \quad \text{2d kind, } \frac{b(a-c)}{a-b}\right).$$

Problem 16. *Find the two contiguous sides of a rectangle; supposing, 1st, that these sides are to each other in the ratio of* m : n; *2d, that if these sides be increased or diminished by the quantities* a *and* b, *the surface will be increased or diminished by the quantity* p.

$$\left(\text{Supposing the sides to be increased, we find } x=\frac{m(p-ab)}{na+mb}, \quad y=\frac{n(p-ab)}{na+mb}.\right)$$

Problem 17. *Find what each of three persons* A, B, C, *are worth, knowing 1st, that what* A *is worth added to* l *times what* B *and* C *are worth is equal to* p; *2d, that, what* B *is worth added to* m *times what* A *and* C *are worth is equal to* q; *3d, that* *what* C *is worth added to* n *times what* A *and* B *are worth is equal to* r.

(This question can be resolved in a very simple manner, by introducing an auxiliary unknown quantity into the calculus. This unknown quantity is equal to what *A*, *B* and *C* are worth.

Problem 18. *Find the values of the estates of six persons,* A, B, C, D, E, F, *from the following conditions:* 1st. *The sum of the estates of* A *and* B *is equal to* a; *that of C and D is equal to* b, *and that of* E *and* F *is equal to* c. 2d. *The estate of* A *is worth* m *times that of* c; *the estate of* D *is worth* n *times that of E, and the estate of* F *is worth* p *times that of B.*

(This problem may be resolved by means of a single equation, involving but one unknown quantity.)

CHAPTER III.

Resolution of Problems and Equations of the Second Degree.

82. *Introduction.* When the enunciation of a problem leads to an equation of the form $ax^2=b$, in which the unknown quantity is multiplied by itself, the equation is said to be of the *second degree*, and the principles established in the two preceding chapters are not sufficient for the resolution of it; but since by dividing the two members by a, it becomes $x^2=\frac{b}{a}$, we see that the question is reduced *to finding a number, which, multiplied by itself, will produce the number expressed* by $\frac{b}{a}$. This is the object *of the extraction of the square root.*

The different procedures for extracting the square root of particular numbers, whether whole numbers or fractions, have been fully exposed in arithmetic; it is only necessary then to develope here the rules relative to the extraction of the roots of numbers, expressed algebraically.

§ 1. *Formation of the Square, and Extraction of the Square Root of Algebraic Quantities.*

83. We will first consider the case of a monomial; and in order to discover the process, see how the square of the monomial is formed.

By the rule for the multiplication of monomials, (16), we have

$$(5a^2b^3c)^2=5a^2b^3c\times 5a^2b^3c=25a^4b^6c^2\ ;$$

that is, in order to square a monomial, it is necessary to *square* *its coefficient, and double each of the exponents of the different letters.* Hence, to find the root of the square of a monomial, it is necessary, 1st. *To extract the square root of the coefficient by the rules given in arithmetic.* 2d. *To take the half of each of the exponents.*

Thus, $\sqrt{64a^6b^4}=8a^3b^2$; for $8a^3b^2\times8a^3b^2=64a^6b^4$.

In like manner,

$$\sqrt{625a^2b^8c^6}=25ab^4c^3, \text{ for } (25ab^4c^3)^2=625a^2b^8c^6.$$

From the preceding rule, it follows, in order that one monomial may be the square of another, *its coefficient must be a perfect square, and all of its exponents even numbers.* Thus, $98ab^4$ is not a perfect square, because 98 is not a perfect square, and a is affected with an uneven exponent.

In this case, the quantity is introduced into the calculus by affecting it with the sign $\sqrt{}$, and it is written thus, $\sqrt{98ab^4}$. Quantities of this kind are called *radical quantities*, or *irrational quantities*, or simply *radicals of the second degree.*

84. These expressions may sometimes be simplified, upon the principle that *the square root of the product of two or more factors is equal to the product of the square roots of these factors;* or, in algebraic language, $\sqrt{abcd}\ldots=\sqrt{a}.\sqrt{b}.\sqrt{c}.\sqrt{d}\ldots$

To demonstrate this principle, we will observe, that from the definition of the square root, we have

$$(\sqrt{abcd}\ldots)^2=abcd\ldots$$

Again,

$$(\sqrt{a}\times\sqrt{b}\times\sqrt{c}\times\sqrt{d}\ldots)^2=(\sqrt{a})^2\times(\sqrt{b})^2\times(\sqrt{c})^2\times(\sqrt{d})^2\ldots$$
$$=abcd\ldots$$

Hence, since the squares of $\sqrt{abcd}\ldots$, and, $\sqrt{a}.\sqrt{b}.\sqrt{c}.\sqrt{d}\ldots$, are equal, these quantities themselves are equal.

This being the case, the above expression, $\sqrt{98ab^4}$, can be put under the form $\sqrt{49b^4\times2a}=\sqrt{49b^4}\times\sqrt{2a}$. Now $\sqrt{49b^4}$ may be reduced to $7b^2$, (83); hence $\sqrt{98ab^4}=7b^2.\sqrt{2a}$.

In like manner,

$$\sqrt{45a^3b^3c^2d}=\sqrt{9a^2b^2c^2\times5bd}=3abc.\sqrt{5bd},$$
$$\sqrt{864a^2b^5c^{11}}=\sqrt{144a^2b^4c^{10}\times6bc}=12ab^2c^5.\sqrt{6bc}.$$

In general, in order to simplify an irrational monomial, take *all of those factors which are perfect squares, and extract the root of them,* (83) ; *then place the product of these roots before the radical sign, under which, leave those factors which are not perfect squares.*

In the expressions $7b^2.\sqrt{2a}$, $3abc.\sqrt{5bd}$, $12ab^2c^5.\sqrt{6bc}$, the quantities $7b^2$, $3abc$, $12ab^2c^5$, are called *coefficients of the radical.*

85. As yet, we have not paid any attention to the sign with which the monomial may be affected : but since in the resolution of questions we are led to consider monomial quantities preceded by the sign + or −, it is necessary to know how to operate upon quantities of this kind. Now the square of a monomial being the product of this monomial by itself, it follows (62) that, *whatever may be its sign, the square of it will be positive.* Thus, the square of $+5a^2b^3$, or $-5a^2b^3$, is $+25a^4b^6$.

Whence we may conclude, that if a monomial is positive, its square root may be affected with either the sign + or —; thus, $\sqrt{9a^4}=\pm 3a^2$, for $+3a^2$ or $-3a^2$, squared, gives $9a^4$. The double sign ± with which the root is affected is called *plus or minus.* If the proposed monomial were *negative* it would be impossible to extract its root, since it has just been shown that the square of every quantity whether positive or negative is essentially positive. Therefore, $\sqrt{-9}$, $\sqrt{-4a^2}$ $\sqrt{-5}$ are algebraic symbols which indicate operations that cannot be performed. They are called *imaginary quantities*, or rather *imaginary expressions :* they are symbols of absurdity, frequently met with in the resolution of equations of the second degree. These symbols can, however, by extending the rules, be simplified in the same manner as those irrational expressions which indicate operations that can be performed. Thus $\sqrt{-9}$ may be reduced to (84)

$$\sqrt{9}\times\sqrt{-1} \text{ or } 3\sqrt{-1}\ ;\quad \sqrt{-4a^2}=\sqrt{4a^2}\times\sqrt{-1}=2a\sqrt{-1}\ ;$$

$$\sqrt{-8a^2b}=\sqrt{4a^2}\times\sqrt{-2b}=2a\sqrt{-2b}=2a\sqrt{2b}\times\sqrt{-1}.$$

86. Let us now try to discover the *law of formation* for the square of any polynomial whatever, from which a process may be deduced for extracting the root of this square.

It has already been shown that the square of a binomial $(a+b)$ is equal to $a^2+2ab+b^2$. (19).

Now to form the square of a trinomial $a+b+c$, denote $a+b$ by the single letter s, and we have

$$(a+b+c)^2=(s+c)^2=s^2+2sc+c^2.$$

But $s^2=(a+b)^2=a^2+2ab+b^2$; and $2sc=2(a+b)c=2ac+2bc$.

Hence $(a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2$; that is, *the square of a trinomial is composed of the sum of the squares of its three terms, and twice the product of these terms multiplied together two and two.*

This *law of formation* is true for any polynomial whatever. For suppose it verified for a polynomial of any number of terms, and see if it is true for one containing one more term.

For this purpose, take the polynomial $a+b+c+d\ldots+i+k$, containing $m-1$ terms, and denote the sum of the m first terms $a+b+c+d\ldots\ldots+i$ by s; then $s+k$ will represent the proposed polynomial, and we have $(s+k)^2=s^2+2sk+k^2$; or substituting for s its value,

$$(s+k)^2=(a+b+c+d+\ldots+i)^2+2(a+b+c+d+\ldots i)k+k^2.$$

But by hypothesis, the first part of this expression is composed of *the squares of all the terms of the first polynomial and the double products of these terms taken two and two;* the second part contains *all of the double products of the terms of the* first polynomial by the additional term k; and the third part *is the square of this term.* Therefore, the *law of composition*, announced above, is true for the new polynomial. But it has been proved to be true for a trinomial; hence it is true for a polynomial containing four terms; being true for *four*, it is necessarily true for *five*, and so on. Therefore it is general. This *law* can be enunciated in another manner: viz. *The square of any polynomial contains the square of the first term, plus twice the product of the first by the second, plus the square of the second; plus twice the product of each of the two first terms by the third, plus the square of the third; plus twice the product of each of the three first terms by the fourth; plus the square of the fourth, and so on.* This enunciation which is evidently comprehended in th-first, shows more clearly the process for extracting the square root of a polynomial.

From this law,

$$(5a^2+4ab^2)^2=25a^6-40a^4b^2+16a^2b^4,$$

$$(3a^2-2ab+4b^2)^2=9a^4-12a^3b+4a^2b^2+24a^2b^2-16ab^3+16b^4,$$

or reducing, $$=9a^4+12a^3b+28a^2b^2-16ab^3+16b^4,$$

$$(5a^2b-4abc+6bc^2-3a^2c)^2=25a^4b^2-40a^3b^2c+76a^2b^2c^2$$
$$-48ab^2c^3+36b^2c^4-30a^4bc+24a^3bc^2-36a^2bc^3+9a^4c^2.$$

We will proceed to the extraction of the square root.

Let the proposed polynomial be designated by N, and its root, which we will suppose is determined, by R; conceive, also, that these two polynomials are arranged with reference to one of the letters which they contain, a, for example.

This being supposed, we will observe in the first place, that the two first terms of N (supposing it arranged) will give immediately the first and second terms of R; for it evidently follows, from the law of formation of the square, (86), 1st. *That the exponent of the letter* a, *in the square of the first term of R, is greater than the exponent of this letter in any other of the parts which enter into the composition of the square of R.* 2d. *That the double product of the first term of R by the second, also contains a higher exponent than any of the following parts.* Therefore the two parts just mentioned could not have been reduced with the others, and are necessarily the two terms of N, affected with the highest exponent of a, and the exponent immediately inferior to it; whence it follows, that if N is really a perfect square, 1st. *Its first term must be a perfect square, and its root, extracted by the rule of No.* 83, *is the first term of R.* 2d. *Its second term must be divisible by twice the first term of R, and performing the division, the quotient will be the second term of R.*

To obtain the following terms, *form the square of the binomial, already found*, and subtract it from N; the remainder, which we will designate by N', contains the double product of the first term of R, by the third, plus some other parts. But *the double product of the first term, by the third, must contain* a, *with a higher exponent than this letter has in any of the following* *parts*, and therefore cannot be reduced with any of these parts. Hence this double product is the first term of N': therefore *this first term must be divisible by twice the first term of R; and performing this division, the quotient will be the third term of R.*

In order to obtain the next term, *form the double products of the first and second terms, by the third, plus the square of the third; then subtract all of these products from the remainder N'.* This gives a remainder N'', which contains the double product of the first term of R, by the fourth, plus some other parts. But it can be proved in the same manner as above, that *the first term of N'' is necessarily* the double product of the first term of R by the fourth; therefore, *dividing the first term of N'' by twice the first term of R, the quotient will be the fourth term of R.* The other terms may be found in the same way.

N. B. After having obtained the two first terms of the root, it is absolutely necessary to subtract the square of this binomial from the polynomial N; for the square of the second term of R commonly contains a, with the same exponent that this letter has in the double product of the first term by the third; consequently it might have been reduced with this double product. Therefore it is only after having subtracted this square from the polynomial N, that we are certain that the first term of the remainder is equal to the double product of the first term of R, by the third. The same remark applies to the third, fourth . . . terms.

From the preceding reasoning, the student can deduce the general rule for the extraction of the square root of a polynomial. To do this, he has only to collect together all the parts in *italics*.

We will moreover make an application of it to a particular example.

Extract the square root of the polynomial

$$49a^2b^2-24ab^3+25a^4-30a^3b+16b^4$$

$$\begin{array}{lll}
 & 25a^4-30a^3b+49a^2b^2-24ab^3+16b^4 & \left\} \begin{array}{l} 5a^2-3ab+4b^2 \\ \hline 10a^2 \end{array} \right. \\
 & -25a^4+30a^3b-9a^2b^2 & \\
\hline
\text{1st Rem.} & 40a^2b^2-24ab^3+16b^4 & \\
 & -40a^2b^2+24ab^3-16b^4 & \\
\hline
\text{2d Rem.} & 0 &
\end{array}$$

After having arranged the polynomials with reference to a, extract the square root of $25a^4$, this gives $5a^2$, which is placed to the right of the polynomial; then divide the second term, $-30a^3b$, by the double of $5a^2$, or $10a^2$, (which is written below $5a^2$); the quotient is $-3ab$, and is placed to the right of $5a^2$. Hence, the two first terms of the root are $5a^2-3ab$. Squaring this binomial, it becomes $25a^4-30a^3b+9a^2b^2$, which, subtracted from the proposed polynomial, gives a remainder, of which the first term is $40a^2b^2$. Dividing this first term by $10a^2$, (the double of $5a^2$), the quotient is $+4b^2$; this is the third term of the root, and is written on the right of the two first terms. Forming the double product of $5a^2-3ab$ by $4b^2$, and the square of $4b^2$, we find the polynomial . . . . . . $40a^2b^2-24ab^3+16b^4$, which, subtracted from the first remainder, gives 0. Therefore $5a^2-3ab+4b^2$ is the required root.

Beginners may exercise themselves upon the squares which have been developed in No. 86.

88. When the proposed polynomial contains two or more terms affected with the same power of the principal letter, it should be arranged in the same manner as in division (29), then apply the above process by regarding the *algebraic sum* of the terms affected with the same power, as one and the same part, and substituting in the enunciation of this process, the expressions: *first part* of the polynomial, or *part affected with the highest power*, *first part* of the remainder, 1st, 2d, 3d . . . *part* of the root, for the words, *first term* of the polynomial, *first term* of the remainder, 1st 2d 3d . . . *term* of the root. Examples of this kind occur very rarely.

89. We will conclude here with the following remarks.

1st. A binomial can never be a perfect square, since we know that the square of the most simple polynomial, viz. a binomial, contains three distinct parts, which cannot experience any reduction amongst themselves. Thus, the expression a^2+b^2 is not a perfect square; it wants the term $\pm 2ab$ in order that it should be the square of $a\pm b$.

2nd. In order that a trinomial, when arranged, may be a perfect square, its two extreme terms must be squares, and the middle term must be the double product of the square roots of the two others. Therefore, to obtain the square root of a trinomial when it is a perfect square; *Extract the roots of the two extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative. To verify it, see if the double product of the two roots gives the middle term of the trinomial.* Thus the square root of $9a^6-48a^4b^2+64a^2b^4$, is, $\sqrt{9a^6}$ or $3a^3-\sqrt{64a^2b^4}$ or $-8ab^2$, that is, $3a^3-8ab^2$, for $3a^3\times-16ab^2=-48a^4b^2$. $4a^2+12ab-9b^2$ cannot be a perfect square, although $4a^2$ and $9b^2$ are the squares of $2a$ and $3b$, and $12ab=2a\times6b$; since $-9b^2$ is not a perfect square.

3d. In the series of operations required in a general problem, when the first term of one of the remainders is not exactly divisible by twice the first term of the root, we may conclude that the proposed polynomial is not a perfect square. This is an evident consequence of the course of reasoning, by which we have arrived at the general rule for extracting the square root.

4th. When the polynomial is *not a perfect square*, it may be simplified. (See No. 84.)

Take, for example, the expression $\sqrt{a^3b+4a^2b^2+4ab^3}$.

The quantity under the radical is not a perfect square; but it can be put under the form $ab(a^2+4ab+4b^2)$. Now, the factor between the parenthesis is evidently the square of $a+2b$, whence we may conclude that,

$$\sqrt{a^3b+4a^2b^2+4ab^3}=(a+2b)\sqrt{ab}. \quad (84).$$

90. *Of the calculus of radicals of the second degree.* As the extraction of the square root gives rise to some new algebraic expressions, such as $\sqrt{a}$, $3\sqrt{b}$, $7\sqrt{2}$, called ***irrational quantities*** or *radicals of the second degree*, it is necessary to establish rules for performing the four fundamental operations upon these expressions.

Definition. Two radicals of the second degree are *similar* when the quantities under the radical are the same in both. Thus, $3a\sqrt{b}$ and $5c\sqrt{b}$, $9\sqrt{2}$ and $7\sqrt{2}$ are *similar radicals.*

Addition and Subtraction. In order to add or subtract similar radicals, *add or subtract the two coefficients, then prefix the sum or difference to the common radical;* thus, we have,

$3a\sqrt{b}+5c\sqrt{b}=(3a+5c)\sqrt{b}$; $3a\sqrt{b}-5c\sqrt{b}=(3a-5c)\sqrt{b}$;

in like manner,

$7\sqrt{2a}+3\sqrt{2a}=10\sqrt{2a}$; $7\sqrt{2a}-3\sqrt{2a}=4\sqrt{2a}$.

Two radicals, which do not appear to be similar at first sight, may become so by simplification, (84).

For example,

$\sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a}$,

$2\sqrt{45}-3\sqrt{5}=6\sqrt{5}-3\sqrt{5}=3\sqrt{5}$.

When the radicals are not similar, the addition or subtraction can only be indicated. Thus, in order to add $3\sqrt{b}$ to $5\sqrt{a}$, we write, $5\sqrt{a}+3\sqrt{b}$.

Multiplication. To multiply one radical by another, *multiply the two quantities under the radical sign together, and place the common radical over the product.* Thus $\sqrt{a}\times\sqrt{b}=\sqrt{a\times b}$; this is the principle of No. 84 announced in an inverse order.

When there are coefficients, *we first multiply them together, and write the product before the radical.*

Thus,

$$3\sqrt{5ab}\times4\sqrt{20a}=12\sqrt{100a^2b}=120a\sqrt{b},$$

$$2a\sqrt{bc}\times3a\sqrt{bc}=6a^2\sqrt{b^2c^2}=6a^2bc,$$

$$2a\sqrt{a^2+b^2}\times-3a\sqrt{a^2+b^2}=-6a^2(a^2+b^2).$$

Division. To divide one radical by another, *divide one of the quantities under the radical sign by the other, and place the common radical over the quotient*; $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$. For the squares of these two expressions are equal to the same quantity $\frac{a}{b}$; hence the expressions themselves must be equal. When there are coefficients, *write their quotient as a coefficient of the radical.*

For example,

$$5a\sqrt{b}:2b\sqrt{c}=\frac{5a}{2b}\sqrt{\frac{b}{c}},$$

$$12ac\sqrt{6bc}:4c\sqrt{2b}=3a\sqrt{\frac{6bc}{2b}}=3a\sqrt{3c}.$$

91. There are *two transformations* of frequent use in finding the numerical values of radicals.

The first consists in passing the coefficient of a radical under this radical. Take, for example, the expression $3a\sqrt{5b}$;

it is equivalent to $\sqrt{9a^2} \times \sqrt{5b}$, or $\sqrt{9a^2 . 5b} = \sqrt{45a^2b}$, by applying the rule for the multiplication of two radicals; therefore, *to pass* the coefficient *of a radical under this radical, it is only necessary to square it.*

The principle use of this transformation, is to find a number which shall differ from the proposed radical, *by a quantity less than unity.* Take, for example, the expression $6\sqrt{13}$; as 13 is not a perfect square, we can only obtain an approximate value for its root. This root is equal to 3, plus a certain fraction; this being multiplied by 6, gives 18, plus the product of the fraction by 6; and the entire part of this result, obtained in this way, cannot be greater than 18. The only method of obtaining the entire part exactly, is to put $6\sqrt{13}$ under the form .... $\sqrt{6^2 \times 13} = \sqrt{36 \times 31} = \sqrt{468}$. Now 468 has 21 for the entire part of its square root; hence, $6\sqrt{13}$ is equal to 21, plus a fraction.

In the same way, we find that $12\sqrt{7} = 31$, plus a fraction.

The object of the second transformation is to convert the denominators of such expressions as $\frac{a}{p+\sqrt{q}}$, $\frac{a}{p-\sqrt{q}}$, into rational quantities, a and p being any numbers whatever, and q *not a perfect square.* Expressions of this kind are often met with in the resolution of equations of the second degree.

Now this object is accomplished by multiplying the two terms of the fraction by $p-\sqrt{q}$, when the denominator is $p+\sqrt{q}$, and by $p+\sqrt{q}$, when the denominator is $p-\sqrt{q}$. For multiplying in this manner, and recollecting that the sum of two quantities, multiplied by their difference, is equal to the difference of their squares, (5), we have

$$\frac{a}{p+\sqrt{q}} = \frac{a(p-\sqrt{q})}{(p+\sqrt{q})(p-\sqrt{q})} = \frac{a(p\sqrt{q})}{p^2-q} = \frac{ap-a\sqrt{q}}{p^2-q},$$

$$\frac{a}{p-\sqrt{q}} = \frac{a(p+\sqrt{q})}{(p-\sqrt{q})(p+\sqrt{q})} = \frac{a(p+\sqrt{q})}{p^2-q} = \frac{ap+a\sqrt{q}}{p^2-q},$$

in which the denominators are rational.

To form an idea of the utility of this method, suppose it is required to find the approximate value of the expression

$\frac{7}{3-\sqrt{5}}$. It becomes $\frac{7(3+\sqrt{5})}{9-5}$, or $\frac{21+7\sqrt{5}}{4}$. Now $7\sqrt{5}$ is equivalent to $\sqrt{49\times5}$, or $\sqrt{245}$, which is equal to 15, *within one* of the true value.

Therefore $\frac{7}{3-\sqrt{5}}=\frac{21+15+\text{a fraction}}{4}=\frac{36}{4}=9$, within a fraction of *one fourth;* that is, it differs from the true value by a quantity less than *one fourth.*

When we wish to have a more exact value for this expression, *extract the square root of 245 to a certain number of decimal places, add 21 to this root, and divide the result by 4, or take a 4th part of it.*

For another example, take $\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$, and find the value of it to within 0.01.

We have,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\frac{7\sqrt{5}(\sqrt{11}-\sqrt{3})}{11-3}=\frac{7\sqrt{55}-7\sqrt{15}}{8}.$$

Now, $7\sqrt{55}=\sqrt{55\times49}=\sqrt{2695}=51.91$, within 0.01,

$$7\sqrt{15}=\sqrt{15\times49}=\sqrt{735}=27.11\ldots;$$

therefore,

$$\frac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}=\frac{51{,}91-27{,}11}{8}=\frac{24{,}80}{8}=3{,}10.$$

Hence, we have 3,10 for the required result. This is exact to within $\frac{1}{800}$.

By a similar process, it will be found that

$$\frac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{6}}=2{,}123, \text{ exact to within } 0{,}001.$$

N. B. Expressions of this kind might be calculated by approximating to the value of each of the radicals which enter the numerator and denominator. But as the value of the denominator would not be exact, we could not form a precise idea of the degree of approximation which would be obtained, whereas by the method just indicated, the denominator becomes *rational,* and we always know to what degree the approximation is made.

The principles for the extraction of the square root of particular numbers and of algebraic quantities, being established, we will proceed to the resolution of problems of the second degree.

§ 2. *Problems and Equations of the Second Degree.*

92. Equations of the second degree are of two kinds, viz. equations involving *two terms*, or *incomplete* equations, and equations involving *three terms* or *complete* equations.

The first are those which contain only terms involving the square of the unknown quantity, and known terms; such are the equations,

$$3x^2=5;\ 4x^2-7=3x^2+9;\ \frac{1}{3}x^2-3+\frac{5}{12}x^2=\frac{7}{24}-x^2+\frac{299}{24}.$$

They are called equations involving *two terms* because they may be reduced to the form $ax^2=b$, by means of the two general transformations (43 and 44). For, let us consider the third equation, which is the most complicated; by clearing the fraction it becomes $8x^2-72+10x^2=7-24x^2+299$; or, transposing and reducing,

$$42x^2=378.$$

Equations involving three terms, or complete equations, are those which contain the square, and also the first power of the unknown quantity; such are the equations

$$5x^2-7x=34;\ \frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}.$$

They can always be reduced to the form $ax^2+bx=c$, by the two transformations already cited.

Equations involving two terms. There is no difficulty in the resolution of the equation $ax^2=b$. We deduce from it $x^2=\frac{b}{a}$, whence $x=\sqrt{\frac{b}{a}}$.

When $\frac{b}{a}$ is a particular number, either entire or fractional, we can obtain the square root of it exactly, or by approximation. If $\frac{b}{a}$ is algebraic, we apply the rules established for algebraic quantities.

But as the *square of* $+m$ or $-m$ is $+m^2$, it follows that

$\left(\pm\sqrt{\frac{b}{a}}\right)^2$ is equal to $\frac{b}{a}$. Therefore the equation is susceptible of two solutions, viz: $x=+\sqrt{\frac{b}{a}}$, and $x=-\sqrt{\frac{b}{a}}$. For, substituting each of these values in the equation $ax^2=b$, it becomes

$$a\times\left(+\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b, \text{ or } b=b,$$

$$\text{and } a\times\left(-\sqrt{\frac{b}{a}}\right)^2=b, \text{ or } a\times\frac{b}{a}=b, \text{ or } b=b.$$

For another example take the equation $4x^2-7=3x^2+9$; by transposing, it becomes, $x^2=16$, whence $x=\pm\sqrt{16}=\pm4$.

Again, take the equation

$$\frac{1}{3}x^2-3+\frac{5}{12}x^2=\frac{7}{24}-x^2+\frac{299}{24};$$

We have already seen (92) that this equation reduces to $42x^2=378$, and dividing by 42, $x^2=\frac{378}{42}=9$; hence $x=\pm3$.

Lastly, from the equation $3x^2=5$; we find

$$x=\pm\sqrt{\frac{5}{3}}=\pm\frac{1}{3}\sqrt{15}.$$

As 15 is not a perfect square, the values of x can only be determined by approximation.

93. *Complete Equations of the second degree.* In order to resolve the general equation $ax^2+bx=c$, we begin by dividing both members of it by the coefficient of x^2, which gives

$$x^2+\frac{b}{a}x=\frac{c}{a}, \text{ or } x^2+px=q.$$

by making $\frac{b}{a}=p$ and $\frac{c}{a}=q$.

Now, if we could make the first member x^2+px the square of a binomial, the equation might be reduced to one of the first degree, by simply extracting the square root. By comparing this member with the square of the binomial $(x+a)$, that is, with $x^2+2xa+a^2$, it is plain that x^2+px is composed of the square of a first term x, plus the double product of this first term x by a second, which must be $\frac{p}{2}$ (since $px=2\frac{p}{2}x$); therefore if the square of $\frac{p}{2}$ or $\frac{p^2}{4}$, is added to x^2+px, the first member of the equation will become the square of $x+\frac{p}{2}$; but in order that the equality may not be destroyed $\frac{p^2}{4}$ must be added to the second member.

By this transformation, it becomes $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}+q$, whence, extracting the square root $x+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}+q}$.

(The double sign $\pm$ is placed here, because either $+\sqrt{\frac{p^2}{4}+q}$, or $-\sqrt{\frac{p^2}{4}+q}$, squared, gives $\frac{p^2}{4}+q$).

Transposing $\frac{p}{2}$, we obtain

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

From this we derive the following general rule for the resolution of complete equations of the second degree: *After reducing the equation to the form* $x^2+px=q$, *add the square of half of the coefficient of* x, *or of the second term, to both members; extract the square root of both members, giving the double sign* $\pm$ *to the second member; then find the value of* x *from the resulting equation.*

The double value of x, obtained in this manner, may be thus enunciated in ordinary language: *The half of the coefficient of* x, *taken with a contrary sign, plus or minus the square root of the known term, increased by the square of half of the coefficient of* x.

94. Take, for an example, the equation

$$\frac{5}{6}x^2-\frac{1}{2}x+\frac{3}{4}=8-\frac{2}{3}x-x^2+\frac{273}{12}.$$

Clearing the fraction, we have

$$10x^2-6x+9=96-8x-12x^2+273,$$

or, transposing and reducing,

$$22x^2+2x=360,$$

and dividing both members by 22,

$$x^2+\frac{2}{22}x=\frac{360}{22}.$$

Add $\left(\frac{1}{22}\right)^2$ to both members, and the equation becomes

$$x^2+\frac{2}{22}x+\left(\frac{1}{22}\right)^2=\frac{360}{22}+\left(\frac{1}{22}\right)^2;$$

whence, by extracting the square root,

$$x+\frac{1}{22}=\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2},$$

Therefore,

$$x=-\frac{1}{22}\pm\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2},$$

which agrees with the enunciation given above for the double value of x.

It remains to perform the numerical operations. In the first place, $\frac{360}{22}+\left(\frac{1}{22}\right)^2$ must be reduced to a single number, having $(22)^2$ for its denominator.

Now, $\frac{360}{22}+\left(\frac{1}{22}\right)^2=\frac{360\times22+1}{(22)^2}=\frac{7921}{(22)^2};$

extracting the square root of 7921, we find it to be 89; therefore, $\sqrt{\frac{360}{22}+\left(\frac{1}{22}\right)^2}=\frac{89}{22}$.

Consequently, $x=-\frac{1}{22}\pm\frac{89}{22}$.

Separating the two values, we have

$$x=-\frac{1}{22}+\frac{89}{22}=\frac{88}{22}=4,$$

$$x=-\frac{1}{22}-\frac{89}{22}=-\frac{45}{11}.$$

Therefore, one of the two values which satisfy the proposed equation, is a positive whole number, and the other a negative fraction.

For another example, take the equation

$$6x^2-37x=-57,$$

which reduces to $x^2-\frac{37}{6}x=-\frac{57}{6}$.

If we add the square of $\frac{37}{12}$, or $\left(\frac{37}{12}\right)^2$ to both members, it becomes $x^2-\frac{37}{6}x+\left(\frac{37}{12}\right)^2=-\frac{57}{6}+\left(\frac{37}{12}\right)^2$; whence, by extracting the square root $x-\frac{37}{12}=\pm\sqrt{-\frac{57}{6}+\left(\frac{37}{12}\right)^2}$.

Consequently, $x=\frac{37}{12}\pm\sqrt{-\frac{57}{6}+\left(\frac{37}{12}\right)^2}$.

In order to reduce $\left(\frac{37}{12}\right)^2-\frac{57}{6}$ to a single number, we will observe, that $(12)^2=12\times12=6\times24$; therefore it is only necessary to multiply 57 by 24, then 37 by itself, and divide the difference of the two products by $(12)^2$.

Now, $37\times37=1369$; $57\times24=1368$;

therefore, $\left(\frac{37}{12}\right)^2-\frac{57}{6}=\frac{1}{(12)^2}$.

the square root of which is $\frac{1}{12}$.

Hence, $x=\frac{37}{12}\pm\frac{1}{12}$, or $\begin{cases} x=\frac{37}{12}+\frac{1}{12}=\frac{38}{12}=\frac{19}{6}. \\ x=\frac{37}{12}-\frac{1}{12}=\frac{36}{12}=3. \end{cases}$

This example is remarkable, as both of the values are positive, and answer directly to the enunciation of the question, of which the proposed equation is the algebraic translation.

Let us now take the literal equation

$$4a^2-2x^2+2ax=18ab-18b^2.$$

By transposing, changing the signs, and dividing by 2, it becomes

$$x^2-ax=2a^2-9ab+9b^2;\text{ whence,}$$

completing the square, $x^2-ax+\frac{a^2}{4}=\frac{9a^2}{4}-9ab+9b^2$.

extracting the square root, $x=\frac{a}{2}\pm\sqrt{\frac{9a^2}{4}-9ab+9b^2}$.

Now, the square root of $\frac{9a^2}{4}-9ab+9b^2$, is evidently, $\frac{3a}{2}-3b$.

Therefore $x=\frac{a}{2}\pm\left(\frac{3a}{2}-3b\right)$, or $\begin{cases} x=2a-3b, \\ x=-a+3b. \end{cases}$

These two values will be positive at the same time, if $2a>3b$, and $3b>a$, that is if the numerical value of b is greater than $\frac{a}{3}$ and less than $\frac{2a}{3}$.

Examples.

$$x^2-7x+10=0 \ldots \text{values} \left\{\begin{matrix} x=2 \\ x=5 \end{matrix}\right\},$$

$$\frac{1}{3}x-4-x^2+2x-\frac{4}{5}x^2=45-3x^2+4x \left\{\begin{matrix} x=7{,}12 \\ x=-5{,}73 \end{matrix}\right\} \text{to within } 0{,}01.$$

$$\left\{\begin{matrix} a^2+b^2-2bx+x^2=\frac{m^2x^2}{n^2} \text{ - - - gives} \\ x=\frac{n}{n^2-m^2}\left(bn\pm\sqrt{a^2m^2+b^2m^2-a^2n^2}\right) \end{matrix}\right\}$$

95. The equation $ax^2+bx=c$, can be resolved without dividing by the coefficient of x^2, but the transformations are more complicated. The term ax^2 can be put under the form $(x\sqrt{a})^2$, and the term bx is equal to $2x\sqrt{a}\times\frac{b}{2\sqrt{a}}$ (multiplying and dividing by $2\sqrt{a}$); from which it follows that ax^2+bx form the two first terms of the square of $x\sqrt{a}+\frac{b}{2\sqrt{a}}$; therefore by adding $\left(\frac{b}{2\sqrt{a}}\right)^2$ or $\frac{b^2}{4a}$ to both members, the first will become a perfect square.

Effecting this transformation the equation becomes

$$ax^2+bx+\frac{b^2}{4a}=c+\frac{b^2}{4a}$$

Extracting the root, $x\sqrt{a}+\frac{b}{2\sqrt{a}}=\pm\sqrt{c+\frac{b^2}{4a}}$;

Transposing, $x\sqrt{a}=-\frac{b}{2\sqrt{a}}\pm\sqrt{c+\frac{b^2}{4a}}$.

Dividing the two members by $\sqrt{a}$, and observing

1st. that $-\frac{b}{2\sqrt{a}}:\ \sqrt{a}=-\frac{b}{2(\sqrt{a})^2}=-\frac{b}{2a}$,

2d. $\sqrt{c+\frac{b^2}{4a}}:\ \sqrt{a}=\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}$. (No. 90), we obtain $x=-\frac{b}{2a}\pm\sqrt{\frac{c}{a}+\frac{b^2}{4a^2}}$ or $x=\frac{-b\pm\sqrt{4ac+b^2}}{2a}$

This result might have been obtained more easily, by putting the equation under the form $x^2+\frac{b}{a}x=\frac{c}{a}$.

96. We will now apply the preceding principles to the resolution of some problems.

Problem 1st. *Find a number such, that twice its square, increased by three times this number, shall give* 65.

Let x be the unknown number, the equation of the problem will be

$$2x^2+3x=65,$$

whence $x=-\frac{3}{4}\pm\sqrt{\frac{65}{2}+\frac{9}{16}}=-\frac{3}{4}\pm\frac{23}{4}$.

Therefore $x=-\frac{3}{4}+\frac{23}{4}=5$, and $x=-\frac{3}{4}-\frac{23}{4}=-\frac{13}{2}$.

The first value satisfies the question in the sense enunciated. For

$$2\times(5)^2+3\times 5=2\times 25+15=65.$$

In order to interpret the second, we will first observe, that when x is replaced by $-x$, in the equation $2x^2+3x=65$, the sign of the second term $3x$ only is changed, because $(-x)^2=x^2$.

Therefore, instead of obtaining $x=-\frac{3}{4}\pm\frac{23}{4}$, we would find $x=\frac{3}{4}\pm\frac{23}{4}$, or $x=\frac{13}{2}$ and $x=-5$, values which only differ from the preceding by their signs. Hence, we may say that the negative solution $-\frac{13}{2}$, considered independently of its sign, satisfies the new enunciation, viz.: *To find a number such, that twice its square, diminished by three times this number, shall give* 65. In fact we have

$$2\times\left(\frac{13}{2}\right)^2-3\times\frac{13}{2}=\frac{169}{2}-\frac{39}{2}=65.$$

Second problem. *A certain person purchased a number of yards of cloth for* 240 *francs. If he had received* 3 *yards less of the same cloth, for the same sum, it would have cost him* 4 *francs more per yard. How many yards did he purchase?*

Let x be the number of yards purchased; then $\frac{240}{x}$ will express the price per yard. If, for 240 francs, he had received 3 yards less, that is $x-3$ yds, the price per yard in this hypothesis would have been represented by $\frac{240}{x-3}$. But, by the enunciation this last price exceeds the first by 4 francs. Therefore, we have the equation.

$$\frac{240}{x-3}-\frac{240}{x}=4\,;$$

whence, by reducing $x^2-3x=180$,

$$x=\frac{3}{2}\pm\sqrt{\frac{9}{4}+180}=\frac{3\pm27}{2};$$

therefore $x=15$, and $x=-12$.

The value $x=15$ satisfies the enunciation; for, 15 yards for 240 francs, gives $\frac{240}{15}$, or 16 francs for the price of one yard, and 12 yards for 240 francs, gives 20 francs for the price of one yard, which exceeds 16 by 4.

As to the second solution, we can form a new enunciation, with which it will agree. For, go back to the equation, and change x into $-x$, it becomes,

$$\frac{240}{-x-3}-\frac{240}{-x}=4, \text{ or } \frac{240}{x}-\frac{240}{x+3}=4,$$

an equation which may be considered the algebraic translation of this problem, viz.: *A certain person purchased a number of yards of cloth for 240 francs: if he had paid the same sum for 3 yards more, it would have cost him 4 francs less per yard. How many yards did he purchase?*

Answer, $x=12$, and $x=-15$.

N. B. Hence the principle of No. 59 is confirmed for two problems of the second degree, as it has been for all problems of the first degree. (See No. 99.)

Problem 3d. *A merchant discounted two notes, one of 8776 francs, payable in 9 months, the other of 7488 francs, payable in 8 months. He paid 1200 francs more for the first than the second. At what rate of interest did he discount them?*

Solution. To simplify the operation, denote the interest of

100 francs for one month by x, or the annual interest by $12x$; $9x$ and $8x$ are the interests for 9 and 8 months. Hence $100+9x$ and $100+8x$, represent what the capital of 100 francs will be at the end of 9 and 8 months. Therefore, to determine the ***actual values*** of the note for 8776 francs, and 7488 francs, make the two proportions,

$$100+9x:100::8776:\frac{877600}{100+9x},$$

$$100+8x:100::7488:\frac{748800}{100+8x};$$

and the fourth terms of these proportions will express what the merchant paid for each note. Hence, we have the equation

$$\frac{877600}{100+9x}-\frac{748800}{100+8x}=1200;$$

or, observing that the two members are divisible by 400,

$$\frac{2194}{100+9x}-\frac{1872}{100+8x}=3.$$

Clearing the fraction, and reducing, it becomes,

$$216x^2+4396x=2200;$$

whence $$x=-\frac{2198}{216}\pm\sqrt{\frac{2200}{216}+\frac{(2198)^2}{(216)^2}}.$$

Reducing the two terms under the radical to the same denominator, $$x=\frac{-2198\pm\sqrt{5306404}}{216},$$

or multiplying by 12, $$12x=\frac{-2198\pm\sqrt{5306404}}{18}.$$

To obtain the value of $12x$ to within 0,01, we have only to extract the square root of 5306404 to within 0,1, since it is afterwards to be divided by 18.

This root is 2303,5;

hence $$12x=\frac{2198-2303,5}{18};$$

and consequently, $$12x=\frac{105,5}{18}=5,86,$$

and $$12x=\frac{-4501,5}{18}=-250,08.$$

The positive value, $12x=5.86$, therefore represents the rate of interest sought.

As to the negative solution, it can only be regarded as connected with the first by an equation of the second degree. By going back to the equation, and changing x into $-x$, we could with some trouble translate the new equation into an enunciation analogous to that of the proposed problem.

Problem 4th. *A man bought a horse, which he sold after some time for 24 dollars. At this sale, he loses as much per cent. upon the price of his purchase, as the horse cost him. What did he pay for the horse?*

Solution. Let x denote the number of dollars that he paid for the horse, $x-24$ will be the expression for the loss he sustained. But as he lost x per cent. by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he loses a sum denoted by $\frac{x^2}{100}$; we have then the equation

$$\frac{x^2}{100}=x-24, \text{ whence } x^2-100x=-2400.$$

$$\text{and } x=50\pm\sqrt{2500-2400}=50\pm10.$$

Therefore, $x=60$ and $x=40$.

Both of these values satisfy the question.

For in the first place suppose the man gave 60 for the horse and sold him for 24, he loses 36. Again, from the enunciation, he should lose 60 *per cent.* of 60, that is, $\frac{60}{100}$ of 60, or $\frac{60\times60}{100}$, which reduces to 36; therefore 60 satisfies the enunciation.

If he paid 40 for the horse, he loses 16 by the sale; Moreover, he should lose 40 *per cent.* of $40\times\frac{40}{100}$, which reduces to 16; therefore 40 verifies the enunciation.

Discussion of the general Equation of the Second Degree.

As yet we have only resolved problems of the second degree, in which the known quantities were expressed by particular numbers. To be able to resolve general problems, and interpret *all of the results* obtained, by attributing particular values to the given quantities, it is necessary to resume the general equation of the second degree, and to examine the circumstances which result from every possible hypothesis made upon its coefficients. This is the object of *the discussion of the equation of the second degree.*

Previous to this discussion, we will explain another method of resolving an equation of the second degree, which will lead to some important properties of the values of the unknown quantity.

It has been shown (93) that every equation of the second degree can be reduced to the form

$$x^2+px=q \quad . \; . \; . \; . \; (1),$$

p and q being numerical or algebraic quantities, whole numbers or fractions, and their signs plus or minus.

If, in order to render the first member a perfect square, we add $\frac{p^2}{4}$ to both members it becomes

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}$$

$$\text{or } \left(x+\frac{p}{2}\right)^2=q+\frac{p^2}{4}.$$

Whatever may be the value of the number expressed by $q+\frac{p^2}{4}$, its root can be denoted by m, and the equation becomes

$$\left(x+\frac{p}{2}\right)^2=m^2, \quad \text{or} \quad \left(x+\frac{p}{2}\right)^2-m^2=0.$$

but as the first member of this equation is the difference between two squares, it can (19) be put under the form

$$\left(x+\frac{p}{2}-m\right)\cdot\left(x+\frac{p}{2}+m\right)=0 \; ; \quad . \; . \; . \; (2).$$

in which the first member is the product of two factors, and the second is 0. Now we can render the product equal to 0, and consequently satisfy the equation (2) in two different ways: viz.

By supposing $x+\frac{p}{2}-m=0$, whence $x=-\frac{p}{2}+m$.

or supposing $x+\frac{p}{2}+m=0$, whence $x=-\frac{p}{2}-m$.

or substituting for m its value, $\begin{cases} x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}, \\ x=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}, \end{cases}$

It is evident that we can render the first member of equation (2) equal to 0, by substituting such a number for x, as will cause either of its factors to vanish.

Therefore as equation (2) is a consequence of equation (1), and reciprocally, *every equation of the second degree admits of two values for the unknown quantity, and cannot have any more.* This method, which is perhaps a little longer than the first, has the advantage of showing more clearly that there are two values of the unknown quantity, and two only.

98. These values have some remarkable properties.

Firstly, since the equation

$$x^2+px=q, \text{ or } x^2+px-q=0$$

can be transformed into $\left(x+\frac{p}{2}-m\right)\left(x+\frac{p}{2}+m\right)=0$, m being equal to $\sqrt{q+\frac{p^2}{4}}$, it follows, that *the first member* x^2+px-q *of every equation of the second degree, of which the second member is* 0, *is composed of the product of two binomials of the first degree with respect to* x, *having* x *for a common term, and the two values of* x *taken with contrary signs for their second terms.*

From this property, by means of which the equation may be obtained when the values of the unknown quantities are found, these values are called the *roots* of the equation.

Thus, the equation $x^2+3x-28=0$ being resolved, gives $x=4, x=-7$. The first member results from the product.... $(x-4)(x+7)$; for it becomes

$$x^2-4x+7x-28=x^2+3x-28.$$

Secondly. If we denote the two roots of the equation by x' and x'', we have from the preceding property

$$x^2+px-q=(x-x')(x-x''),$$

or by performing the operation indicated,

$$x^2+px-q=x^2-(x'+x'')x+x'x''.$$

By comparing the analogous terms, we find

$$x'+x''=-p,\ x'x''=-q.$$

Hence, 1st. *The algebraic sum of the two roots is equal to the coefficient of the second term of the equation, taken with a contrary sign.* 2d. *The product of the two roots is equal to the last term of the equation;* i. e. *the known quantity, when transposed into the first number.*

These two properties can be verified from the general expressions for the two roots.

For by adding

$$x'=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}} \text{ to } x''=-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}},$$

we obtain,

$$x'+x''=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}=-p;$$

and by multiplying them together, we have

$$x'x''=\left(-\frac{p}{2}+\sqrt{q^2+\frac{p^2}{4}}\right)\left(-\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}\right)$$

$$=\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)=-q.$$

N. B. The preceding properties suppose that the equation has been reduced to the form $x^2+px-q=0$; *i. e.* 1st. That every term of the equation has been divided by the coefficient of x^2. 2d. That all the terms have been transposed and arranged in the first member.

Discussion.

99. The general equation $x^2+px=q$ being resolved, gives

$$x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

In order that the value of this expression may be estimated, either exactly, or by approximation, it is necessary (85) that the quantity under the radical, *i. e.* $q+\frac{p^2}{4}$, should be positive. Now $\frac{p^2}{4}$ being necessarily positive, whatever may be the sign of

p, it follows that the sign of $q+\frac{p}{4}$ will principally depend upon that of q, or the known term of the equation.

This being the case, let us first *suppose* q *positive*, then the equation will be of the form $x^2 \pm px = +q$. It is visible that the two values of x are real, and may be determined exactly, if $q+\frac{p^2}{4}$ is a perfect square, or to any degree of approximation we please, if it is not. The *first* of these *two values* is positive, and answers directly to the equation, or to the problem of which this equation is the algebraic translation; for the radical $\sqrt{q+\frac{p^2}{4}}$ being numerically greater than $\frac{p}{2}$, the expression . . . $\mp\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}$ is necessarily of the same sign as that of the radical.

For the same reason, *the second value is essentially negative*, since it should have the same sign as that with which the radical is affected. Considered independent of its sign, it does not answer to the equation in the sense in which it was stated, but to this equation when x has been replaced by $-x$; that is, to $x^2 \mp px = q$. In fact, this new equation gives

$$x = \pm\frac{p}{2} \pm \sqrt{q+\frac{p^2}{4}},$$

values which only differ from the preceding in their signs.

100. Suppose q *essentially negative*, in which case the equation is of the form $x^2 \pm px = -q$, and the values of x are

$$x = \mp\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

In order that this root may be extracted, it is *necessary* that $q < \frac{p^2}{4}$. When this condition is satisfied, the two values are *real*.

Moreover, since $\sqrt{\frac{p^2}{4}-q}$ is numerically less than $\frac{p}{2}$, it follows that these values are *both negative*, when p is positive in the equation; that is, when the equation is of the form . . . . . . $x^2+px=-q$; and they are *both positive*, when p is negative; *that is, when* the equation is of the form $x^2-px=-q$.

The same consequences would result from the two following properties of the equation $x^2+px-q=0$, viz.: *The algebraic sum of the roots is equal to the coefficient of the second term, taken with a contrary sign, and their product is equal to the last term, or the known term transposed into the first member.*

For *suppose* q *positive* in this second member, and consequently negative in the first, it follows that the product of the two roots is negative; hence, *they have contrary signs.*

Moreover, their sum will be *negative* or *positive* according as p is positive or negative; or the root of the greatest numerical value will have its sign contrary to that of the coefficient p. But when q *is negative* in the second member, and consequently positive in the first, the product of the two roots is positive; hence *they are of the same sign*, viz. *both negative* when p is positive, and both positive when p is negative, for in the first case their algebraic sum is negative, and in the second it is positive.

It might be shown *à priori* that, when q is negative in the second member, and p, negative in the first, the problem admits of two direct solutions, provided $q<\frac{p^2}{4}$.

The equation $x^2-px=-q$, may, by changing the signs of both members, be put under the form

$$px-x^2=q, \text{ or } x(p-x)=q.$$

Now this new equation is evidently the algebraic translation of this enunciation, viz.: *Divide a given number* p, *into two such parts that their product shall be equal to another given number* q; for if we denote one of the parts by x, the other will be denoted by $p-x$, and their product will be expressed by $x\,(p-x)$.

This being the case, I say at once, that the enunciation of the problem admits of two direct solutions. To prove it, we will remark that the equation is always the same, whether x denotes the greater or less part; therefore when the equation is resolved, there is no reason why it should give one part rather than the other; it must therefore give them both at the same time.

Otherwise, the two required parts are such, that their sum shall be equal to p, and their product equal to q. Now, these relations exist between the roots and coefficients of the equation $x^2-px=-q$, or $x^2-px+q=0$; hence the two required parts are equal to the two roots of this equation.

In the second place, in order that the problem may be possible, we must have $q<\frac{p^2}{4}$.

For whatever may be the two required parts, their difference can be denoted by d; and as their sum is p, we will have, by the theorem (4) . . $\frac{p}{2}+\frac{d}{2}$, for the greater part;

and . . $\frac{p}{2}-\frac{d}{2}$, for the less part.

Taking the product of these expressions, we have

$$\frac{p^2}{4}-\frac{d^2}{4}.$$

which is essentially less than $\frac{p^2}{4}$, unless the two parts are supposed to be equal, in which case $d=0$, and the product reduces to $\frac{p^2}{4}$. It is therefore absurd to require that the product, which is represented by q, should be greater than $\frac{p}{4}$. Whence we may conclude, that *when the known quantity is negative in the second member, but numerically greater than the square of half of the coefficient of the second term, the proposed question is absurd.*

Remark. It also follows, that *the greatest product that can be obtained, by decomposing a number into two parts, and multiplying these two parts together, is the square of half of the number.* For we have just seen that this product can be expressed by $\frac{p^2}{4}-\frac{d^2}{4}$, which is less than $\frac{p^2}{4}$, but which becomes equal to it when we suppose $d=0$, or the two parts equal.

101. 1st. If, when q is negative, that is, when the equation is of the form $x^2+px=-q$ (p having either sign), we suppose $q=\frac{p^2}{4}$, the radical part of the two values of x becomes 0, and both of these values reduce to $x=-\frac{p}{2}$: *the two roots are then said to be equal.*

In fact, by substituting $\frac{p^2}{4}$ for q in the equation, it becomes

$x^2+px=-\frac{p^2}{4}$, whence

$$x^2+px+\frac{p^2}{4}=0, \text{ or } \left(x+\frac{p}{2}\right)^2=0.$$

In this case, the first member is the *product of two equal factors.* Hence we may also say, that the roots of the equation are equal, since in this case the two factors being placed equal to zero, give the same value for x.

2d. If, in the general equation, $x^2+px=q$, we suppose $q=0$, the two values of x reduce to $x=-\frac{p}{2}+\frac{p}{2}$, or $x=0$, and to $x=-\frac{p}{2}-\frac{p}{2}$, or $x=-p$.

In fact, the equation is then of the form $x^2+px=0$, or $x(x+p)=0$, which can be satisfied either by supposing $x=0$, or $x+p=0$, whence $x=-p$.

3d. If in the general equation $x^2+px=q$, we suppose $p=0$, there will result $x^2=q$, whence $x=\pm\sqrt{q}$; that is, in this case *the two values of* x *are equal, and have contrary signs, real* when q is positive, and *imaginary* when q is negative. The equation then belongs to the class of equations involving two terms, treated of in No. 92.

4th. Suppose we have at the same time $p=0$, $q=0$; the equation reduces to $x^2=0$, and gives two values of x, equal to 0.

102. There remains a singular case to be examined, which is often met with in the resolution of problems of the second degree.

For this purpose, take the equation $ax^2+bx=c$. This equation gives $x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$.

Suppose now, that from a particular hypothesis made upon the given quantities of the question, we have $a=0$; the expression for x becomes $x=\frac{-b\pm b}{0}$, whence $\begin{cases} x=\frac{0}{0}, \\ x=-\frac{2b}{0}. \end{cases}$

The second value is presented under the form of *infinity*, and may be considered as an answer when the proposed questions will admit of infinite solutions.

As to the first $\frac{0}{0}$, we must endeavour to interpret it.

In the first place, by examining the equation, we see that the hypothesis $a=0$ reduces it to $bx=c$, whence $x=\frac{c}{b}$, a *finite* and *determinate* expression, which ought to be considered as representing the true value of $\frac{0}{0}$ in the present case.

But to leave no doubt upon this subject, we will resume the equation $ax^2+bx=c$, and suppose $x=\frac{1}{y}$; there will result

$$\frac{a}{y^2}+\frac{b}{y}=c, \text{ or } cy^2-by-a=0,$$

(by clearing the fraction, and transposing.)

Now, suppose $a=0$; this last equation becomes $cy^2-by=0$, and gives two values, $y=0, y=\frac{b}{c}$.

Substituting these values in the relation $x=\frac{1}{y}$, we have

$$\text{1st. } x=\frac{1}{0}, \text{ 2d. } x=\frac{c}{b}.$$

If, besides the hypothesis $a=0$, we have also $b=0$, the value $x=\frac{c}{b}$ becomes $\frac{c}{0}$, or *infinite*.

And in fact, the equation $cy^2-by-a=0$, in this double hypothesis reduces to $cy^2=0$, in which the two values of y are equal to 0; therefore the corresponding values of x are *infinite*.

If we had at the same time $a=0$, $b=0$, $c=0$, the proposed equation would be altogether indeterminate.

This is the only case of indetermination that the equation of the second degree presents.

We are now going to apply the principles of this general discussion to problems which will give rise to all the circumstances which are commonly met with in problems of the *second degree*.

Problem of the Lights.

C″ A C B C′

103. Problem 15. ***Find upon the line which joins two lights, A and B, of different intensities, the point which is equally illuminated;*** admitting the following principle of physics, viz.: ***The intensity of the same light at two different distances is in the inverse ratio of the squares of these distances.***

Solution. Let a be the distance AB between the two lights, b the intensity of the light A, at the units distance, c that of the light B, at the same distance. Let C be the required point, and make $AC=x$, whence $BC=a-x$.

From the principle of physics, the intensity of A, at the *unity* of distance, being b, its intensity at the distances 2, 3, 4 ...., is $\frac{b}{4}$, $\frac{b}{9}$, $\frac{b}{16}$ ...., hence at the distance x it will be expressed by $\frac{b}{x^2}$. In like manner, the intensity of B at the distance $a-x$, is $\frac{c}{(a-x)^2}$; but, by the enunciation, these two intensities are equal to each other, therefore we have the equation

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}.$$

Whence, by developing and reducing,

$$(b-c)x^2-2abx=-a^2b.$$

This equation gives $x=\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}}$,

or reducing, $\quad x=\frac{a(b\pm\sqrt{bc})}{b-c}.$

This expression may be simplified by observing, 1st. that $b\pm\sqrt{bc}$ can be put under the form $\sqrt{b}.\sqrt{b}\pm\sqrt{b}.\sqrt{c}$, or $\sqrt{b}(\sqrt{b}\pm\sqrt{c})$; 2d. that $b-c=(\sqrt{b})^2-(\sqrt{c})^2=(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c}.)$ Therefore, by first considering the superior sign of the above expression, we have

$$x=\frac{a\sqrt{b}(\sqrt{b}+\sqrt{c})}{(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})}=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}.$$

In like manner we obtain for the second value,

$$x=\frac{a\sqrt{b}(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})}=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}.$$

These simplified values might be obtained immediately from the proposed equations. For the equation

$$\frac{b}{x^2}=\frac{c}{(a-x)^2}, \text{ becomes } \frac{(a-x)^2}{x^2}=\frac{c}{b}.$$

Now by extracting the square root of both members, we have

$$\frac{a-x}{x}=\pm\sqrt{\frac{c}{b}}=\frac{\pm\sqrt{c}}{\sqrt{b}}.$$

Whence, by clearing the fraction and transposing,

$$a\sqrt{b}-x\sqrt{b}=\pm x\sqrt{c}; \text{ therefore, } x=\frac{a\sqrt{b}}{\sqrt{b}\pm\sqrt{c}}.$$

N. B. The values first obtained were more complicated, because the equation of the second degree was resolved by the general method, which is less simple than that just employed. We will now discuss the two simplified values. We have

$$\left.\begin{array}{l}\text{1st} \ldots x=\dfrac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}},\\ \text{2d} \ldots x=\dfrac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}},\end{array}\right\} \text{ from which we obtain } \left\{\begin{array}{l}a-x=\dfrac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}},\\ a-x=\dfrac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}},\end{array}\right.$$

In the first place, suppose that $b>c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is positive and less than a, because $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is a fraction; thus this value gives for the required point, a point C, situated between the points A, and B. We see moreover, that the point is nearer to B than A; for since $b>c$, we have $\sqrt{b}+\sqrt{b}$ or $2\sqrt{b}>\sqrt{b}+\sqrt{c}$; whence $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and consequently, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. In fact this ought to be the case, since the intensity of A was supposed to be greater than that of B.

The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$ is positive, and less than $\frac{a}{2}$, as may easily be verified.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is also positive, but greater than a; because $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}>1$. Hence this second value gives a second point C', situated upon the prolongation of AB, and to the right of the two lights. We may in fact conceive that the two lights, exerting their influence in every direction, should have upon the prolongation of AB, another point equally illuminated; but this point must be nearest that light whose intensity is the least. We can discover, *à posteriori*, why these two values are connected by the same equation. If, instead of taking AC for the unknown quantity x, we take AC', there will result $BC'=x-a$; therefore we have the equation $\frac{b}{x^2}=\frac{c}{(x-a)^2}$; Now, as $(x-a)^2$ is identical with $(a-x)^2$, the new equation is the same as that already established, which consequently should not give AC rather than AC'.

The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it should be, since we have $x>a$; but by changing the signs in the equation $a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$ it becomes $x-a=\frac{a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$; and this value of $x-a$ represents the absolute value of BC'.

Let $b<c$.

The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ is always positive, but less than $\frac{a}{2}$, since we have

$$\sqrt{b}+\sqrt{c}>\sqrt{b}+\sqrt{b}>2\sqrt{b}.$$

The corresponding value of $a-x$, or $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$ is positive, and greater than $\frac{a}{2}$.

Therefore in this hypothesis, the point C, situated between A and B, must be nearer A than B.

The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$ or $\frac{-a\sqrt{b}}{\sqrt{c}-\sqrt{b}}$, is essentially *negative*. To interpret it, let us resume the equation, which becomes . . . $\frac{b}{x^2}=\frac{c}{(a+x)^2}$ by replacing x by $-x$. Now, as $a-x$ in the first place expressed the distance from the point B to the required point, $a+x$ ought now to express this same distance, which requires that the point should be to the left of A, at C''. for example. In fact, since the intensity of the light B, is greater than that of A, the second required point ought to be nearer A than B.

The corresponding value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, or $\frac{a\sqrt{c}}{\sqrt{c}-\sqrt{b}}$, is positive, which it should be, since when x is negative, $a-x$ is an arithmetical sum.

Let $b=c$.

The two first values of x and $a-x$ reduce to $\frac{a}{2}$, which gives the middle of $A\ B$ for the first required point. This result agrees with the hypothesis.

The other two values reduce to $\frac{a\sqrt{b}}{0}$, or *infinity*; that is, the second required point is situated at a distance from the two points A and B, greater than any assignable quantity. This result agrees perfectly with the present hypothesis, because, by supposing the difference $b-c$ to be extremely small, without being absolutely nothing, the second point must be at a very great distance from the lights; this is indicated by the expression $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, the denominator of which is extremely small with respect to the numerator. And if we finally suppose $b=c$, or $\sqrt{b}-\sqrt{c}=0$, the required point cannot exist, or is situated at an *infinite* distance.

We will observe, that in the case of $b=c$, if we should consider the values before they were simplified, viz.

$$x=\frac{a(b+\sqrt{bc})}{b-c}, \text{ and } x=\frac{a(b-\sqrt{bc})}{b-c}.$$

The first, which corresponds to $x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, would become $\frac{2ab}{0}$, and the second, which corresponds to $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, would become $\frac{0}{0}$. But $\frac{0}{0}$ would be obtained in consequence of the existence of a common factor, $\sqrt{b}-\sqrt{c}$, between the two terms of the value of x. (See No. 72.)

Let $b=c$, and $a=0$.

The first system of values for x and $a-x$, reduces to 0, and the second to $\frac{0}{0}$. This last symbol is that of *indetermination;* for, resuming the equation of the problem, ... $(b-c)x^2-2abx=-a^2b$, it reduces in the present hypothesis to $0.x^2-0.x=0$, which may be satisfied by giving x any value whatever. In fact, since the two lights have the same intensity, and are placed at the same point, *they ought to illuminate equally each point of the line A B.*

The solution 0, given by the first system, is one of those solutions *in infinite numbers,* of which we have spoken.

Finally, suppose $a=0$, b being different from c.

Each of the two systems reduce to 0, which proves that there is but one point in this case equally illuminated, and *that is the point in which the two lights are placed.*

In this case, the equation reduces to $(b-c)x^2=0$, and gives the two equal values, $x=0$, $x=0$.

The preceding discussion presents another example of the precision with which algebra responds to all the circumstances of the enunciation of a problem.

104. Problem 16. *Find two numbers, such that the difference of their products by the numbers* a *and* b, *respectively, may be equal to a given number* s, *and that the difference of their squares may be equal to another given number,* q.

Solution. Let x and y be the required numbers; we evidently have the two equations, $\begin{cases} ax-by=s, \\ x^2-y^2=q. \end{cases}$

From the first, we obtain $x=\frac{by+s}{a}$, which, substituted in the second, gives

$$(a^2-b^2)y^2-2bsy=s^2-a^2q \;.\;.\;.\;.\; (1);$$

hence,

$$y=\frac{bs\pm a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}.$$

Substituting the value of y in the expression for x, it becomes

$$x=\frac{b\left(\frac{bs\pm a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}\right)+s}{a},$$

whence $$x=\frac{as\pm b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}.$$

(It must be observed here that, in the two values of y and x, the superior signs correspond with each other, as also the inferior signs).

Discussion. In what follows, we will suppose that a, b, q, s, are absolute numbers; if they were not, certain terms in the values of x and y would change their signs, and it would be necessary to make these changes before discussing them.

Suppose $a>b$, whence a^2-b^2 is positive.

In order that the two values of x and of y may be real, we must have

$$q(a^2-b^2)<s^2, \text{ whence } q<\frac{s^2}{a^2-b^2}.$$

Suppose this last condition fulfilled, and let us determine the signs with which the two systems of values are affected.

The first system is $$\begin{cases} x=\dfrac{as+b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}. \\ y=\dfrac{bs+a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}. \end{cases}$$

The two values of this system are necessarily positive, and consequently form a *direct solution* of the problem in the sense in which it was stated.

The second system is $$\begin{cases} x=\dfrac{as-b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}, \\ y=\dfrac{bs-a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}. \end{cases}$$

The value of x is essentially positive; for $a>b$ we have $as>bs$, and *à fortiori*, $as>b\sqrt{s^2-q(a^2-b^2)}$, since the radical is less than s.

As to the value of q, it may be positive or negative.

In order that it may be positive, we must have

$$b\ s > a\sqrt{s^2 - q(a^2 - b^2)};$$

Whence by squaring $b^2s^2 > a^2s^2 - a^2q(a^2 - b^2);$

or, adding $a^2q\ (a^2 - b^2)$ to both members, and subtracting b^2s^2,

$$a^2q(a^2 - b^2) > s^2(a^2 - b^2).$$

Whence, dividing by $a^2(a^2 - b^2)$, $q > \frac{s^2}{a^2}$.

Therefore, in order that the second system may be a *real and direct solution*, it is necessary that $q < \frac{s^2}{a^2 - b^2}$, but $q > \frac{s^2}{a^2}$, that is, that q may be comprised between the two numbers $\frac{s^2}{a^2}$ and $\frac{s^2}{a^2 - b^2}$.

The condition $q > \frac{s^2}{a^2}$ might be more easily obtained from the equation involving q.

This equation being $(a^2 - b^2)y^2 - 2bsy = s^2 - a^2q$, we see that, in the hypothesis $a > b$, it will be of the form ... $x^2 - px = -q$ when $a^2q > s^2$ or $q > \frac{s^2}{a^2}$; and we know (100) that the two roots are then both positive. If, on the contrary, we had $q < \frac{s^2}{a^2}$, in which case we would have $q < \frac{s^2}{a^2 - b^2}$, the value of y for the second system would be *negative*, and this system (considered independently of the sign for y) would no longer be a solution of the problem, in the sense in which it was stated, but of the problem whose equations would be $\left\{ \begin{matrix} ax + by = s \\ x^2 - y^2 = q \end{matrix} \right\}$, and would only differ from the proposed problem in this, that s would express a *sum* instead of a *difference*. Therefore, in the case where $a > b$, the problem admits of *two real and direct solutions*, so long as we have

$$q > \frac{s^2}{a^2} \text{ and } q < \frac{s^2}{a^2 - b^2};$$

and it admits of but *one* when we have $q < \frac{s^2}{a^2}$.

By taking any absolute numbers whatever, for, a, b, s, provided however that a is $> b$, and afterward taking for q, any number comprised between the two limits $\frac{s^2}{a^2}$ and $\frac{s^2}{a^2-b^2}$, we will be certain of obtaining *two direct solutions.*

For example, let $a=6, b=4, s=15$, whence we deduce

$$\frac{s^2}{a^2}=\frac{225}{36}=6\frac{1}{4},\ \frac{s^2}{a^2-b^2}=\frac{225}{20}=11\frac{1}{4}.$$

We can now suppose $q=10$, and we have

$$x=\frac{6\times 15\pm 4\sqrt{225-20\times 10}}{20}=\frac{90\pm 20}{20}=\frac{11}{2}\text{ and }\frac{7}{2},$$

$$y=\frac{4\times 15\pm 6\sqrt{225-20\times 10}}{20}=\frac{60\pm 30}{20}=\frac{9}{2}\text{ and }\frac{3}{2}.$$

The solutions $x=\frac{11}{2}, y=\frac{9}{2} \mid x=\frac{7}{2}, y=\frac{3}{2}$, evidently form *two direct solutions* of the equations

$$6x-4y=15,$$
$$x^2-y^2=10.$$

But if we suppose $a=6, b=4, s=15, q=5$, it will be easily ascertained that, of the two systems, the first only will give a direct solution.

Particular cases which are involved in the hypothesis $a>b$.

Let $q=\frac{s^2}{a^2-b^2}$, whence $q(a^2-b^2)=s^2$.

The two systems of values of x and y reduce to

$$x=\frac{as}{a^2-b^2},\ y=\frac{bs}{a^2-b^2}.$$

Therefore, in this hypothesis there is but one solution of the problem, and it is *direct.*

Again, take $q=\frac{s^2}{a^2}$; whence $s^2=a^2q$, and $s=a\sqrt{q}$.

The first system becomes $\begin{cases} x=\dfrac{as+b\sqrt{b^2q}}{a^2-b^2}=\dfrac{a^2+b^2}{a^2-b^2}\sqrt{q}, \\ y=\dfrac{bs+a\sqrt{b^2q}}{a^2-b^2}=\dfrac{2ab}{a^2-b^2}\sqrt{q}; \end{cases}$

The second - - - - - - $\begin{cases} x=\dfrac{as-b\sqrt{b^2q}}{a^2-b^2}=\sqrt{q}, \\ y=\dfrac{-bs-a\sqrt{b^2q}}{a^2-b^2}=0. \end{cases}$

In fact, suppose $s^2=a^2q$ in the equation involving only y; it reduces to $(a^2-b^2)y^2-2bsy=0$, whence we deduce $y=0$, $y=\dfrac{2bs}{a^2-b^2}=\dfrac{2ab}{a^2-b^2}\sqrt{q}$. Substituting each of these values in the equation $x=\dfrac{by+s}{a}$, we have

$$x=\frac{s}{a}=\sqrt{q},\quad x=\frac{a^2+b^2}{a^2-b^2}\sqrt{q}.$$

Suppose that $a<b$, whence a^2-b^2 is negative.

The expressions for x and y can be put under the form

$$x=\frac{-as\mp b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$$

$$y=\frac{-bs\mp a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}.$$

All of these values are real, since the quantities under the radical are essentially positive.

With respect to the signs, the first value of x (and that of y also) is essentially negative. Therefore these values, independent of their signs, are not answers to the proposed equations, but to the equations $by-ax=s$, $x^2-y^2=q$, in the first of which, *the order of the difference between the products* ax *and* by *is reversed.*

The second value of x is necessarily positive, for from $b>a$ we deduce $b\sqrt{s^2+q(b^2-a^2)}>as$, since the radical is numerically greater than s.

But the second value of y is not always positive. In order that it may be, we must have the relation

$$a\sqrt{s^2+q(b^2-a^2)}>bs.$$

whence, by squaring . . . . . $a^2s^2+a^2q(b^2-a^2)>b^2s^2$,

or, transposing a^2s^2 . . . . . $a^2q(b^2-a^2)>(b^2-a^2)s^2$,

and dividing by $a^2(b^2-a^2)$ $\quad q>\dfrac{s^2}{a^2}$.

By giving to a, b, s, q, particular values, such that we may have $b>a$, and $q>\frac{s^2}{a^2}$, the problem will still be susceptible *of a direct solution.*

Lastly, let $a=b$; *whence* $a^2-b^2=0$.

In this hypothesis, the first system of values becomes

$$x=\frac{2as}{0},\ y=\frac{2as}{0},$$

and the second $\qquad x=\frac{0}{0},\qquad y=\frac{0}{0}.$

But resuming the equation $\qquad (a^2-b^2)y-2bsy=s^2-a^2q$,
which, when we make $a=b$, reduces to $\qquad -2asy=s^2-a^2q$,
from which we deduce - - - - - - - - - - - - - - $y=\frac{a^2q-s^2}{2as}$;

and the expression for x in terms of y, $x=\frac{by+s}{a}$, gives $x=\frac{a^2q+s^2}{2as}$.

(By a procedure analogous to that of No. 102, viz. making $y=\frac{1}{z}$ in the equation involving only y, we would obtain the same results).

In order that the solution $x=\frac{a^2q+s^2}{2as}$, $y=\frac{a^2q-s^2}{2as}$, may be *direct*, we must have $q>\frac{s^2}{a^2}$.

Of Inequalities.

105. In discussing the two preceding problems, we have had occasion to suppose several *inequalities*, and we have performed transformations upon them, analogous to those executed upon *equalities*. We are often obliged to do this, when, in discussing a problem, we wish to establish the necessary relations between the given quantities, in order that the problem may be susceptible of a direct, or at least a real solution, and to fix, with the aid of these relations, the limits between which the particular values of certain given quantities must be found, in order that the enunciation may agree with such or such a circumstance. Now, although the principles established for equations are in general applicable to inequalities, there are nevertheless some exceptions,

of which it is necessary to speak, in order to put the beginner upon his guard against some errors that he might commit, in making use of the sign of inequality. These exceptions arise from the introduction of *negative expressions* into the calculus, as *quantities*.

In order that we may be clearly understood, we will take examples of the different transformations that inequalities may be subjected to, taking care to point out the exceptions to which these transformations are liable.

TRANSFORMATION BY ADDITION AND SUBTRACTION. *We may, without any exception, add the same quantity to, or subtract it from, the two members of any inequality whatever. The inequality always subsists in the same sense.*

Thus, take $8>3$, we still have $8+5>3+5$, and $8-5>3-5$. In like manner, $-3<-2$ gives $-3+6<-2+6$, and ..... $-3-6<-2-6$. This is evident from No. 63.

This principle serves to transpose certain terms from one member of the inequality to the other. Take, for example, the inequality $a^2+b^2>3b^2-2a^2$; there will result from it ..... $a^2+2a^2>3b^2-b^2$, or $3a^2>2b^2$.

We can, without exception, add together, member to member, two or more inequalities established in the same sense; and the resulting inequality will subsist in the same sense as the proposed. Thus, from $a>b$, $c>d$, $e>f$ ....., there results $a+c+e>b+d+f$.

But this is not always the case, when we subtract, member from member, two inequalities established in the same sense.

Let there be the two inequalities $4<7$ and $2<3$, we have $4-2$ or $2<7-3$ or 4.

But if we have the inequalities $9<10$ and $6<8$, by subtracting we have $9-6$ or $3>10-8$ or 2.

We should then avoid this transformation as much as possible, or if we employ it, determine in what sense the resulting inequality exists.

TRANSFORMATION BY MULTIPLICATION AND DIVISION. *The two members of an equality may be multiplied by a* positive *or* absolute *number, and the resulting inequality will subsist in the same sense.*

Thus, from $a<b$ we deduce $3a<3b$; and from $-a<-b$, $-3a<-3b$.

This principle serves to make the denominators disappear.

From the inequality $\frac{a^2-b^2}{2d} > \frac{c^2-d^2}{3a}$, we deduce, by multiplying by $6ad$,

$$3a(a^2-b^2) > 2d(c^2-d^2).$$

The same principle is true for division.

But *when the two members of an inequality are multiplied or divided by a negative quantity, the inequality subsists in a contrary sense.*

Take, for example, $8>7$; multiplying by -3, we have $-24<-21$.

In like manner, $8>7$ gives $\frac{8}{-3}$, or $-\frac{8}{3}<-\frac{7}{3}$.

Therefore, when the two members of an inequality are multiplied or divided by a number expressed algebraically, it is necessary to ascertain whether the *multiplier* or *divisor* is negative; for, in this case, the inequality would exist in a contrary sense.

In the problem of No. 104, from the inequality

$$a^2q(a^2-b^2) > s^2(a^2-b^2),$$

We have been able to deduce $q>\frac{s^2}{a^2}$, by dividing by .... $a^2(a^2-b^2)$, because we have supposed $a>b$, or a^2-b^2, positive.

It is not permitted to change the signs of the two members of an inequality unless we establish the resulting inequality in a contrary sense; for the transformation is evidently the same as multiplying the two members by -1.

Transformation by squaring. *Both members of an inequality between absolute numbers can be squared, and the inequality will exist in the same sense.*

Thus from $5>3$, we deduce $25>9$; from $a+b>c$, we find $(a+b)^2>c^2$.

But *when both members of the inequality are not positive, we cannot tell before the operation is performed, in what sense the resulting inequality will exist.*

For example, $-2<3$ gives $(-2)^2$ or $4<9$; but $-3>-5$ gives, on the contrary, $(-3)^2$ or $9<(-5)^2$ or 25.

We must then, before squaring, ascertain whether the two members can be considered as *absolute numbers.*

Transformation by extracting the square root. *The square root of the two members of an inequality between absolute numbers may be extracted, and the inequality will exist in the same sense between the numerical values of the square roots.*

We will observe, in the first place, that it cannot be proposed to extract the square root of the two members of an inequality, *unless they are essentially positive*, for otherwise *imaginary expressions* would be obtained.

But when we have $9<25$; we deduce from it $\sqrt{9}$ or $3<\sqrt{25}$ or 5.

From $a^2>b^2$ we deduce $a>b$, if a and b express absolute numbers.

In like manner the inequality $a^2>(c-b)^2$ gives $a>c-b$ if we suppose c greater than b, and $a>b-c$ when we suppose that b is greater than c.

When the two members of an inequality are composed of additive and subtractive terms, *care should be taken to write for the square root of each member, a polynomial in which the subtractions will be possible.*

Problems.

106. Problem 7. *Two merchants each sold the same kind of stuff; the second sold* 3 *yards more of it than the first, and together, they receive* 35 *crowns. The first said to the second, I would have received* 24 *crowns for your stuff; the other replied, and I would have received* $12\frac{1}{2}$ *crowns for yours. How many yards did each of them sell?*

$$\left\{\text{Ans.}\begin{array}{l}\text{1st merchant } x=15\\ \text{2d} \;.\;.\;.\;.\;.\;.\; y=18\end{array}\text{ or }\begin{array}{l}x=5.\\ y=8.\end{array}\right\}$$

Problem 8. *A merchant has two notes out, one for* 6240 *francs payable in* 8 *months, the other* 7632 *francs payable in* 9 *months. He draws in these two notes, and in place of them gives one for* 14256 *francs payable in one year. What is the rate of interest?*

(Ans. 10.33 per cent.)

Problem 9. *A widow possessed* 13000 *francs, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part* 360 *francs interest, and if she had placed* *the second out at the same rate as the first, she would have drawn for it 490 francs interest. What were the two rates of interest?*

N. B. The equation of this problem can be resolved in a more simple way than by the general method.

Problem 10. *Find two rectangles, knowing the sum* (q) *of their surfaces, the sum* (a) *of their bases, and also the surfaces* (p *and* p′) *when to the base of each of them we give the altitude of the other.*

Resolve this problem, and discuss it.

$$\left\{ \text{Ans.} \quad \text{Base of the first.} \quad x=\frac{a[2p+q\pm\sqrt{q^2-4pp'}]}{2(p+p'+q)} \right\}$$

Problem 11. *Divide two numbers* a, *and* b, *each into two parts, such that the product of one part of* a *by a part of* b, *may be equal to a given number* p, *and that the product of the remaining parts of* a *and* b *may be equal to another given number* p′.

Resolve this problem and discuss it.

Problem 12. *Find a number, such that its square may be to the product of the differences between this number and two other numbers,* a *and* b, *in the ratio of* p : q.

This last problem is recommended, not only because its discussion presents new applications of the rules for inequalities, but because the formulas obtained from it contain implicitly the solutions of a great many analogous questions, the enunciations of which only differ as to the sense of certain conditions.

Questions of Maximum and Minimum. Properties of Trinomials of the Second Degree.

107. There is a certain class of problems, frequently met with in the application of algebra to geometry, which refer to the theory of equations of the second degree. The object of these questions is to *determine the greatest or least values that the result of certain arithmetical operations effected upon numbers may receive.*

Suppose the question proposed is: *To divide a given number,* 2a, *into two parts, the product of which shall be the greatest possible, or a maximum.*

Denote one of the parts by x, the other will be $2a-x$, and their product $x(2a-x)$. By giving x different values, this product will pass through different *states of magnitude*, and it is required to assign that value to x, which will render this product *the greatest possible.*

Denote this greatest product, which is unknown, by y; we will have from the enunciation the equation

$$x(2a-x)=y.$$

Proceeding as though y was known, we find the value of x to be

$$x=a\pm\sqrt{a^2-y}.$$

Now this result shows that the two values of x cannot be real, unless we have $y<a^2$, or at most $y=a^2$; whence we may conclude that the greatest value that y, or the product of the two parts, can receive, is a^2. But by making $y=a^2$, we find $x=a$.

Therefore, *to obtain the greatest product, it is necessary to divide the given number into two equal parts, and the maximum obtained is the square of half of the number*, a result which has already been obtained by another method. (No. 100).

More simple solution. Let $2x$ be the difference between the two parts; since their sum is expressed by $2a$, the greater will be represented by $\frac{2a+2x}{2}$, (No. 4), or $a+x$, the less by $a-x$, and the equation will be $(x+a)\ (x-a)=y$; or, performing the operation indicated, $a^2-x^2=y$; whence,

$$x=\pm\sqrt{a^2-y}.$$

In order that this value of x may be real, it is necessary that the value of y should not be greater than a^2, and making $y=a^2$, we have $x=0$, which proves that *the two parts must be equal to each other.*

This solution has the advantage of leading to an equation of the second degree, involving two terms.

108. N. B. In the above equations, $x(2a-x)=y$, and . . . $(a+x)\ (a-x)=y$, the quantity x is called *a variable*, and the expression $x(2a-x)$, or $(a+x)\ (a-x)$, is said to be a function of the variable. This function, represented by y, is itself another *variable*, the value of which depends upon that attributed to the first. For this reason, analysts sometimes call

the first the *independent variable*, whilst the second, or y, receives values depending upon those attributed to x.

By resolving the two equations $x(2a-x)=y$, and ...... $(a+x)(a-x)=y$, with reference to x, which gives

$$x=a\pm\sqrt{a^2-y},$$

and
$$x=\pm\sqrt{a^2-y},$$

we can consider y as an *independent variable*, and x as a certain function of this variable.

109. Suppose that it is required to *divide a number* 2a, *into two parts, such that the sum of their square roots shall be a maximum.*

Call one of these parts x^2, the other will be, $2a-x^2$, and the expression for the sum of their square roots will be $x+\sqrt{2a-x^2}$; it is required to determine the *maximum* of this expression.

$$\text{Suppose}\quad x+\sqrt{2a-x^2}=y.$$

In order to resolve this equation, it is necessary to get clear of the radical. By transposing the term x, into the second member, we have

$$\sqrt{2a-x^2}=y-x,$$

whence, by squaring, $2a-x^2=y^2-2yx+x^2$,

or, arranging it with reference to x, $2x^2-2xy=2a-y^2$,

from which we find $x=\dfrac{y}{2}\pm\sqrt{\dfrac{y^2}{4}+\dfrac{2a-y^2}{2}}$,

or, simplifying, $x=\dfrac{y}{2}\pm\dfrac{1}{2}\sqrt{4a-y^2}$.

In order that these two values of x may be real, it is necessary that y^2 should not exceed $4a$; hence $2\sqrt{a}$ is the *greatest value* that y can receive.

But by making $y=2\sqrt{a}$ we find $x=\sqrt{a}$, whence $x^2=a$, and $2a-x^2=a$.

Therefore, *the given number* 2a *must be divided into two equal parts, in order that the sum of the square roots of these parts may be a* maximum. This maximum is moreover equal to $2\sqrt{a}$.

For example, let 72, be the proposed number; we have $72=36+36$; whence $\sqrt{36}+\sqrt{36}=12$; this is the *maximum* of all the values that can be obtained for the sum of the square roots of the two parts of 72.

In fact, decompose 72, into 64+8; we have $\sqrt{64}=8$, $\sqrt{8}=2+$ a fraction; whence $\sqrt{64}+\sqrt{8}=10+$ a fraction; again $72=49+23$; we have $\sqrt{49}=7$, $\sqrt{23}=4+$ a fraction; hence $\sqrt{49}+\sqrt{23}=11+$ a fraction.

For a third example we will consider the expression $\frac{m^2x^2+n^2}{(m^2-n^2)x}$, which it is required to render a minimum, (m being supposed $>n$.)

Suppose $\frac{m^2x^2+n^2}{(m^2-n^2)x}=y$; whence $m^2x^2-(m^2-n^2)yx=-n^2$; we deduce from it $x=\frac{(m^2-n^2)y}{2m^2}\pm\frac{1}{2m^2}\sqrt{(m^2-n^2)^2y^2-4m^2n^2}$.

Now, in order that the two values of x, corresponding to a value of y may be real, it is necessary that $(m^2-n^2)^2y^2$, should not be less than $4m^2n^2$, and consequently that y should not be less than $\frac{2mn}{m^2-n^2}$. Therefore $\frac{2mn}{m^2-n^2}$ is the *minimum* value that the function y can receive.

But by making $y=\frac{2mn}{m^2-n^2}$, in the expression for x, the radical will disappear, and the value of x becomes

$$x=\frac{m^2-n^2}{2m^2}\times\frac{2mn}{m^2-n^2}=\frac{n}{m}.$$

Hence this value $x=\frac{n}{m}$ will render the proposed expression a *minimum*.

110. *These examples will suffice to point out the course to be followed, in the solution of questions of this kind. After having formed the algebraic expression of the quantity which is susceptible of becoming a maximum or minimum, place it equal to some letter, as* y. *If the equation thus obtained is of the second degree in* x (x *denoting the variable quantity which enters into the expression), resolve it with reference to* x, *then place the quantity under the radical equal to zero, and from this last equation find the value of* y, *which will represent the required maximum or minimum. Substitute this value of* y *in the expression for* x, *and we will have the value of this variable, which will satisfy the enunciation.*

N. B. If it should happen that the quantity under the radical remains essentially positive, whatever value be given to y, we may conclude that *the proposed expression can pass through every possible state of magnitude;* in other words, *it would have infinity for its* maximum, *and zero for the* minimum.

For another example take the expression $\frac{4x^2+4x-3}{6(2x+1)}$; and find whether it is susceptible of a *maximum* or *minimum.*

$$\text{Take } \frac{4x^2+4x-3}{6(2x+1)}=y, \text{ there will result}$$

$$4x^2-4(3y-1)x=6y+3, \text{ whence } x=\frac{3y-1}{2}\pm\frac{1}{2}\sqrt{9y^2+4}.$$

Now whatever value is given to y, the quantity under the radical will always be positive. Therefore y, or the proposed expression, can pass through every state of magnitude.

In the preceding examples, the quantity under the radical, in the value of x, contained but two parts, one affected with y or y^2, and the other entirely known; and the *maximum* or *minimum* of which the function was susceptible was easily obtained. But it may happen that the quantity under the radical is a trinomial of the second degree of the form my^2+ny+p. In this case the question becomes more difficult, and to be able to resolve it, it is necessary to demonstrate some properties relative to these trinomials.

Properties of Trinomials of the Second Degree.

111. Every algebraic expression which can be reduced to the form my^2+ny+p, is called a *trinomial of the second degree;* m, n, and p, being *known quantities*, + or −, y designating a *variable*, that is, a quantity which may pass through different states of magnitude.

Thus,

$$3y^2-5y+7, \quad -9y^2+2y+5,$$
$$(a-b+2c)y^2+4b^2y-2ac^2+3a^2b,$$

are called trinomials of the second degree.

Placing the trinomial my^2+ny+p equal to zero, we have $my^2+ny+p=0$, whence $y=-\frac{n}{2m}\pm\frac{1}{2m}\sqrt{n^2-4mp}$.

Three principal hypotheses can be made with respect to the nature of the values of y. We may have $n^2-4mp>0$, or *positive;* in which case, the two roots are *real* and *unequal*, + or —.

Or we may have $n^2-4mp=0$; in which case the two roots are *real* and *equal.*

Or, lastly, we may have $n^2-4mp<0$, or *negative;* then the two roots are *imaginary.*

Therefore the following properties will result from these different cases:

1st. When a trinomial of the second degree is such, that by placing it equal to 0, and resolving the resulting equation, we obtain two real and unequal roots, *every quantity* (positive or negative) *comprehended between the two roots, and substituted for* y *in the trinomial, will necessarily give a result, with a sign contrary to that affecting the coefficient of* y^2; *but every quantity not comprised between the two roots, and substituted for* y, *will give a result with the same sign as that of the coefficient of* y^2.

For, let y' and y'' denote the two roots (supposed *real*) of the equation $my^2+ny+p=0$, or $m\left(y^2+\frac{n}{m}y+\frac{p}{m}\right)=0$.

The first member $m\left(y^2+\frac{n}{m}y+\frac{p}{m}\right)$ can be put under the form (No. 98) $m(y-y')(y-y'')$. Therefore we have the identity

$$my^2+ny+p=m(y-y')(y-y'').$$

Let α be a quantity comprised between y' and y'', that is, such that $\alpha>$ or $<y'$, but $\alpha<$ or $>y''$; there will result $\alpha-y'>$ or <0, but $\alpha-y''<$ or >0; hence the two factors $\alpha-y'$, $\alpha-y''$ have contrary *signs*, therefore their product is *negative.* Consequently, $m(\alpha-y')(\alpha-y'')$, or its value, $m\alpha^2+n\alpha+p$, has a sign contrary to that of m.

If, on the contrary, we suppose at the same time $\alpha>$ or $<y'$, and $\alpha>$ or $<y''$, whence $\alpha-y'>$ or <0, and $\alpha-y''>$ or <0, the two factors will have the same sign; hence their product, ... $(\alpha-y')(\alpha-y'')$, is positive, and consequently $m(\alpha-y')(\alpha-y'')$, or $m\alpha^2+n\alpha+p$, is of the same sign as m.

2d. When the two roots are real and equal, *every quantity different from that which reduces the trinomial to* 0, *will, when* *substituted in the trinomial, give a result of the same sign as the coefficient of* y^2.

For since the two roots are equal, we have the relation . . . . $n^2-4mp=0$, whence $p=\frac{n^2}{4m}$; and the trinomial my^2+ny+p, or $m\left(y^2+\frac{n}{m}y+\frac{p}{m}\right)$ can be put under the form . . . . $m\left(y^2+\frac{n}{m}y+\frac{n^2}{4m^2}\right)=m\left(y+\frac{n}{2m}\right)^2$. Now it is evident that the quantity $\left(y+\frac{n}{m}\right)^2$ will be positive for every value substituted for y, except $-\frac{n}{2m}$. Therefore $m\left(y+\frac{n}{2m}\right)^2$, or my^2+ny+p, will be of the same sign as m.

3d. When the two roots are imaginary, every real positive or negative quantity, substituted in the place of y, will give a result of the same sign as that of the coefficient of y^2.

For since the two roots are imaginary, we have the relation $n^2-4mp<0$, whence $4mp>n^2$; or (105) dividing by $4m^2$ . . . .
$\frac{p}{m}>\frac{n^2}{4m^2}$.

Take $\frac{p}{m}=\frac{n^2}{4m^2}+k^2$ . . . ., k^2 denoting a quantity essentially positive. There will result

$$my^2+ny+p, \text{ or } m\left(y^2+\frac{n}{m}y+\frac{p}{m}\right)=m\left(y^2+\frac{n}{m}y+\frac{n^2}{4m^2}+k^2\right)$$
$$=m\left(y+\frac{n}{2m}\right)^2+mk^2,$$

which always has the same sign as m, whatever value may be substituted for y.

112. The second case naturally leads us to speak of a proposition of frequent use in analysis.

When a trinomial of the second degree, my^2+ny+p, *is a perfect square, there is between the coefficients the relation* $n^2-4mp=0$.

For if this trinomial is a perfect square, and of the form $(m'y+n')^2$, by placing it equal to 0, the two roots of the resulting equation will be *equal*. Now in order that they may be equal, the quantity under the radical, or n^2-4mp, *must be nothing*. Therefore we have the relation $n^2-4mp=0$.

Reciprocally. When there is, between the coefficients the relation $n^2-4mp=0$, the trinomial is a perfect square; for we deduce from this relation, $p=\frac{n^2}{4m}$; whence

$$my^2+ny+p=my^2+ny+\frac{n^2}{4m}=\left(y\sqrt{m}+\frac{n}{2\sqrt{m}}\right)^2$$

113. The following examples will show the use of these properties in the solution of questions of *maximum and minimum.*

Let it be proposed to determine whether, in making x vary, the function $\frac{x^2-2x+21}{6x-14}$ may pass through every state of magnitude.

Take $\frac{x^2-2x+21}{6x-14}=y$; whence

$$x^2-2(3y+1)x=-21-14y.$$

From which we have $x=3y+1\pm\sqrt{9y^2-8y-20}$.

In order that x may be real, it is necessary that ..... $9y^2-8y-20$ should be positive. Now, placing this equal to 0, we have

$$y^2-\frac{8}{9}y-\frac{20}{9}=0;\text{ whence } y=2 \text{ and } y=-\frac{10}{9}.$$

These two values of y being real, it follows, from the first of the above properties, that by giving y values comprised between 2 and $-\frac{10}{9}$, such as $1, 0, -1, \ldots$ the value of the trinomial will be *negative*, since the coefficient of y^2 is positive; but by giving to y values which are not comprised between 2 and $-\frac{10}{9}$, as $3, 4 \ldots$, or $-2, -3, -4, \ldots$, the result of the substitution will be *positive*. Hence we perceive that 2 is *in absolute numbers*, the *minimum* value that y should have, in order that x may be real. If, in the above expression we make $y=2$, the radical will disappear, and we find $x=7$.

In fact, the expression

$$\frac{x^2-2x+21}{6x-14} \text{ becomes } \frac{49-14+21}{42-14}=\frac{56}{28}=2;$$

when $x=7$.

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The root $y=-\frac{10}{9}$ is the maximum value for y in *negative* numbers; and the value of x, corresponding to this maximum, is $x=3\times-\frac{10}{9}+1=-\frac{7}{3}$.

When, after having expressed x in terms of y, the coefficient of y^2 under the radical is negative, and the two values of y deduced from the trinomial placed equal to zero, are one positive and the other negative, *the positive value is a maximum*, since every greater value would give a result of the same sign as the coefficient of y^2; *and the negative value is a minimum* of all the negative values for y.

The student may investigate the other circumstances which may occur. For example, the case in which, the coefficient of y^2 being positive, the two values of y are positive; that in which, the same coefficient being positive, the two values are imaginary.

Questions for Practice.

Divide a given number 2a *into two parts, such that the sum of the quotients obtained by dividing them by each other may be a minimum.*

(Ans. The two parts are equal, and the minimum is 2).

Let a and b, be two given numbers, of which a is the greater, find the greatest possible value for the expression $\frac{(x+a)(x-b)}{x^2}$.

(Ans. *Maximum* $=\frac{(a-b)^2}{4ab}$, and the corresponding value of x is $\frac{2ab}{a-b}$.

Find the least possible value for $\frac{(a+x).(b+x)}{x}$.

(Ans. *Minimum* $=(\sqrt{a}+\sqrt{b})^2$; $x=\sqrt{ab}$.)

§ III. *Of Problems and Equations of the Second Degree, involving two or more unknown quantities.*

114. A complete theory of this subject cannot be given here, because the resolution of two equations of the second degree involving two unknown quantities, in general depends upon the

solution of an equation of the fourth degree involving one unknown quantity; but we will propose some questions, which depend only upon the solution of an equation of the second degree involving one unknown quantity.

Problem 1. *Find two numbers such that the sum of their products by the respective numbers* a *and* b *may be equal to* 2s, *and that their product may be equal to* p.

Let x and y be the required numbers, we have the equations,

$$ax+by=2s.$$
$$xy=p.$$

From the first $y=\frac{2s-ax}{b}$; whence, by substituting in the second, and reducing, $ax^2-2sx=-bp$.

Therefore, $x=\frac{s}{a}\pm\frac{1}{a}\sqrt{s^2-abp}$,

and consequently, $y=\frac{s}{b}\mp\frac{1}{b}\sqrt{s^2-abp}$.

This problem is susceptible of two direct solutions, because s is evidently $>\sqrt{s^2-abp}$, but in order that they may be real, it is necessary that $s^2>$ or $=abp$.

Let $a=b=1$; the values of x, and y, reduce to

$$x=s\pm\sqrt{s^2-p} \text{ and } y=s\mp\sqrt{s^2-p}.$$

Whence we see, that the two values of x, are equal to those of y, taken in an inverse order; which shows, that if $s+\sqrt{s^2-p}$ represents the value of x, $s-\sqrt{s^2-p}$ will represent the corresponding value of y, and reciprocally.

This circumstance is accounted for, by observing, that in this particular case the equations reduce to $\begin{cases} x+y=2s, \\ xy=p; \end{cases}$ and then the question is reduced to, *finding two numbers of which the sum is* $2s$, *and their product* p, or in other words, *to divide a number* $2s$, *into two such parts, that their product may be equal to a given number* p.

It has been seen (100) that the two parts are necessarily connected together by the same equation of the second degree, $x^2-2sx+p=0$, in *which the coefficient of the second term is the sum* 2s *taken with a contrary sign, and the last term is the product* p *of the two parts.*

115. Problem 2. *Find four numbers in proportion, knowing the sum* 2s *of their extremes, the sum* 2s′ *of the means, and the sum* 4c² *of their squares.*

Let u, x, y, z, denote the four terms of the proportion; the equations of the problem will be

$$u+z=2s$$
$$x+y=2s'$$
$$uz=xy$$
$$u^2+x^2+y^2+z^2=4c^2.$$

At first sight, it may appear difficult to find the values of the unknown quantities, but with the aid of an *unknown auxiliary* they are easily determined.

Let p, be the unknown product of the extremes or means, we have

1st. The equations $\begin{cases} u+z=2s, \\ uz=p, \end{cases}$ which give $\begin{cases} u=s+\sqrt{s^2-p}, \\ z=s-\sqrt{s^2-p}. \end{cases}$

(See the preceding problem.)

2d. The equations $\begin{cases} x+y=2s', \\ xy=p, \end{cases}$ which give $\begin{cases} x=s'+\sqrt{s'^2-p}, \\ y=s'-\sqrt{s'^2-p}. \end{cases}$

Hence we see that the determination of the four unknown quantities depends only upon that of the product p.

Now, by substituting these values of u, x, y, z in the last of the equations of the problem, it becomes

$$(s+\sqrt{s^2-p})^2+(s-\sqrt{s^2-p})^2+(s'+\sqrt{s'^2-p})^2$$
$$+(s'-\sqrt{s'^2-p})^2=4c^2;$$

or, developing and reducing,

$$4s^2+4s'^2-4p=4c; \text{ hence } p=s^2+s'^2-c^2.$$

Substituting this value for p, in the expressions for u, x, y, z, we find

$$\begin{cases} u=s+\sqrt{c^2-s'^2}, \\ z=s-\sqrt{c^2-s'^2}, \end{cases} \qquad \begin{cases} x=s'+\sqrt{c^2-s^2}, \\ y=s'-\sqrt{c^2-s^2}. \end{cases}$$

These four numbers evidently form a proportion; for we have

$$uz=(s+\sqrt{c^2-s'^2})(s-\sqrt{c^2-s'^2})=s^2-c^2+s'^2,$$
$$xy=(s'+\sqrt{c^2-s^2})(s'-\sqrt{c^2-s^2})=s'^2-c^2+s^2.$$

This problem shows how much the introduction of an *unknown auxiliary* facilitates the determination of the principal unknown quantities. There are other problems of the same kind, which lead to equations of a degree superior to the second, and yet they may be resolved by the aid of equations of the first and second degrees, by introducing *unknown auxiliaries*.

116. We will now consider the case in which a problem leads to two equations of the second degree, involving two unknown quantities.

An equation involving two unknown quantities is said to be of the *second degree*, when it contains terms in which *the sum of the exponents of the two unknown quantities is equal to* 2. Thus, $3x^2-4x+y^2-xy-5y+6=0$, $7xy-4x+y=0$, are equations of the second degree.

Hence, every equation of the second degree, involving two unknown quantities, is of the form

$$ay^2+bxy+cx^2+dy+fx+g=0,$$

a, a, c, . . . representing known quantities, either numerical or algebraic.

Take the two equations

$$ay^2+bxy+cx^2+dy+fx+g=0,$$
$$a'y^2+b'xy+c'x^2+d'y+f'x+g=0.$$

Arranging them with reference to x, they become

$$cx^2+(by+f)x+ay^2+dy+g=0,$$
$$c'x^2+(b'y+f')x+a'y^2+d'y+g'=0.$$

Now if the coefficients of x^2 in the two equations were the same, we could, by subtracting one equation from the other, obtain an equation of the first degree in x, which could be substituted for one of the proposed equations; from this equation the value of x could be found in terms of y, and by substituting this value in one of the proposed equations, we would obtain an equation involving only the unknown quantity y.

By multiplying the first equation by c', and the second by c, they become

$$cc'x^2+(by+f)c'x+(ay^2+dy+g)c'=0,$$
$$cc'x^2+(b'y+f')cx+(a'y^2+d'y+g')c=0,$$

and these equations, in which the coefficients of x^2 are the same, may take the place of the preceding.

Subtracting one from the other, we have

$$[(bc'-cb')y+fc'-cf']x+(ac'-ca')y^2+(dc'-cd')y+gc'-cg'=0,$$

which gives

$$x=\frac{(ca'-ac')y^2+(cd'-dc')y+cg'-gc'}{(bc'-cb')y+fc'-cf'}.$$

This expression for x, substituted in one of the proposed equations, will give *a final equation*, involving y.

But without effecting this substitution, which would lead to a very complicated result, it is easy to perceive that the equation involving y will be of the fourth degree; for the numerator of the expression for x being of the form my^2+ny+p, its square, or the expression for x^2, is of the fourth degree. Now this square forms one of the parts of the result of the substitution.

Therefore, in general, *the resolution of two equations of the second degree, involving two unknown quantities, depends upon that of an equation of the fourth degree, involving one unknown quantity.*

117. There is a class of equations of the fourth degree, that can be resolved in the same way as equations of the second degree; these are equations of the form $x^4+px^2+q=0$. They are called *trinomial equations*, because they contain but three kinds of terms; terms involving x^4, those involving x^2, and terms entirely known.

In order to resolve the equation $x^4+px^2+q=0$, suppose $x^2=y$, we have

$$y^2+py+q=0,\ \text{whence}\ y=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q};$$

But the equation $x^2=y$, gives $x=\pm\sqrt{y}$.

Hence, $$x=\pm\sqrt{-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}}.$$

We perceive that the unknown quantity has four values, since each of the signs + and —, which affect the first radical, can be combined successively with each of the signs which affect the second; *but these values taken two and two are equal, and have contrary signs.*

Take for example the equation $x^4-25x^2=-144$;
by supposing $x^2=y$, it becomes $y^2-25y=-144$;
whence $y=16,\ y=9$.

Substituting these values in the equation $x^2=y$ there will result

1st. $x^2=16$, whence $x=\pm4$; 2d. $x^2=9$, whence $x=\pm3$.

Therefore the four values are $+4$, -4, $+3$ and -3.

Again, take the equation $x^4-7x^2=8$. Supposing $x^2=y$, the equation becomes $y^2-7y=8$; whence $y=8$, $y=-1$.

Therefore 1st. $x^2=8$, whence $x=\pm2\sqrt{2}$; 2d. $x^2=-1$; whence $x=\pm\sqrt{-1}$; the two last values of x are imaginary.

Let there be the algebraic equation $x^4-(2bc+4a^2)x^2=-b^2c^2$; taking $x^2=y$, the equation becomes $y^2-(2bc+4a^2)y=-b^2c^2$;

from which we deduce $\quad y=bc+2a^2\pm2a\sqrt{bc+a^2}$.

And consequently $\quad x=\pm\sqrt{bc+2a^2\pm2a\sqrt{bc+a^2}}$.

Of the Extraction of the square root of Binomials of the form a$\pm\sqrt{}$b.

118. The resolution of *trinomial equations of the fourth degree*, gives rise to a new species of algebraic operation; viz.: *the extraction of the square root of a quantity of the form* $a\pm\sqrt{b}$, a and b being numerical or algebraic quantities.

By squaring the expression $3\pm\sqrt{5}$, we have $(3\pm\sqrt{5})^2=9\pm6\sqrt{5}+5=14\pm6\sqrt{5}$; hence *reciprocally*, $\sqrt{14\pm6\sqrt{5}}=3\pm\sqrt{5}$.

In like manner, $(\sqrt{7}\pm\sqrt{11})^2=7\pm2\sqrt{7}\times\sqrt{11}+11$
$=18\pm2\sqrt{77}$.

Hence reciprocally $\sqrt{13\pm2\sqrt{77}}=\sqrt{7}\pm\sqrt{11}$.

Whence we see that an expression of the form $\sqrt{a\pm\sqrt{b}}$, may sometimes be reduced to the form $a'\pm\sqrt{b'}$ or $\sqrt{a'}\pm\sqrt{b'}$; and when this transformation is possible it is advantageous to effect it, since in this case we have only to extract two simple square roots, whereas the expression $\sqrt{a\pm\sqrt{b}}$ requires the extraction of the square root of the square root.

119. This naturally leads to the following question: *Having given a quantity of the form* a$\pm\sqrt{}$b, *to ascertain whether it is the square of a quantity of the form* a'$\pm\sqrt{}$b, *or* $\sqrt{}$a'$\pm\sqrt{b'}$, *and determine this root.*

Let p and q represent the two parts of which the square root of $a\pm\sqrt{b}$ is supposed to be composed; p and q will be *irrational* *monomials*, or one will be a *rational*, and the other an *irrational monomial.*

We will observe, in the first place, that if we have

$$\sqrt{a+\sqrt{b}}=p+q \ldots\ldots (1),$$

there will necessarily result

$$\sqrt{a-\sqrt{b}}=p-q \ldots\ldots (2).$$

These equations multiplied together give

$$\sqrt{a^2-b}=p^2-q^2 \ldots\ldots (3).$$

Now, since p and q are irrational monomials, or one rational and the other irrational, it follows that p^2 and q^2 are rational; therefore, p^2-q^2, or its value, $\sqrt{a^2 b}$, is necessarily a rational quantity.

Whence we may conclude, that when $a\pm\sqrt{b}$ is the square of a quantity of the form $a'\pm\sqrt{b'}$, or $\sqrt{a'}\pm\sqrt{b'}$, the expression a^2-b will be a *perfect square.* From this characteristic the possibility of the operation may be discovered.

Therefore, take a^2-b, a perfect square, and suppose ..... $\sqrt{a^2-b}=c$; the equation (3) becomes

$$p^2-q^2=c.$$

Moreover, the equations (1) and (2) being squared, give

$$p^2+q^2+2pq=a+\sqrt{b},$$
$$p^2+q^2-2pq=a-\sqrt{b};$$

whence, by adding member to member,

$$p^2+q^2=a \text{ - - - - - - } (4);$$

but $$p^2-q^2=c \text{ - - - - - - } (5).$$

Hence, by adding these last equations, and subtracting the second from the first, we obtain

$$2p^2=a+c,$$
$$2q^2=a-c;$$

and consequently, $$\begin{cases} p=\pm\sqrt{\dfrac{a+c}{2}}, \\ q=\pm\sqrt{\dfrac{a-c}{2}}. \end{cases}$$

Therefore, $\sqrt{a+\sqrt{b}}$, or $p+q=\pm\sqrt{\dfrac{a+c}{2}}\pm\sqrt{\dfrac{a-c}{2}}$,

$\sqrt{a-\sqrt{b}}$, or $p-q=\pm\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}}$;

or
$$\sqrt{a+\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}\right) \text{ - - - - } (6),$$

$$\sqrt{a-\sqrt{b}}=\pm\left(\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}\right) \text{ - - - - } (7).$$

These two formulas can be verified *à posteriori;* for by squaring both members of the first, it becomes

$$a+\sqrt{b}=\frac{a+c}{2}+\frac{a-c}{2}+2\sqrt{\frac{a^2-c^2}{4}}=a+\sqrt{a^2-c^2};$$

but the relation $\sqrt{a^2-b}=c$, gives $c^2=a^2-b$.

Hence, $a+\sqrt{b}=a+\sqrt{a^2-a^2+b}=a+\sqrt{b}$.

The second formula can be verified in the same manner.

120. *Remark.* As the accuracy of the formulas (6) and (7) is proved, whatever the quantity c or $\sqrt{a^2-b}$ may be, it follows, that when this quantity is not a perfect square, we may still replace the expressions $\sqrt{a+\sqrt{b}}$ and $\sqrt{a-\sqrt{b}}$, by the second members of the equalities (6) and (7); but then we would not simplify the expression, since the quantities p and q would be of the same form as the proposed expression.

We would not, therefore, in general, effect this transformation, unless a^2-b is a perfect square.

Examples.

121. Take the numerical expression $94+42\sqrt{5}$, which reduces to $94+\sqrt{8820}$. We have

$$a=94,\ b=8820,$$

whence $c=\sqrt{a^2-b}=\sqrt{8836-8820}=4$,

a rational quantity; therefore the formula (1) is applicable to this case.

It becomes

$$\sqrt{94+42\sqrt{5}}=\pm\left(\sqrt{\frac{94+4}{2}}+\sqrt{\frac{94-4}{2}}\right),$$

or, reducing, $=\pm(\sqrt{49}+\sqrt{45})$;

therefore, $\sqrt{94+42\sqrt{5}}=\pm(7+3\sqrt{5})$.

In fact, $(7+3\sqrt{5})^2=49+45+42\sqrt{5}=94+42\sqrt{5}$.

Again, take the expression

$$\sqrt{np+2m^2-2m\sqrt{np+m^2}};$$

we have $a=np+2m^2$, $b=4m^2(np+m^2)$,

whence $a^2-b=n^2p^2$,

and c or $\sqrt{a^2-b}=np$;

therefore the formula (7) is applicable. It gives for the required root

$$\pm\left(\sqrt{\frac{np+2m^2+np}{2}}-\sqrt{\frac{np+2m^2-np}{2}}\right),$$

or, reducing, $\pm(\sqrt{np+m^2}-m)$.

In fact, $(\sqrt{np+m^2}-m)^2=np+2m^2-2m\sqrt{np+m^2}$.

For another example, take the expression

$$\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}},$$

and reduce it to its simplest terms. By applying the preceding formulas, we find

$$\sqrt{16+30\sqrt{-1}}=5+3\sqrt{-1}\qquad\sqrt{16-30\sqrt{-1}}=5-3\sqrt{-1}.$$

Hence $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}=10$.

This last example shows, better than any of the others, the utility of the general problem; because it proves that *imaginary expressions* combined together, may produce *real*, and even *rational results*.

$$\sqrt{28+10\sqrt{3}}=5+\sqrt{3};\quad\sqrt{1+4\sqrt{-3}}=2+\sqrt{-3},$$

$$\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}=\pm 2b;$$

$$\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}}=\sqrt{ab}+\sqrt{4c^2-d^2}.$$

CHAPTER IV.

Formation of Powers, and Extraction of Roots of any degree whatever.

Introduction. The resolution of equations of the second degree supposes the process for extracting the square root to be known; in like manner the resolution of equations of the third, fourth...... degree, requires that we should know how to extract the third, fourth...... root of any numerical or algebraic quantity.

The raising of powers, extraction of roots, and the calculus of radicals, will be the principal object of this chapter, which, with the first, and a part of the third, constitute all the operations that are required to be performed upon numbers expressed algebraically.

Although any power of a number can be obtained from the rules of multiplication, yet this power is subjected to a certain *law of composition* which it is absolutely necessary to know, in order *to deduce the root from the power.* Now, as the law of composition of the square of a numerical or algebraic quantity, is founded (86) upon the expression for the square of a binomial, so likewise the law relative to a power of any degree, is deduced from the same power of a binomial. We will therefore determine *the development of any power of a binomial.*

§ 1. *Newton's Theorem for the Binomial, and consequences to be derived from it.*

146. By multiplying the binomial $x+a$ into itself several times, the following results are obtained;

$(x+a)=x+a,$

$(x+a)^2=x^2+2ax+a^2,$

$(x+a)^3=x^3+3ax^2+3a^2x+a^3$,
$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4$,
$(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$.

By inspecting these developments it is easy to discover *a law* according to which the exponents of x, and a decrease and increase in the successive terms of each of them; it is not so easy to discover a law for the coefficients. Newton discovered *one*, by means of which, the degree of a power being given, this power of a binomial can be formed, without first obtaining all of the inferior powers. He did not however explain the course of reasoning which led him to the discovery of it; but the existence of this law has since been demonstrated in a rigorous manner. Of all the known demonstrations of it, the most elementary is that which is founded upon the *theory of combinations*. However, as it is rather complicated, we will, in order to simplify the exposition of it, begin by resolving some problems relative to combinations, from which it will be easy to deduce the *formula for the binomial*, or the development of any power of a binomial.

147. We know (Arith. no. 127.) that the product of a number n, of factors, a, b, c, d . . . is the same, in whatever order the multiplication is performed. Now it may be proposed to determine the *whole number* of ways in which these letters can be written one after the other. The results corresponding to each change in the position of these letters, are called *permutations*.

Thus, the two letters a and b, give a single product ab, but furnish the two *permutations* ab and ba.

In like manner, the three letters a, b, c, give a single product abc, but furnish the six *permutations* $abc, acb, cab, bac, bca, cba$.

Suppose we have a number m, of letters a, b, c, d . . . if they are written one after the other, 2 and 2, 3 and 3, 4 and 4 . . . . in every possible order, in such a manner, however, that the number of letters in each result may be less than the number of given letters, we may demand the *whole number* of results thus obtained. These results are called *arrangements*.

Thus ab, ac, ad . . . . ba, bc, bd, ... ca, cb, cd, . . . are *arrangements* of m letters taken 2 and 2.

In like manner, abc, abd, . . . bac, bad . . . acb, acd, . . . are *arrangements* taken 3 and 3.

Finally, when the letters are thus disposed one after the other

2 and 2, 3 and 3, 4 and 4, . . . it may be required that no two of the results thus formed shall be composed of the same letters, that is, that they shall differ from each other by at least one of the letters; and we may then demand the whole number of results thus obtained. In this case, the results are called *combinations.*

Thus, *ab*, *ac*, *bc* . . ., *ad*, *bd* . . ., are *combinations* of the letters taken 2 and 2.

In like manner, *abc*, *abd* . . ., *acd*, *bcd* . . ., are *combinations* of the letters taken 3 and 3.

Hence, there is an essential difference in the signification of the words, *permutations*, *arrangements*, and *combinations*.

Permutations *are the results obtained by writing a certain number of letters one after the other in every possible order, in such a manner that all the letters enter into each result, and each letter enters but once.*

Arrangements *are the results obtained by writing a number* m *of letters one after the other in every possible order, in sets of* 2 *and* 2, 3 *and* 3, 4 *and* 4 . . . n *and* m; m *being* > n, that is, the number of letters in each result being less than the whole number of letters considered. However, if we suppose $n = m$, the *arrangements* taken *n* and *n* will become simple *permutations.*

Combinations *are arrangements in which any two will differ from each other by at least one of the letters which enter them.*

148. Problem 1. *Determine the whole number of permutations of which* n *letters are susceptible.*

In the first place, two letters *a* and *b* evidently give the two permutations *ab* and *ba*. Therefore, *the number of permutations of* 2 *letters is* 2, *or* 1×2.

Take the 3 letters *a*, *b*, *c*. Take any one of them, *c*, for example, and write it to the right of the two arrangements *ab* and *ba*; there will result the two permutations of three letters, *abc*, *bac*. Now, as the same thing may be done with each of the three letters, it follows that *the total number of permutations of three letters is equal to* 2×3, or $1 \times 2 \times 3$.

In general, let there be a number *m* of letters *a*, *b*, *c*, *d* . . ., and suppose *the total number of permutations of* n—1 *letters* known; let *Q* denote this number.

Take one of the *n* letters, and write it to the right of each of the *Q* permutations given by the *n*—1 remaining letters; there will result Q permutations of n letters, terminated by the letter first taken. Now, as the same thing can be done with each of the n letters, it follows that the *total number of permutations of* n *letters* is equal to - - - - - - - $Q\times n$.

Let $n=2$. Q will then denote the number of permutations that can be made with a single letter; hence $Q=1$, and in this particular case we have $Q\times n=1\times 2$.

Let $n=3$. Q will then express the number of permutations of 3—1 or 2 letters, and is equal to 1×2. Therefore $Q\times n$ is equal to $1\times 2\times 3$.

Let $n=4$. Q in this case denotes the number of permutations of 3 letters, and is equal to $1\times 2\times 3$. Hence, $Q\times n$ becomes $1\times 2\times 3\times 4$.

149. Problem 2. *Having given a number* m *of letters* a, b, c, d . . ., *to determine the total number of arrangements that may be formed of them by taking them* n *at a time;* m *being supposed greater than* n.

In order to resolve this question in a general manner, suppose the total number of *arrangements* of the m letters taken $n-1$ at a time to be known, and denote this number by P.

Take any one of these arrangements, and annex to it each of the letters which do not enter it, and of which the number is $m-(n-1)$, or $m-n+1$; it is evident that we would thus form a number $m-n+1$ of arrangements of n letters, differing from each other by the last letter.

Now take another arrangement of $n-1$ letters, and annex to it each of the $m-n+1$ letters which do not make a part of it; we again obtain a number $m-n+1$ of arrangements of n letters, differing from each other, and from those obtained as above, at least in the disposition of one of the $n-1$ first letters. Now, as we may in the same manner consider each of the P arrrangements of the m letters taken $n-1$ at a time, and annex to them successively the $m-n+1$ other letters, it follows that the total number of arrangements of m letters taken n at a time, is expressed by

$$P(m-n+1).$$

To apply this to the particular cases of the number of arrangements of m letters taken 2 and 2, 3 and 3, 4 and 4, make $n=2$, whence $m-n+1=m-1$; P will in this case express the total number of arrangements taken $2-1$ and $2-1$, or 1 and 1, and is consequently equal to m; therefore the formula becomes $m(m-1)$.

Let $n=3$, whence $m-n+1=m-2$; P will then express the number of arrangements taken 2 and 2, and is equal to . . . . $m(m-1)$; therefore the formula becomes $m(m-1)(m-2)$.

Again, take $n=4$, whence $m-n+1=m-3$; P will express the number of arrangements taken, 3 and 3, or is equal to . . . . . $m(m-1)(m-2)$; therefore the formula becomes . . . . $m(m-1)(m-2)(m-3)$.

N. B. From the manner in which the particular cases have been deduced from the general formula, we may conclude that it reduces to

$$m(m-1)(m-2)(m-3) \cdots (m-n+1);$$

that is, *it is composed of the product of the* n *consecutive numbers comprised between* m *and* $m-n+1$, *inclusively.*

From this formula, that of the preceding No. can easily be deduced, viz. the developement of the value of $Q \times n$.

For we have seen (123) that the *arrangements* become permutations when the number of letters composing each arrangement is supposed equal to the total number of letters considered.

Therefore, to pass from the total number of arrangements of m letters, taken n and n, to the number of permutations of n letters, it is only necessary to make $m=n$ in the above developement, which gives

$$n(n-1)(n-2)(n-3) \cdots 1.$$

By reversing the order of the factors, observing that the last is 1, the next to the last 2, which is preceded by 3 . . . . , it becomes

$$1, 2, 3, 4 \cdots (n-2)(n-1)n,$$

for the developement of $Q \times n$.

This is nothing more than the series of natural numbers comprised between 1 and n, inclusively.

150. Problem 3. *To determine the total number of different combinations that can be formed of* m *letters, taken* n *at a time.*

Let X denote the total number of arrangements that can be formed of m letters, taken n and n, Y the number of permutations of n letters, and Z the total number of *different combinations* taken n and n.

It is evident that all the possible arrangements of m letters, taken n at a time, can be obtained, by subjecting the n letters of each of the Z *combinations*, to all the *permutations* of which these letters are susceptible. Now a single combination of n letters gives, by hypothesis Y permutations; therefore Z combinations will give $Y\times Z$ . . . arrangements, taken n and n; and as X denotes the total number of arrangements, it follows that the three quantities X, Y, and Z, give the relations $X=Y\times Z$; whence $Z=\frac{X}{Y}$.

But we have (No. 149.) $X=P(m-n+1)$
and (148) $Y=Q\times n$

Therefore, $Z=\frac{P(m-n+1)}{Q\times n}=\frac{P}{Q}\times\frac{m-n+1}{n}$.

Since P expresses the total number of arrangements, taken $n-1$ and $n-1$, and Q the number of permutations of $n-1$ letters, it follows that $\frac{P}{Q}$ expresses the number of different combinations of m letters taken $n-1$ and $n-1$.

To apply this to the particular case of combinations of m letters taken 2 and 2, 3 and 3, 4 and 4 . . .

Make $n=2$, in which case $\frac{P}{Q}$ expresses the number of combinations of the letters taken 2—1 and 2—1 or 1 and 1, and is equal to m; the above formula becomes $m\times\frac{m-1}{2}$ or $\frac{m(m-1)}{1\times 2}$.

Let $n=3$, $\frac{P}{Q}$ will express the number of combinations taken 2 and 2, and is equal to $\frac{m(m-1)}{1.2}$; and the formula becomes $\frac{m(m-1)(m-2)}{1.2.3}$.

In like manner, we would find the number of combinations of m letters taken 4 and 4, to be $\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}$; and in general the number of combinations of m letters taken n and n is expresssed by

$$\frac{m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)}{1.2.3.4.\ldots\ldots(n-1).n};$$

which is the development of the expression $\dfrac{P(m-n+1)}{Q\times n}$.

Demonstration of the Binomial Theorem.

151. In order to discover more easily the law for the development of the mth power of the binomial $x+a$, we will observe the law of the product of several binomial factors $x+a$, $x+b$, $x+c$, $x+d\ldots\ldots$ of which the first term is the same in each, and the second terms different.

$$\begin{array}{ll}
 & x+a \\
 & x+b \\ \hline
\text{1st. product}\ldots & \begin{array}{l|l} x^2+a & x+ab \\ +b & \end{array} \\
 & x+c \\ \hline
\text{2d.}\ldots\ldots\ldots & \begin{array}{l|l|l} x^3+a & x^2+ab & x+abc \\ +b & +ac & \\ +c & +bc & \end{array} \\
 & x+d \\ \hline
 & \begin{array}{l|l|l|l} x^4+a & x^3+ab & x^2+abc & x+abcd \\ +b & +ac & +abd & \\ +c & +ad & +acd & \\ +d & +bc & +bcd & \\ & +bd & & \\ & +cd & & \end{array}
\end{array}$$

From these products, obtained by the common rule for algebraic multiplication, we discover tho following laws:

1st. With respect to the exponents; the exponent of x in the first term is equal to the number of binomial factors employed. In the following terms this exponent diminishes by unity to the last term, when it is 0.

2d. With respect to the coefficients of the different powers of x: that of the first term is unity; the coefficient of the second term is equal to the sum of the second terms of the binomials; the coefficients of the third term is equal to the sum of the products of the different second terms taken two and two; the coefficient of the fourth term is equal to the sum of their different products taken three and three. Reasoning from analogy, we may conclude that the coefficient of the term which has n terms before it, is equal to the sum of the different products of the m second terms of the binomials taken n and n. The last term is equal to the continued product of the second terms of the binomials.

In order to be certain that this law of composition is general, suppose that it has been proved to be true for a number m of binomials, and see if it be true when a new factor is introduced into the product.

For this purpose, suppose

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \ldots + Mx^{m-n+1} + Nx^{m-n} + \ldots + U,$$

to be the product of m binomial factors (Nx^{m-n} representing the term which has n terms before it, and Mx^{m-n+1} that which immediately precedes it).

Let $x+k$ be the new factor, the product when arranged according to the powers of x, will be

$$x^{m+1} \begin{array}{l|l|l|l|l} +A & x^m_r +B & x^{m-1} +C & x^{m-2} + \ldots +N & x^{m-n+1} + \ldots \\ +K & \quad +AK & \quad +BK & \quad\quad +MK & \quad +UK. \end{array}$$

From which we perceive that *the law of the exponents* is evidently the same.

With respect to the coefficients, 1st. That of the first term *is unity*. 2d. $A+K$, or the coefficient of x^m, is also the *sum of the second terms of the* m+1 *binomials*.

3d. B is by hypothesis the sum of the different products of the second terms of the m binomials, and AK expresses the sum of the products of each of the second terms of the m first binomials, by the new second term K; therefore $B+AK$ *is the sum of the different products of the second terms of the* m+1 *binomials, taken two and two.*

In general, since N expresses the sum of the products of the second terms of the m first binomials, taken n and n; and as MK represents the sum of the products of these second terms, taken $n-1$ and $n-1$, multiplied by the new *second term* K, it follows that $N+MK$, or the coefficient of the term which has n terms before it, is equal to the sum of the different products of the second terms of the $m+1$ binomials, taken n and n. The last term is equal to the continued product of the $m+1$ second terms.

Therefore, the law of composition, supposed true for a number m of binomial factors, is also true for a number denoted by $m+1$. It is therefore general.

Let us suppose, that in the product resulting from the multiplication of the m binomial factors, $x+a$, $x+b$, $x+c$, $x+d$.... we make $a=b=c=d$...., the indicated expression of this product, $(x+a)(x+b)(x+c)$, will be changed into $(x+a)^m$. With respect to its development, the coefficients being $a+b+c+d$...., $ab+ac+ad+$...., $abc+abd+acd+$...., the coefficient of x^{m-1}, or $a+b+c+d$..., becomes $a+a+a+a+$..., that is, a taken as many times as there are letters a, b, c..., and is therefore equal to ma. The coefficient of x^{m-2}, or...... $ab+ac+ad+$..., reduces to $a^2+a^2+a^2$..., or to as many times a^2 as we can form different combinations with m letters, taken two and two, or (250) to $m.\frac{m-1}{2}a^2$.

The coefficient of x^{m-3} reduces to the product of a^3, multiplied by the number of different combinations of m letters, taken 3 and 3, or to $m.\frac{m-1}{2}\cdot\frac{m-2}{3}a^3$, &c.

In general, if the term which has n terms before it is denoted by Nx^{m-n}, the coefficient, which in the hypothesis of the second terms being different, is equal to the sum of their products, taken n and n, reduces, when all of the terms are supposed equal, to a^n, multiplied by the number of different combinations that can be made with m letters, taken n and n. Therefore (150)

$$N=\frac{P(m-n+1)}{Q\times n}a^n.$$

From which we have the formula

$$(x+a)^m=x^m+max^{m-1}+m.\frac{m-1}{2}a^2x^{m-2}$$

$$+m.\frac{m-1}{2}\cdot\frac{m-2}{3}a^3x^{m-3}\ldots\ldots+\frac{P(m-n+1)}{Q.n}a^nx^{m-n}\ldots+a^m.$$

152. By inspecting the different terms of this development, a *simple law* will be perceived, by means of which the coefficient of any term is formed from the coefficient of the preceding term.

The coefficient of any term is formed by multiplying the coefficient of the preceding term by the exponent of x *in that* *term, and dividing the product by the number of terms which precede the required term.*

For, take the *general term* $\frac{P(m-n+1)}{Q\times n}a^n x^{m-n}$. (This is called the *general term*, because by making $n=2, 3, 4\ldots$, all of the others can be deduced from it). The term which immediately precedes it, is evidently $\frac{P}{Q}a^{n-1}x^{m-n+1}$, since (150) $\frac{P}{Q}$ expresses the number of combinations of m letters taken $n-1$ and $n-1$. Here we see that the coefficient $\frac{P(m-n+1)}{Q\times n}$ is equal to the coefficient $\frac{P}{Q}$ which precedes it, multiplied by $m-n+1$, the exponent of x in this term, and divided by n, the number of terms preceding the required term. This law serves to develope a particular power, without our being obliged to have recourse to the general formula.

For example, let it be required to develope $(x+a)^6$. From this law we have.

$(x+a)^6=x^6+6ax^5+15a^2x^4+20a^3x^3+15a^4x^2+6a^5x+a^6$.

After having formed the two first terms from the terms of the general formula $x^m+max^{m-1}+\ldots$, multiply 6, the coefficient of the second term, by 5, the exponent of x in this term, then divide the product by 2, which gives 15 for the coefficient of the third term. To obtain that of the fourth, multiply 15 by 4, the exponent of x in the third term, and divide the product by 3, the number of terms which precede the fourth, this gives 20; and the other terms are found in the same way.

In like manner we find

$(x+a)^{10}=x^{10}+10ax^9+45a^2x^8+120a^3x^7+210a^4x^6,$
$+252a^5x^5+210a^6x^4+120a^7x^3+45a^8x^2+10a^9x+a^{10}$.

Consequences of the binomial formula, and theory of Combinations.

153. *First.* The expression $(x+a)^m$ being such that x may be substituted for a and a for x without altering its value, it follows that the same thing can be done in the development of it; therefore if this development contains a term of the form Ka^nx^{m-n}, it must have another equal to Kx^na^{m-n} or $Ka^{m-n}x^n$.

These two terms are evidently at equal distances from the two extremes, in the development; for the number of terms which precede any term, being indicated by the exponent of a in that term, it follows that the term Ka^nx^{m-n} has n terms before it, and that the term $Ka^{m-n}x^n$ has $m-n$ terms before it, and consequently n terms after it (since the whole number of terms is denoted by $m+1$).

Therefore, *in the development of any power of a binomial, the coefficients at equal distances from the two extremes are equal to each other.*

N. B. In the terms Ka^nx^{m-n}, $Ka^{m-n}x^n$, the coefficients express the number of different combinations that can be formed with m letters taken n and n and $m-n$ and $m-n$; we may therefore conclude that *the number of different combinations of* m *letters taken* n *and* n *is equal to the number of combinations of* m *quantities taken* m—n *and* m—n.

For example, *twelve* quantities combined 5 and 5, give the same number of combinations as these twelve quantities taken 12—5 and 12—5, or 7 and 7. Five quantities combined 2 and and 2, give the same number of combinations as five quantities combined 5—2 and 5—2, or 3 and 3.

154. *Second.* If in the general formula,

$$(x+a)^m=x^m+max^{m-1}+m\frac{m-1}{2}a^2x^{m-2}+\&c.$$

we suppose $x=1$, $a=1$, it becomes

$$(1+1)^m \text{ or } 2^m=1+m+m\frac{m-1}{2}+m\frac{m-1}{2}\cdot\frac{m-2}{3}+\&c.$$

That is, *the sum of the coefficients of the different terms of the formula for the binomial, is equal to the* m*th power of* 2.

Thus, in the particular case

$$(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5,$$

the sum of the coefficients $1+5+10+10+5+1$ is equal to 2^5 or 32. In the 10th power developed in No. 152, the sum of the coefficients is equal to 2^{10} or 1024.

155. *Third. In a series of numbers decreasing by unity, of which the first term is* m *and the last* m—p (m *and* p *being entire numbers,) the continued product of all these numbers is divisible by the continued product of all the natural numbers from* 1 *to* p+1 *inclusively.*

that is $\frac{m(m-1)(m-2)(m-3)\ldots(m-p)}{1\;.\;2\;.\;3\;.\;4\ldots\ldots(p+1)}$ is a whole number. For, from what has been said in No. 150. this expression represents the number of different combinations that can be formed of m letters taken $p+1$ and $p+1$. Now this number of combinations is, from its nature, an entire number; therefore the above expression is necessarily a whole number.

§ II. *Of the extraction of the Roots of particular numbers.*

156. The *third power* or *cube* of a number, is the product of this number multiplied by itself twice; and the *third* or *cube root* is the number which, raised to the third power, will produce the proposed number.

The first ten numbers being

	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
their cubes are	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000.

Reciprocally, the numbers of the second line have the numbers of the first line for their *cube roots.*

By inspecting these lines, we perceive that there are but nine *perfect cubes* among numbers composed of one, two, or three figures; each of the others has for its cube root a whole number, plus a fraction *which cannot be expressed exactly by means of unity.* For suppose that the irreducible fraction $\frac{a}{b}$ can have the whole number N for its root, it would follow that the cube of $\frac{a}{b}$, or $\frac{a^3}{b^3}$, would be equal to N. Now this is impossible, for a and b being prime with respect to each other, a^3 and b^3 will also be prime with respect to each other; therefore $\frac{a^3}{b^3}$ cannot be equal to an entire number.

157. The greater the roots of *two consecutive perfect cubes* are, the greater will be the difference between these cubes.

Let a and $a+1$ be two consecutive whole numbers; we have (127)

$$(a+1)^3=a^3+3a^2+3a+1\,;$$

whence

$$(a+1)^3-a^3=3a^2+3a+1.$$

That is, *the difference between the cubes of two consecutive*

***whole numbers is equal to three times the square of the least number, plus three times this number, plus* 1.**

Thus, the difference between the cube of 90 and the cube of 89, is equal to $3(89)^2+3\times 89+1=24031$.

158. In order to extract the cube root of an entire number, we will observe, that when the number does not contain more than three figures, *its root is obtained by merely inspecting the cubes of the nine first numbers.* Thus, the cube root of 125 is 5; the cube root of 72 is 4 plus a fraction, or is within one of 4; the cube root of 841 is within one of 9, since 841 falls between 729, or the cube of 9, and 1000, or the cube of 10.

When the number contains more than three figures, the process will be as follows. Let the proposed number be 103823.

103.823	47
64	48
398. 23	

48	47
48	47
384	329
192	188
2304	2209
48	47
18432	15463
9216	8836
110592	103823

This number being comprised between 1000, which is the cube of 10, and 1000000, which is the cube of 100, its root must be composed of two figures, or of tens and units. Denoting the tens by a, and the units by b, we have (46)

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

Whence it follows, that the cube of a number composed of tens and units, contains *the cube of the tens, three times the product of the square of the tens by the units, three times the product of the square of the units by the tens, plus the cube of the units.*

This being the case, the cube of the tens, giving at least, ***thousands***, the three last figures to the right cannot form a part of it, the cube of the tens must therefore be found in the part 103 which is separated from the three last figures by a point. Now the root of the greatest cube contained in 103 being 4, this is the number of tens in the required root; for 103823 is evidently comprised between $(40)^3$ or 64000, and $(50)^3$ or 125,000; hence the required root is composed of 4 tens, plus a certain number of units less than *ten.*

Having found the number of tens, subtract its cube 64 from from 103; there remains 39, and bringing down the part 823, we have 39823, which contains *three times the square of the tens by the units*, plus the two parts before mentioned. Now as the square of a number of tens gives at least hundreds, it follows that three times the squares of the tens by the units, must be found in the part 398 to the left of 23, which is separated from it by a point. Therefore, dividing 348 by three times the square of the tens, which is 48, the quotient 8, will be the unit of the root, or something greater, since 398 hundreds is composed of three times the square of the tens by the units, together with the two other parts. We may ascertain whether the figure 8 is too great, by forming the three parts which enter in 39823, by means of the figure 8 and the number of tens 4; but it is much easier to cube 48, as has been done in the above table. Now the cube of 48 is 110592, which is greater than 103823; therefore 8 is too great. By cubing 47 we obtain 103823; hence the proposed number is a perfect cube, and 47 is the cube root of it.

N. B. The units figures could not be obtained first; because the cube of the units might give tens, and even hundreds, and the tens and hundreds would be confounded with those which arise from other parts of the cube.

Again, extract the cube root of 47954

```
47.954|36
27    |27
------
209             36
                36
                ---
47954          216
46656         108
-----         ----
 1298         1296
                36
              ----
              7776
             3888
             -----
             46656
```

The number 47954 being below 1000000, its root contains only two figures, viz. tens and units. The cube of the tens is found in 47 *thousands*, and we can prove, as in the preceding example, that 3, the root of the greatest cube contained in 47, expresses the tens. Subtracting the cube of 3 or 27 from 47 there remains 20; bringing down to the right of this remainder the figure 9 from the part 954, the number 209 hundreds is composed of three times the square of the tens by the units, plus the number arising from the other two parts. Therefore by forming three times the square of the tens, 3, which is 27, and dividing 209 by it, the quotient 7 will be the units of the root, or something greater. Cubing 37, we have 50653, which is greater than 47954; then cubing 36, we obtain 46656, which subtracted from 47954, gives 1298 for a remainder. Hence the proposed number is not a perfect cube; but 36 is its root to *within unity*. In fact, the difference between the proposed number and the cube of 36, is, as we have just seen, 1298, which is less than $3(36)^2+3\times36+1$, for in verifying the result we have obtained 3888 for three times the square of 36.

159. Again, take for another example, the number, 43725658 containing more than 6 figures.

43.725.658	352		
27	27.........3675		
167		35	352
		35	352
43 725		175	704
42 875		105	1760
8506		1225	1056
		35	123904
43725658		6125	352
43614208		3675	247808
Rem. . . . 111450		42875	619520
			371712
			43614208

Whatever may be the required root, it contains more than one figure, and it may be considered as composed of units and tens only, (the tens being expressed by one or more figures.)

Now the cube of the tens gives at least thousands; it must therefore be found in the part which is to the left of the three last figures 658. I say now that if we extract the root of the greatest cube contained in the part 43725, considered with reference to its absolute value, we will obtain the whole number of tens of the root; for let a be the root of 43725, to within unity, that is, such that 43725 shall be comprised between a^3 and $(a+1)^3$; then will 43725000 be comprehended between $a^3\times1000$ and $(a+1)^3\times1000$; and as these last two numbers differ from each other by more than 1000, it follows that the proposed number itself, 43725658, is comprised between $a^3\times1000$ and $(a+1)^3\times1000$; therefore the required root is comprised between that of $a^3\times1000$, and $(a+1)^3\times1000$, that is, between $a\times10$ and $(a+1)\times10$. It is therefore composed of a tens, plus a certain number of units less than *ten*.

The question is then reduced to extracting the cube root of 43725; but this number having more than three figures, its root will contain more than one, that is, it will contain tens and units. To obtain the tens, point off the three last figures, 725, and extract the root of the greatest cube contained in 43.

The greatest cube contained in 43 is 27, the root of which is 3; this figure will then express the tens of the root of 43725, (or the figure in the place of hundreds in the total root). Subtracting the cube of 3, or 27, from 43, we obtain 16 for a remainder, to the right of which bring down the first figure 7, of the second period 725, which gives 167.

Taking three times the square of the tens, 3, which is 27, and dividing 166 by it, the quotient 6 is the unit figure of the root of 43725, or something greater. It is easily seen that this number is in fact too great; we must therefore try 5. The cube of 35 is 42875, which, subtracted from 43725, gives 850 for a remainder, which is evidently less than $3\times(35)^2+3\times35+1$. Therefore 35 is the root of the greatest cube contained in 43725; hence it is the number of tens in the required root.

To obtain the units, bring down to the right of the remainder 850, the first figure, 6, of the last period, 658, which gives 8506; then take 3 times the square of the tens, 35, which is 3675, and divide 8506 by it; the quotient is 2, which we try by cubing 352: this gives 43614208, which is less than the proposed number, and subtracting it from this number, we obtain 111450 *for a remainder*. Therefore 352 is the cube root of 43725658, *to within unity.*

General Rule.

In order to extract the cube root of an entire number, divide it into periods of three figures each, commencing on the right, until you arrive at a period containing but one, two, or three figures; extract the root of the greatest cube contained in the first period on the left, and subtract this cube from the first period; bring down to the right of the remainder the first figure of the second period, and divide the number thus formed by three times the square of the figure already found in the root; write the quotient to the right of this figure, and cube the number thus formed; if this cube is greater than the number expressed by the two first periods, diminish the quotient by one or more units, until you obtain a number which can be subtracted from the two first periods. After performing the subtraction, bring down to the right of the remainder the first figure of the third period, then divide the number thus formed by three times the number expressed by the two figures already found; the quotient, if it is not too great, will be such that in writing it to the right of the two figures of the root, and cubing the number thus formed, the result can be subtracted from the number expressed by the three first periods. After this subtraction, bring down to the right of the remainder the first figure of the fourth period, and continue the same operations until you have brought down all of the periods.

Remark. In the course of the operation, we may suspect that the quotient of which we have just spoken is much too great, and would wish to diminish it at once by two or more units; but in cubing the root already found, prefixed to this figure, and subtracting this cube from the number expressed by the periods already considered in the given number, we might obtain a very great remainder, which would lead us to suppose that the last figure of the root is too small. But if this is the case, then (157) *the remainder must be equal to, or greater than three times the square of the root obtained, plus three times this same root, plus one.* In this case the root must be increased by one or more units, of the order of the last figure obtained.

Examples.

$\sqrt[3]{483249}=78$, with a remainder 8697;

$\sqrt[3]{91632508641}=4508$, with a remainder 20644129;

$\sqrt[3]{32977340218432}=32068$.

To extract the n^{th} root of a whole number.

160. In order to generalize the process for *the extraction of roots*, we will denote the proposed number by N, and the degree of the root to be extracted by n. When N has not more than n figures, its root has but one, and it is obtained immediately by forming the n^{th} power of each of the whole numbers comprised between 1 and 10, of which the n^{th} power is 10^n, or the smallest number which contains $n+1$ figures.

When N contains more than n figures, its root has more than one figure, and may then be considered as composed of tens and units. Designating the tens by a, and the units by b, we have (151)

$$N=(a+b)^n=a^n+na^{n-1}b+n\frac{n-1}{2}a^{n-2}b^2+, \&c.;$$

that is, the proposed number contains *the* n[th] *power of the tens, plus* n *times the product of the* n—1 *power of the tens by the units*, plus a series of other parts which it is not necessary to consider.

Now as the n^{th} power of the tens cannot give units of an order inferior to unity followed by n ciphers, the n last figures on the right cannot make a part of it. They must then be pointed off, and the root of the greatest n^{th} power contained in the figures on the left should be extracted; this root will be *the tens of the required root.*

If this part on the left should contain more than n figures, the n figures on the right of it must be separated from it, and the root of the greatest n^{th} power contained in the part on the left extracted, and so on.

After having, in this manner, divided the number N *into periods of* n *figures each, extract the root of the greatest* n[th] *power contained in the left hand period;* this gives the units of the highest order contained in the total root, or the tens in the root of the number formed by the two first periods on the left. *Subtract the* n^{th} *power of this figure from the left hand period;* the remainder, followed by the second period, contains n times the product of the $n-1$ power of the figure found into the following figure, plus a series of other parts. But it is evident that this first part cannot give units of an order inferior to 10^{n-1}; therefore the $n-1$ last figures of the second period cannot make a part of it. Hence *it is only necessary to bring down to the right of the remainder corresponding to the first period, the first figure of the second period; and if, after having formed* n *times the* n—1 *power of the first figure of the root, we divide by this result, the remainder, followed by the first figure of the second period*, the quotient will be the second figure of the root, or something greater. To ascertain whether it is too great, we should write it to the right of the first figure, *and raise the number thus formed to the* n^{th} *power; then subtract this result from the two first periods*, which will give a new remainder, to the right of which *bring down the first figure of the third period; then divide the number thus formed by* n *times the* n—1 *power of the two figures of the root already found.*

This operation is continued until all the periods are brought down.

161. *Remark.* When the degree of the root to be extracted is a multiple of two or more numbers, as 4, 6, ..., *the root can be obtained by extracting the roots of more simple degrees, successively.* To explain this, we will remark that

$$(a^3)^4=a^3\times a^3\times a^3\times a^3=a^{3+3+3+3}=a^{3\times 4}=a^{12}.$$

and that in general $(a^m)^n=a^m\times a^m\times a^m\times a^m...=a^{m\times n}$(16). Hence *the* n^{th} *power of the* m^{th} *power of a number, is equal to the* mn*th power of this number.*

Reciprocally, *the* mn^{th} *root of a number is equal to the* n^{th} *root of the* m^{th} *root of this number*, or algebraically ....

$$\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}.$$

For, let ... $\sqrt[n]{\sqrt[m]{a}}=a'$, raising both members to the n^{th} power there will result .... $\sqrt[m]{a}=a'^n$; (for from the definition of a root (2), we have $(\sqrt[n]{K})^n=K$).

Again, by raising both members to the mth power, we obtain $a=(a'^{n})^{m}=a'^{mn}$. Extracting the mn^{th} root of both members, $\sqrt[mn]{a}=a'$; but we already have $\sqrt[n]{\sqrt[m]{a}}=a'$; hence $\sqrt[mn]{a}=\sqrt[n]{\sqrt[m]{a}}$. By this method we find that

$$\sqrt[4]{256}=\sqrt{\sqrt{256}}=\sqrt{16}=4;$$

$$\sqrt[6]{2985984}=\sqrt[3]{\sqrt{2985984}}=\sqrt[3]{1728}=12;$$

$$\sqrt[6]{1771561}=\sqrt[3]{\sqrt{1771561}}=11;$$

$$\sqrt[8]{1679616}=\sqrt[4]{1296}=\sqrt{\sqrt{1296}}=6.$$

N. B. Although the successive roots may be extracted in any order whatever, it is better to first extract the roots of the lowest degree, for then the extraction of the roots of the higher degrees, which is a more complicated operation, is effected upon numbers containing fewer figures than the proposed number.

Extraction of Roots by approximation.

162. When it is required to extract the nth root of a number which is not a *perfect power*, the method of No. 160 will give only the entire part of the root, or the root to within unity. As to the fraction which is to be added in order to complete the root, it cannot be obtained exactly; for the nth power of a fraction $\frac{a}{b}$, $\left(\frac{a^n}{b^n}\right)$, cannot be reduced to a whole number, but we can approximate as near as we please to the required root.

Let it be required to extract the nth root of the whole number a, to within a fraction $\frac{1}{p}$; that is, so near it, that the error shall be less than $\frac{1}{p}$.

We will observe that a can be put under the form $\frac{a\times p^n}{p^n}$. If we denote the root of ap^n to within unity, by r, the number $\frac{a\times p^n}{p^n}$ or a, will be comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$;

therefore the $\sqrt[n]{a}$ is comprised between these two numbers, that is, between $\frac{r}{p}$ and $\frac{r+1}{p}$. Hence $\frac{r}{p}$ is the required root, to within the fraction $\frac{1}{p}$.

Rule. *To extract the root of a whole number to within a fraction* $\frac{1}{p}$, *multiply the number by* p^m*; extract the* m^{th} *root of the product to within unity, and divide the result by* p.

163. Again, suppose it is required to extract the n^{th} root of the fraction $\frac{a}{b}$.

Multiply each term of the fraction by b^{n-1}; it becomes $\frac{a}{b} = \frac{ab^{n-1}}{b^n}$. Let r denote the n^{th} root of ab^{n-1}, to within unity; $\frac{ab^{n-1}}{b^n}$ or $\frac{a}{b}$, will be comprised between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$. Therefore, *after having made the denominator of the fraction a perfect power of the* n^{th} *degree, extract the* n^{th} *root of the numerator, to within unity, and divide the result by the root of the new denominator.* When a greater degree of exactness is required than that indicated by $\frac{1}{b}$, extract the root of ab^{n-1} to within any fraction $\frac{1}{p}$; designate this root by $r'+\frac{m}{p}$, then $\frac{r'+\frac{m}{p}}{b}$ will represent the required root to within a fraction $\frac{1}{pb}$.

164. Suppose it is required to extract the cube root of 15, to within $\frac{1}{12}$. We have $15 \times 12^3 = 15 \times 1728 = 25920$. Now the cube root of 25920, to within unity, is 29; hence the required root is $\frac{29}{12}$ or $2\frac{5}{12}$. (See No. 162.)

Again, extract the cube root of 47, to within $\frac{1}{20}$.

We have $47 \times 20^3 = 47 \times 8000 = 376000$. Now the cube root of 376000, to within unity, is 72; hence $\sqrt[3]{47}=\frac{72}{20}=3\frac{12}{20}$, to within $\frac{1}{20}$.

Find the value of $\sqrt[3]{25}$ to within 0,001.

To do this, multiply 25 by the cube of 1000, or by 1000000000, which gives 25000000000. Now, the cube root of this number, is 2920; hence $\sqrt[3]{25}=2,920$ to within 0,001. (See No. 162).

In general, *in order to extract the cube root of a whole number to within a given decimal fraction, annex three times as many ciphers to the number, as there are decimal places in the required root; extract the cube root of the number thus formed to within unity, and point off from the right of this root the required number of decimals.*

165. We will now explain the method for *extracting the cube root of a decimal fraction.* Suppose it is required to extract the cube root of 3,1415.

As the denominator 10000 of this fraction is not a perfect cube, it is necessary to make it one, by multiplying it by 100, *which amounts to annexing two ciphers to the proposed decimal,* and we have 3,141500. Extract the cube root of 3141500 (that is, of the number considered independent of the comma,) to within unity; this gives 146. Then divide by 100, or $\sqrt[3]{1000000}$, and we find $\sqrt[3]{3.1415}=1,46$ to within 0.01.

If it be required to approximate still nearer, annex *three times* as many ciphers more, as there are *additional decimal places required in the root.*

To extract the cube root of a vulgar fraction to within a given decimal fraction, the most simple method is to *reduce the proposed fraction to a decimal fraction, continuing the operation until the number of decimal places is equal to three times the number required in the root.* The question is then reduced to extracting the cube root of a decimal fraction.

166. Suppose it is required to find the sixth root of 23, to within 0,01.

Applying the rule of No. 162 to this example, we multiply

23 by 100^6, or annex *twelve* ciphers to 23, extract the sixth root of the number thus formed to within unity, and divide this root by 100, or point off two decimals on the right.

In this way we will find that $\sqrt[6]{23}=1,68$, to within 0,01.

Examples.

$\sqrt[3]{473}$, to within $\frac{1}{20}=\frac{155}{20}$; $\sqrt[3]{79}=4,2908$, to within 0,0001;

$\sqrt[6]{13}=1,53$, to within 0,01; $\sqrt[3]{3,00415}=1,4429$, to within 0,0001;

$\sqrt[3]{0,00101}=0,01$, to within 0,01; $\sqrt{\frac{14}{25}}=0,824$, to within 0,001.

§ III. *Formation of Powers, and Extraction of Roots of Algebraic Quantities. Calculus of Radicals.*

We will first consider monomials.

167. Let it be required to form the fifth power of $2a^3b^2$. We have (No. 2)

$$(2a^3b^2)^5=2a^3b^2\times2a^3b^2\times2a^3b^2\times2a^3b^2\times2a^3b^2,$$

from which it follows, 1st. That the coefficient 2 must be multiplied by itself four times, or raised to the fifth power. 2d. That each of the exponents of the letters must be added to itself four times, or multiplied by 5.

Hence, $\qquad (2a^3b^2)^5=2^5.a^{3\times5}b^{2\times5}=32a^{15}b^{10}.$

In like manner, $(8a^2b^3c)^3=8^3.a^{2\times3}b^{3\times3}c^3=512a^6b^9c^3.$

Therefore, in order to raise a monomial to a given power, *raise the coefficient to this power, and multiply the exponent of each of the letters by the exponent of the power.*

Hence, reciprocally, to extract any root of a monomial, 1st. *Extract the root of the coefficient.* 2d. *Divide the exponent of each letter by the exponent of the root.*

$$\sqrt[3]{64a^9b^3c^6}=4a^3bc^2\ ;\quad \sqrt[4]{16a^8b^{12}c^4}=2a^2b^3c.$$

From this rule, we perceive, that in order that a monomial may be a perfect power of the degree of the root to be extracted, its coefficient *must be* a perfect power, and the exponent of the *letters must be divisible* by the exponent or index of the root to be extracted. It will be shown hereafter how the expression for the root of a quantity which is not a perfect power is reduced to its simplest terms.

168. Hitherto we have not paid any attention to the sign with which the monomial may be affected; but if we observe, that whatever may be the sign of a monomial, *its square is always positive*, and that every power of an even degree, $2n$, can be considered as the n^{th} power of the square, that is, $a^{2n}=(a^2)^n$, *every power of a quantity, of an even degree, whether positive or negative, is essentially positive.*

Thus, $$(\pm 2a^2b^3c)^4=+16a^8b^{12}c^4.$$

Again, as a power of an uneven degree, $2n+1$, is the product of a power of an even degree, $2n$, by the first power, it follows that *every power of an uneven degree of a monomial, is affected with the same sign as the monomial.*

Hence, $(+4a^2b)^3=+64a^6b^3$; $(-4a^2b)^3=-64a^6b^3$.

From this it is evident, 1st. That when the degree of the root of a monomial is uneven, the root will be affected with the same sign as the quantity.

Therefore,

$$\sqrt[3]{+8a^3}=+2a;\ \sqrt[3]{-8a^3}=-2a;\ \sqrt[5]{-32a^{10}b^5}=-2a^2b.$$

2d. When the degree of the root is even, and the monomial a positive quantity, the root is affected with either + or —.

Thus, $$\sqrt[4]{81a^4b^{12}}=\pm 3ab^3;\ \sqrt[6]{64a^{18}}=\pm 2a^3.$$

3d. *When the degree of the root is even, and the monomial uneven, the root is impossible;* for there is no quantity which, raised to a power of an even degree, can give a negative result. Therefore $\sqrt[4]{-a}$, $\sqrt[6]{-b}$, $\sqrt[8]{-c}$, are symbols of operations which it is impossible to execute. They are, like $\sqrt{-a}$, $\sqrt{-b}$, (No. 85) *imaginary expressions.*

169. It has already been seen in what manner a binomial $x+a$ is raised to a power of any degree whatever; but it may happen that the terms of the binomial are affected with coefficients and exponents.

Let it be required, for example, to develope $(2a^2+3ab)^3$. Take $2a^2=x$, and $3ab=y$, we have

$$(2a^2+3ab)^3=(x+y)^3=x^3+3x^2y+3xy^2+y^3.$$

Substituting $2a^2$ and $3ab$, for x and y, we have
$(2a^2+3ab)^3=(2a^2)^3+3(2a^2)^2.(3ab)+3(2a^2).(3ab)^2+(3ab)^3$,
or performing the operations indicated, by the rules given in No. 167, and those for the multiplication of monomials,

$$(2a^2+3ab)^3=8a^6+36a^5b+54a^4b^2+27a^3b^3.$$

In like manner, we will find

$$(4a^2b-3abc)^4=(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$
$$=(4a^2b)^4+4(4a^2b)^3(-3abc)+6(4a^2b)^2(-3abc)^2$$
$$+4(4a^2b)(-3abc)^3+(-3abc)^4$$
$$=256a^8b^4-768a^7b^4c+864a^6b^4c^2-432a^5b^4c^3+81a^4b^4c^4.$$

(The signs are alternately positive and negative.)

In order to develope $(x+y+z)^3$, we will take $x+y=u$, and we have

$$(u+z)^3=u^3+3u^2z+3uz^2+z^3;$$

or, replacing u by its value $x+y$,

$$(x+y+z)^3=(x+y)^3+3(x+y)^2.z+3(x+y)z^2+z^3,$$

and performing the operations indicated,

$$(x+y+z)^3=x^3+3x^2y+3xy^2+y^3+3x^2z+3xz^2+6xyz+3y^2z+3yz^2+z^3.$$

This expression is composed of *the cubes of the three terms, plus three times the square of each term by the first powers of the two others, plus six times the product of all three terms.* It is easily proved that this *law* is true for any polynomial.

To apply the preceding formula to the development of the cube of a trinomial, in which the terms are affected with coefficients and exponents, *designate each term by a single letter, then replace the letters introduced, by their values, and perform the operations indicated.*

From this rule, we will find that

$$(2a^2-4ab+3b^2)^3=8a^6-48a^5b+132a^4b^2-208a^3b^3+198a^2b^4-108ab^5+27b^6.$$

The fourth, fifth, &c. powers of any polynomial can be developed in an analogous manner.

170. As to the extraction of roots of polynomials, it will be sufficient to explain the method for the cube root; it will afterwards be easy to generalize.

Let N be the polynomial, and R its cube root. Conceive the two polynomials to be arranged with reference to some letter, a, for example. It results from the law of composition of the cube of a polynomial, (169), that the cube of R contains two parts, which cannot be reduced with the others; these are, the cube of the first term, and three times the square of the first term by the second. For it is evident, that these two terms contain the letter a, with an exponent greater than the exponent of this same letter in three times the square of the second term by the first; or in the cube of the second, three times the square of the first term by the third, &c. Hence these two parts necessarily form the first and second terms of N. Therefore, *by extracting the cube root of the first term of* N, *we obtain the first term of* R; *then dividing the second term of* N *by* 3 *times the square of the first term of* R, *we obtain the second term of* R. *Knowing the two first terms of* R, *we can form the cube of the binomial, and subtract it from* N. The remainder N' contains the product of three times the square of the first term in R, by the third, plus a series of other terms, involving a with a less exponent than it has in this product, which is consequently the first term of the remainder N'. Hence, *by dividing the first term of* N′ *by three times the square of the first term of* R, *we will obtain the third term of* R. *Cubing the trinomial found in the root, and subtracting this cube from* N, *we will obtain a new remainder*, N″, *upon which we can operate in the same manner as we did upon* N′; *and so on.*

By connecting those parts of the preceding demonstration, which are written in italics, we will form a general rule for extracting the cube root of any polynomial, and can apply it to the polynomials in No. 169.

Calculus of Radicals.

171. When it is required to extract a certain root of a monomial or polynomial which is not a perfect power, it can only be indicated by writing the proposed quantity after the sign $\sqrt{}$, placing over this sign the number which denotes the degree of the root to be extracted. This number is called the *index of the radical.*

Radical expressions may be reduced to their simplest terms by observing (84) that, *the* n[th] *root of a product is equal to the product of the* n[th] *roots of its different factors.*

Or in algebraic terms:

$$\sqrt[n]{abcd}=\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}.$$

For raising both members to the n^{th} power, we have for the first

$$(\sqrt[n]{abcd})^n=abcd.., \text{ and for the second}$$

$$(\sqrt[n]{a}\times\sqrt[n]{b}\times\sqrt[n]{c}\times\sqrt[n]{d}...)^n=(\sqrt[n]{a})^n.(\sqrt[n]{b})^n.(\sqrt[n]{c})^n.(\sqrt[n]{d})^n...=abcd.$$

Therefore, since the n^{th} powers of these quantities are equal, the quantities themselves must be equal. (No. 176)

This being the case, take the expression $\sqrt[3]{54a^4b^3c^2}$, which cannot be replaced by a rational monomial, since 54 is not a perfect cube, and the exponents of a and c are not divisible by 3, we have

$$\sqrt[3]{54a^4b^3c^2}=\sqrt[3]{27a^3b^3}.\sqrt[3]{2ac^2}=3ab\sqrt[3]{2ac^2}$$

In like manner, $\sqrt[3]{8a^2}=2\sqrt[3]{a^2}$; $\sqrt[4]{48a^5b^8c^6}=2ab^2c\sqrt[4]{3ac^2}$;

$$\sqrt[6]{192a^7bc^{12}}=\sqrt[6]{64a^6c^{12}}\times\sqrt[6]{3ab}=2ac^2\sqrt[6]{3ab}.$$

In the expressions, $3ab\sqrt[3]{2ac}$, $2\sqrt[3]{a^2}$, $2ab^2c\sqrt[3]{3ac^2}$, the quantities placed before the radical, *with the sign of multiplication*, are called *coefficients* of the radical.

172. The rule of No. 161 gives rise to another kind of simplification.

Take for example the radical expression, $\sqrt[6]{4a^2}$; from this rule we have, $\sqrt[6]{4a^2}=\sqrt[3]{\sqrt{4a^2}}$, and as the quantity affected with the radical of the second degree $\sqrt{}$, is a perfect square, its root can be extracted, hence

$$\sqrt[6]{4a^2}=\sqrt[3]{2a}.$$

In like manner, $\sqrt[4]{36a^2b^2}=\sqrt{\sqrt{36a^2b^2}}=\sqrt{6ab}.$

In general, $\sqrt[mn]{a^n}=\sqrt[m]{\sqrt[n]{a^n}}=\sqrt[m]{a}$; that is, when the index of a radical is multiplied by a certain number n, and the quantity under the radical sign is an exact n^{th} power, *we can, without changing the value of the radical, divide its index by* n, *and extract the* n^{th} *root of the quantity under the sign.*

This proposition is the inverse of another, not less important, viz. *we can multiply the index of a radical by a certain number, provided we raise the quantity under the sign, to a power of which this number denotes the degree.*

Thus, $\sqrt[m]{a}=\sqrt[mn]{a^n}$. For a is the same thing as $\sqrt[n]{a^n}$; hence,

$$\sqrt[m]{a}=\sqrt[m]{\sqrt[n]{a^n}}=\sqrt[mn]{a^n}.$$

This last principle serves to reduce two or more radicals to the same index.

For example, let it be required to reduce the two radicals $\sqrt[3]{2a}$ and $\sqrt[4]{(a+b)}$ to the same index.

By multiplying the index of the first by 4, the index of the second, and raising the quantity $2a$ to the fourth power; then multiplying the index of the second by 3, the index of the first, and cubing $a+b$, the values of the radicals will not be changed, and they will become

$$\sqrt[3]{2a}=\sqrt[12]{2^4a^4}=\sqrt[12]{16a^4};\quad \sqrt[4]{(a+b)}=\sqrt[12]{(a+b)^3}.$$

General Rule for Reducing Radicals to a Common Index.

Multiply the index of each radical by the product of the indices of all the other radicals, and raise the quantity under the sign to a power denoted by this product.

This rule, which is analogous to that given for the reduction of fractions to a common denominator, is susceptible of some modifications.

For example, reduce the radicals $\sqrt[4]{a}$, $\sqrt[6]{5b}$, $\sqrt[8]{a^2+b^2}$, to the same index.

As the numbers 4, 6, 8, have common factors, and 24 is the most simple multiple of the three numbers, it is only necessary to multiply the first by 6, the second by 4, and the third by 3, provided we raise the quantities under each radical sign to the 6th, 4th, and 3d powers respectively, which gives

$$\sqrt[4]{a}=\sqrt[24]{a^6}\,;\ \ \sqrt[6]{5b}=\sqrt[24]{5^4b^4},\ \ \sqrt[8]{a^2+b^2}=\sqrt[24]{(a^2+b^2)^3}.$$

In applying the above rules to numerical examples, beginners very often make mistakes similar to the following, viz.: In reducing the radicals $\sqrt[3]{2}$ and $\sqrt{3}$ to a common index, after having multiplied the index of the first (3) by that of the second, (2), and the index of the second by that of the first, then, instead of multiplying the *exponent* of the quantity under the first sign by 2, and the *exponent* of that under the second by 3, they often multiply the *quantity* under the first sign by 2, and the *quantity* under the second by 3. Thus, they would have

$$\sqrt[3]{2}=\sqrt[6]{2\times2}=\sqrt[6]{4},\ \text{and}\ \sqrt{3}=\sqrt[6]{3\times3}=\sqrt[6]{9}.$$

Whereas, they should have, by following the rule,

$$\sqrt[3]{2}=\sqrt[6]{(2)^2}=\sqrt[6]{4},\ \text{and}\ \sqrt{3}=\sqrt[6]{(3)^3}=\sqrt[6]{27}.$$

Reduce $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt[5]{\frac{1}{2}}$, to the same index.

Addition and Subtraction of Radicals.

173. Two radicals are ***similar***, when they have the same index, and the same quantity, under the sign. Thus, $3\sqrt{ab}$ and $7\sqrt{ab}$, are similar radicals, as also $3a^2\sqrt[3]{b^2}$, and $9c^3\sqrt[3]{b^2}$.

Therefore, to add or subtract similar radicals, ***add or subtract their coefficients, and prefix the sum or difference to the common radical.***

Thus, $\quad 3\sqrt[3]{b}+2\sqrt[3]{b}=5\sqrt[3]{b},\ \ 3\sqrt[3]{b}-2\sqrt[3]{b}=\sqrt[3]{b},$

$$3a\sqrt[4]{b}\pm2c\sqrt[4]{b}=(3a\pm2c)\sqrt[4]{b}.$$

Sometimes when two radicals are dissimilar, they can be reduced to similar radicals by Nos. 171 and 172. For example,

$$\sqrt{48ab^2}+b\sqrt{75a}=4b\sqrt{3a}+5b\sqrt{3a}=9b\sqrt{3a}.$$

$$\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}=2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$$
$$=(2a-b)\sqrt[3]{b+2a};$$

$$3\sqrt[6]{4a^2}+2\sqrt[3]{2a}=3\sqrt[3]{2a}+2\sqrt[3]{2a}=5\sqrt[3]{2a}.$$

When the radicals are dissimilar, they can only be added or subtracted by means of the signs + or —.

Multiplication and Division.

174. We will first suppose that the radicals have a common index.

Let it be required to multiply or divide $\sqrt[n]{a}$ by $\sqrt[n]{b}$. We have

$$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab},\text{ and }\sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}.$$

For by raising $\sqrt[n]{a}$. $\sqrt[n]{b}$ and $\sqrt[n]{ab}$ to the n^{th} power, we obtain the same result ab; hence the two expressions are equal.

In like manner, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\sqrt[n]{\frac{a}{b}}$ raised to the n^{th} power give $\frac{a}{b}$; hence these two expressions are equal. Therefore we have the following

Rule for the Multiplication or Division of Radicals having a common Index.

Multiply or divide the quantities under the sign by each other, and give to the product, or quotient, the common radical sign. If they have coefficients, first multiply or divide them separately.

Thus,

$$2a\sqrt[3]{\frac{a^2+b^2}{c}}\times-3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}}=-6a^2\sqrt[3]{\frac{(a^2+b^2)^3}{cd}}.$$

or, reducing to its simplest terms,

$$-\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}};$$

$$3a\sqrt[4]{8a^2}\times2b\sqrt[4]{4a^2c}=6ab\sqrt[4]{32a^4c}=12a^2b\sqrt[4]{2c}.$$

$$\frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\frac{a^2-b^2}{8b}}}=\sqrt[3]{\frac{8b(a^2b^2+b^4)}{a^2-b^2}}=2b\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$$

When the radicals have not a common index, they should be reduced to one if possible.

For example, $3a\sqrt[6]{b}\times 5b\sqrt[8]{2c}=15ab\sqrt[24]{8b^4c^3}$.

Examples.

Multiply $\sqrt{2}\times\sqrt[3]{3}$, $\sqrt[4]{\tfrac{1}{2}}\times\sqrt[3]{\tfrac{1}{3}}$.

$2\sqrt[3]{15}$ by $3\sqrt[3]{10}$, $4\sqrt[3]{\tfrac{2}{3}}$ by $2\sqrt{\tfrac{3}{4}}$.

Divide $3\sqrt[3]{4}$ by $7\sqrt{6}$, $2\sqrt{\tfrac{1}{2}}$ by $\tfrac{1}{2}\sqrt[3]{\tfrac{3}{5}}$.

Reduce $\dfrac{2\sqrt{3}\times\sqrt[3]{4}}{\tfrac{1}{2}\sqrt[4]{2}\times\sqrt[3]{3}}$ to its simplest terms.

Reduce $\sqrt{\dfrac{\sqrt{\tfrac{1}{2}}\times 2\sqrt[3]{3}}{4\sqrt[3]{2}\times\sqrt{3}}}$ to its simplest terms.

Formation of Powers, and Extraction of Roots.

175. By raising $\sqrt[m]{a}$ to the n^{th} power, we have........

$(\sqrt[m]{a})^n=\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}\ldots.=\sqrt[m]{a^n}$, by the rule just given for the multiplication of radicals. Hence the rule for raising a radical to any power.

Rule.

Raise the quantity under the sign to the given power, and affect the result with the radical sign, having the primitive index. If it has a coefficient, first raise it to the given power.

Thus $(\sqrt[4]{4a^3})^2=\sqrt[4]{(4a^3)^2}=\sqrt[4]{16a^6}=2a\sqrt[4]{a^2}$;

$(3\sqrt[3]{2a})^5=3^5.\sqrt[3]{(2a)^5}=243\sqrt[3]{32a^5}=46a\sqrt[3]{4a^2}$.

When the index of the radical is a multiple of the power formed under the radical, the result can be reduced.

Thus, in squaring $\sqrt[4]{2a}$, *since* (161) $\sqrt[4]{2a}=\sqrt{\sqrt{2a}}$, *in order to square this last, it is only necessary to suppress the first radi-*

cal sign; therefore we have $(\sqrt[4]{2a})^2=\sqrt{2a}$.

Again, squaring $\sqrt[6]{3b}$, we have $\sqrt[6]{3b}=\sqrt{\sqrt[3]{3b}}$; hence..... $(\sqrt[6]{3b})^2=\sqrt[3]{3b}$.

Consequently, *when the index of the radical is divisible by the exponent of the power, perform this division, leaving the quantity under the radical as it was.*

To extract the root of a radical, *multiply the index of the radical by the index of the root to be extracted, leaving the quantity under the sign as it was.*

Thus, $\sqrt[3]{\sqrt[4]{3c}}=\sqrt[12]{3c}$; $\sqrt{\sqrt[3]{5c}}=\sqrt[6]{5c}$.

This rule is nothing more than the principle of No. 161, enunciated in an inverse order.

When the quantity under the radical is a perfect power of the degree of the root to be extracted, the result can be reduced.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}}$ being equal to $\sqrt[4]{\sqrt[3]{8a^3}}$ it reduces to $\sqrt[4]{2a}$.

In like manner, $\sqrt{\sqrt[5]{9a^2}}=\sqrt[5]{\sqrt{9a^2}}=\sqrt[5]{3a}$.

It is evident that $\sqrt[n]{\sqrt[m]{a}}=\sqrt[m]{\sqrt[n]{a}}$; because both expressions are equal to $\sqrt[mn]{a}$. (161.)

176. The rules just demonstrated for the calculus of radicals, principally depend upon the fact that the n^{th} root of the product of several factors is equal to the product of the n^{th} roots of these factors; and the demonstration of this principle depends upon this: *When the powers (of the same degree) of two expressions are equal, the expressions are also equal.* Now this last proposition, which is true for absolute numbers, is not always true for algebraic expressions.

To prove this, we will show that the same number can have *more than one square root, cube root, fourth root, &c.*

For, denote the general expression of the square root of a by

x, and the *arithmetical* value of it by p; we have the equation $x^2=a$, or $x^2=p^2$, whence $x=\pm p$. Hence we see that the square of p (which is the root of a) will give a, whether its sign be + or —.

In the second place, let x be the general expression of the cube root of a, and p the numerical value of this root; we have the equation

$$a^3=a, \text{ or } x^3=p^3.$$

This equation is satisfied by making $x=p$.

Observing that the equation $x^3=p^3$ can be put under the form $x^3-p^3=0$, and that the expression x^3-p^3 is divisible (31) by $x-p$, which gives the exact quotient, x^2+px+p^2, the above equation can be transformed into

$$(x-p)\,(x^2+px+p^2)=0,$$

which can be verified by supposing $x-p=0$, whence $x=p$; or by supposing

$$x^2+px+p^2=0,$$

from which we have

$$x=-\frac{p}{2}\pm\frac{p}{2}\sqrt{-3}, \text{ or } x=p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

Hence, *the cube root of* a *admits of three different algebraic values, viz.*

$$p,\ p\left(\frac{-1+\sqrt{-3}}{2}\right), \text{ and } p\left(\frac{-1-\sqrt{-3}}{2}\right).$$

Again, resolve the equation $x^4=p^4$, in which p denotes the arithmetical value of $\sqrt[4]{a}$. This equation can be put under the form $x^4-p^4=0$. Now this expression reduces to (19) ...... $(x^2-p^2)\,(x^2+p^2)$. Hence the equation reduces to .... $(x^2-p^2)\,(x^2+p^2)=0$, and can be satisfied by supposing $x^2-p^2=0$, whence $x=\pm p$; or by supposing $x^2+p^2=0$, whence $x=\pm\sqrt{-p^2}=\pm p\sqrt{-1}$.

We therefore obtain four *different algebraic* expressions for the fourth root of a.

For another example, resolve the equation $x^6=p^6$,
which can be put under the form $x^6-p^6=0$.
Now x^6-p^6 reduces to $(x^3-p^3)(x^3+p^3)$,
therefore the equation becomes $(x^3-p^3)\,(x^3+p^3)=0$,

But $x^3-p^3=0$, gives

$$x=p, \text{ and } x=p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

And if in the equation $x^3+p^3=0$, we make $p=-p'$, it becomes $x^3-p'^3=0$, from which we deduce $x=p'$, and

$$x=p'\left(\frac{-1\pm\sqrt{-3}}{2}\right);$$

or, substituting for p' its value, $-p$,

$$x=-p \text{ and } x=-p\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

Therefore the equation $x^6-p^6=0$, and consequently the 6th root of a admits of *six* values, p, αp, $\alpha' p$, $-p$, $-\alpha p$ $-\alpha' p$, by making

$$\alpha=\frac{-1+\sqrt{-3}}{2},\quad \alpha'=\frac{-1-\sqrt{-3}}{2}.$$

We may then conclude from analogy, that every equation of the form $x^m-a=0$, or $x^m-p^m=0$, is susceptible of m different values, that is, the m^{th} root of a number admits of m *different algebraic values:*

177. If in the preceding equations and the results corresponding to them, we suppose as a particular case $a=1$, whence $p=1$, we will obtain the second, third, fourth, &c. roots of unity. Thus $+1$ and -1 are the two *square roots of unity*, because the equation $x^2-1=0$, gives $x=\pm 1$.

In like manner $+1$, $\dfrac{-1+\sqrt{-3}}{2}$, $\dfrac{-1-\sqrt{3}}{2}$, are the three cube roots of unity, or the roots of $x^3-1=0$.

$+1$, -1, $+\sqrt{-1}$, $-\sqrt{-1}$, are the four fourth roots of unity, or the roots of $x^4-1=0$.

178. It results from the preceding analysis, that the rules for the calculus of radicals which are exact when applied to absolute numbers, are susceptible of some *modifications*, when applied to *expressions or symbols which are purely algebraic;* these are more particularly necessary when applied to *imaginary expressions*, and are a consequence of what has been said in No. 176.

For example, the product of $\sqrt{-a}$ by the rule of No. 174, *would be*

$$\sqrt{-a}\times\sqrt{-a}=\sqrt{+a^2}.$$

Now, $\sqrt{a^2}$ is equal to $\pm a$ (No. 176.); there is, then, apparently, an uncertainty as to the sign with which a should be affected. Nevertheless, the true answer is $-a$; for, in order to square $\sqrt{m}$, it is only necessary to suppress the radical; but the $\sqrt{-a}\times\sqrt{-a}$ reduces to $(\sqrt{-a})^2$, and is therefore equal to $-a$.

Again, let it be required to form the product $\sqrt{-a}\times\sqrt{-b}$, by the rule of No. 174, we will have

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{+ab}.$$

Now, $\sqrt{ab}=\pm p$ (No. 176), p being the arithmetical value of the square root of ab; but I say that the true result should be $-p$ or $-\sqrt{ab}$, so long as both the radicals $\sqrt{-a}$ and $\sqrt{-b}$ are considered to be affected with the sign $+$.

For, $\sqrt{-a}=\sqrt{a}.\sqrt{-1}$ and $\sqrt{-b}=\sqrt{b}.\sqrt{-1}$; hence

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{a}.\sqrt{-1}\times\sqrt{-b}\times\sqrt{-1}=\sqrt{ab}(\sqrt{-1})^2$$
$$=\sqrt{ab}\times-1=-\sqrt{ab}.$$

Upon this principle we find the different powers of $\sqrt{-1}$ to be, as follows:

$$\sqrt{-1}=\sqrt{-1},\ (\sqrt{-1})^2=-1,$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2.\sqrt{-1}=-\sqrt{-1},$$
and $$(\sqrt{-1})^4=(\sqrt{-1})^2.(\sqrt{-1})^2=-1\times-1=+1.$$

Again, let it be proposed to determine the product of $\sqrt[4]{-a}$ by the $\sqrt[4]{-b}$ which, from the rule, will be $\sqrt[4]{+ab}$, and consequently will give the four values (No. 176).

$$+\sqrt[4]{ab},\ -\sqrt[4]{ab},\ +\sqrt[4]{ab}.\sqrt{-1},\ -\sqrt[4]{ab}.\sqrt{-1}.$$

To determine the true product, observe that

$$\sqrt[4]{-a}=\sqrt[4]{a}.\sqrt[4]{-1},\ \sqrt[4]{-b}=\sqrt[4]{b}.\sqrt[4]{-1}$$

But $\sqrt[4]{-1}\times\sqrt[4]{-1}=(\sqrt[4]{-1})^2=(\sqrt{\sqrt{-1}})^2=\sqrt{-1}$;

hence $\sqrt[4]{-a}.\sqrt[4]{-b}=\sqrt[4]{ab}.\sqrt{-1}$.

We will apply the preceding calculus to the verification of the expression $\frac{-1+\sqrt{-3}}{2}$, considered as a root of the equation $x^3-1=0$, that is, as the cube root of 1. (See No. 177.)

From the formula $(a+b)^3=a^3+3a^2b+3ab^2+b^3$,

we have $\left(\frac{-1+\sqrt{-3}}{2}\right)^3$

$$=\frac{(-1)^3+3(-1)^2.\sqrt{-3}+3(-1).(\sqrt{-3})^2+(\sqrt{-3})^3}{8}.$$

$$=\frac{-1+3\sqrt{-3}-3\times-3-3\sqrt{-3}}{8}=\frac{8}{8}=1.$$

The second value, $\frac{-1-\sqrt{-3}}{2}$, may be verified in the same manner.

§ IV. *Theory of Exponents. Of Series.*

179. In extracting the n^{th} root of a quantity a^m, we have seen that when m is a multiple of n, we should divide the exponent m by the index of the root n; but when m is not divisible by n, in which case the root cannot be extracted algebraically, it has been agreed to indicate this operation by indicating the division of the two exponents.

Hence $\sqrt[n]{a^m}=a^{\frac{m}{n}}$, from *a convention founded upon the rule for the exponents, in the extraction of the roots of monomials.*

Therefore, $\sqrt[3]{a^2}=a^{\frac{2}{3}}$; $\sqrt[4]{a^7}=a^{\frac{7}{4}}$.

In like manner, suppose it is required to divide a^m by a^n. We know that the exponent of the divisor should be subtracted from the exponent of the dividend, when $m>n$, which gives .... $\frac{a^m}{a^n}=a^{m-n}$. But when $m<n$, in which case the division cannot be effected algebraically, it has been agreed to subtract the exponent of the divisor from that of the dividend. Let p be the absolute difference between n and m; then will $n=m+p$, whence $\frac{a^m}{a^{m+p}}=a^{-p}$; but $\frac{a^m}{a^{m+p}}$ reduces to $\frac{1}{a^p}$; hence $a^{-p}=\frac{1}{a^p}$.

Therefore the *expression* a^{-p} is the symbol of a division *which it has* been impossible to perform; and its true value is the quotient represented by unity divided by the letter a, affected with the exponent p, taken positively. Thus,

$$a^{-3}=\frac{1}{a^3};\ a^{-5}=\frac{1}{a^5}.$$

The notation of fractional exponents has the advantage of giving an entire form to fractional expressions.

From the combination of the extraction of a root, and an impossible division, there results another notation, viz. *negative fractional exponents.*

In extracting the n^{th} root of $\frac{1}{a^m}$, we have first $\frac{1}{a^m}=a^{-m}$, hence $\sqrt[n]{\frac{1}{a^m}}=\sqrt[n]{a^{-m}}=a^{-\frac{m}{n}}$, by substituting the fractional exponent for the radical sign.

Hence $a^{\frac{m}{n}}$, a^{-p}, $a^{-\frac{m}{n}}$, *are conventional expressions, founded upon preceding rules,* and equivalent to $\sqrt[n]{a^m}$, $\frac{1}{a}$, $\sqrt[n]{\frac{1}{a^m}}$.

We may therefore substitute the second for the first, or reciprocally.

As a^p is called a to the p power, when p is a positive whole number, so by analogy, $a^{\frac{m}{n}}$, a^{-p}, $a^{-\frac{m}{n}}$, is called a to the $\frac{m}{n}$ power, a to the $-p$ power, a to the $-\frac{m}{n}$ power, which has induced algebraists to generalize the word *power*; but it would, perhaps, be more convenient to say, a, exponent $-\frac{m}{n}$, exponent $-p$, exponent $-\frac{m}{n}$, using the word *power* only when we wish to designate the product of a number multiplied by itself two or more times. (See No. 2.)

180. *Multiplication of Quantities affected with any Exponents whatever.*

In order to multiply $a^{\frac{3}{5}}$ by $a^{\frac{2}{3}}$, it is only necessary to *add the two exponents*, and we have

$$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=a^{\frac{3}{5}+\frac{2}{3}}=a^{\frac{19}{15}}.$$

For by No. 179, $a^{\frac{3}{5}}=\sqrt[5]{a^3}$; $a^{\frac{2}{3}}=\sqrt[3]{a^2}$;

hence, $$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=\sqrt[5]{a^3}\times\sqrt[3]{a^2};$$

or, performing the multiplication by the rule of No. 174,

$$a^{\frac{3}{5}}\times a^{\frac{2}{3}}=\sqrt[15]{a^{19}}=a^{\frac{19}{15}}.$$

Again, multiplying $a^{-\frac{3}{4}}$ by $a^{\frac{5}{6}}$, we have

$$a^{-\frac{3}{4}}\times a^{\frac{5}{6}}=a^{-\frac{3}{4}+\frac{5}{6}}=a^{-\frac{9}{12}+\frac{10}{12}}=a^{\frac{1}{12}};$$

for $a^{-\frac{3}{4}}=\sqrt[4]{\frac{1}{a^3}}$, $a^{\frac{5}{6}}=\sqrt[6]{a^5}$; hence

$$a^{-\frac{3}{4}}\times a^{\frac{5}{6}}=\sqrt[4]{\frac{1}{a^3}}\times\sqrt[6]{a^5}=\sqrt[12]{\frac{1}{a^9}}\times\sqrt[12]{a^{10}}=\sqrt[12]{\frac{a^{10}}{a^9}}=\sqrt[12]{a}=a^{\frac{1}{12}}.$$

In general, multiplying $a^{-\frac{m}{n}}$ by $a^{\frac{p}{q}}$; we have

$$a^{-\frac{m}{n}}\times a^{\frac{p}{q}}=a^{-\frac{m}{n}+\frac{p}{q}}=a^{\frac{np-mq}{nq}};$$

Therefore, in order to multiply two monomials affected with any exponents whatever, *add together the exponents of the same letters;* this rule is the same as that given in No. 16, for quantities affected with entire exponents.

From this rule we will find that

$$a^{\frac{3}{4}}b^{-\frac{1}{2}}c^{-1}\times a^2b^{\frac{2}{3}}c^{\frac{3}{5}}=a^{\frac{11}{4}}b^{\frac{1}{6}}c^{-\frac{2}{5}};$$

$$3a^{-2}b^{\frac{2}{3}}\times 2a^{-\frac{4}{5}}b^{\frac{1}{2}}c^2=6a^{-\frac{14}{5}}b^{\frac{7}{6}}c^2.$$

Division.

To divide one monomial by another when both are affected with any exponent whatever, follow the rule given in No. 22 for quantities affected with entire and positive exponents; that is, *subtract the exponent of the letters in the divisor from the exponents of the same letters in the dividend.*

For the exponent of each letter in the quotient must be such, that added to that of the same letter in the divisor, the sum will be equal to the exponent of the dividend; hence the exponent of the quotient is equal to the difference between the exponent of the dividend and that of the divisor.

Examples.

$$a^{\frac{2}{3}} : a^{-\frac{3}{4}} = a^{\frac{2}{3}-\left(-\frac{3}{4}\right)} = a^{\frac{17}{12}};$$

$$a^{\frac{3}{4}} : a^{\frac{4}{5}} = a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}};$$

$$a^{\frac{2}{5}} \times b^{\frac{3}{4}} : a^{-\frac{1}{2}} b^{\frac{7}{8}} = a^{\frac{9}{10}} b^{-\frac{1}{8}}.$$

Formation of powers.

To form the n^{th} power of a monomial affected with any exponent whatever, observe the rule given in No. 167, viz. *multiply the exponent of each letter by the exponent* m *of the power;* for, to raise a quantity to the m^{th} power, is the same thing as to multiply it by itself m—1 times; therefore, by the rule for multiplication, the exponent of each letter must be added to itself m—1 times, or multiplied by m.

$$\text{Thus } \left(a^{\frac{3}{4}}\right)^{5} = a^{\frac{15}{4}};\ \left(a^{\frac{2}{3}}\right)^{3} = a^{\frac{6}{3}} = a^{2};$$

$$\left(2a^{-\frac{1}{2}} b^{\frac{3}{4}}\right)^{6} = 64a^{-3} b^{\frac{9}{2}};\ \left(a^{-\frac{5}{6}}\right)^{12} = a^{-10}.$$

Extraction of Roots.

To extract the n^{th} root of a monomial, follow the rule given in No. 167, viz. *divide the exponent of each letter by the exponent of the root.*

For the exponent of each letter in the result should be such, that multiplied by n, the exponent of the root to be extracted, it will reproduce the exponent with which the letter is affected in the proposed monomial; therefore, the exponents in the result must be *respectively* equal to the quotients arising from the division of the exponents in the proposed monomial, by n, the exponent of the root.

Thus $\sqrt[3]{a^{\frac{2}{3}}}=a^{\frac{2}{9}}$; $\sqrt[4]{a^{\frac{8}{11}}}=a^{\frac{2}{11}}$; $\sqrt{a^{-\frac{3}{4}}}=a^{-\frac{3}{8}}$;

$$\sqrt[3]{a^{\frac{3}{5}}b^{-2}}=a^{\frac{1}{5}}b^{-\frac{2}{3}}.$$

The three last rules have been easily deduced from the rule for multiplication; but we might give a direct demonstration for them, by going back to the origin of quantities affected with fractional and negative exponents.

We will terminate this subject by an operation which contains implicitly the demonstration of the two preceding rules.

Let it be required to raise $a^{\frac{m}{n}}$ to the $-\frac{r}{s}$ power;

I say that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}}=a^{\frac{m}{n}\times-\frac{r}{s}}=a^{-\frac{mr}{ns}}.$$

For by going back to the origin of these notations, we find that

$$\left(a^{\frac{m}{n}}\right)^{-\frac{r}{s}}=\sqrt[s]{\frac{1}{\left(a^{\frac{m}{n}}\right)^r}}=\sqrt[s]{\frac{1}{(\sqrt[n]{a^m})^r}}=\sqrt[s]{\frac{1}{\sqrt[n]{a^{mr}}}}$$

$$=\sqrt[s]{\sqrt[n]{\frac{1}{a^{mr}}}}=\sqrt[ns]{a^{-mr}}=a^{-\frac{mr}{ns}}$$

The advantage derived from the use of exponents consists principally in this: The operations performed upon expressions of this kind require no other rules than those established for the calculus of quantities affected with entire exponents. Besides, this calculus is reduced to simple operations upon fractions, with which we are already familiar.

181. *Remark.* In the resolution of certain questions, we will be led to consider quantities affected with *incommensurable exponents.* Now, it would seem that the rules just established for commensurable exponents, ought to be demonstrated for the

case in which the exponents were incommensurable; but we will observe, that an incommensurable, such as $\sqrt{3}$, $\sqrt[3]{11}$, is by its nature composed of an entire part, and a fraction which cannot be expressed exactly, but *to which it is possible to approximate as near as we please*, so that we may always conceive the incommensurable to be replaced by an exact fraction, which only differs from it by a quantity less than any given quantity; and in applying the rules to the symbol which designates the incommensurable, it is necessary to understand that we apply it to the exact fraction which represents it approximately.

Examples.

Reduce $\dfrac{2\sqrt{2}\times(3)^{\frac{1}{3}}}{\frac{1}{2}\sqrt{2}}$ to its simplest terms.

Reduce $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}\,(3)^{\frac{1}{2}}}\right\}^4$ to its simplest terms.

Reduce $\sqrt{\left\{\dfrac{(\frac{1}{2})^3+\sqrt{3}.\frac{1}{2}}{2\sqrt{2}.(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$ to its simplest terms.

Demonstration of the Binomial Theorem in the case of any Exponent whatever.

182. Since the rules for the calculus of entire and positive exponents may be extended to the case of any exponent whatever, it is natural to suppose that the binomial formula, which serves to develope the m^{th} power of a binomial when m is entire and positive, will also effect this when m is any exponent whatever. In fact, analysts have discovered that this is the case, and they have deduced important consequences from it, *both for the extraction of roots by approximation, and the development of algebraic expressions into series.*

The following is a modification of Euler's demonstration.

We will remark, in the first place, that the binomial $x+a$ can be put under the form $x\left(1+\dfrac{a}{x}\right)$; whence there results

$$(x+a)^m=x^m\left(1+\frac{a}{x}\right)^m=x^m(1+z)^m, \text{ by making } \frac{a}{x}=z.$$

Therefore, if the formula

$$(1+z)^m=1+mz+m\frac{m-1}{2}z^2+m.\frac{m-1}{2}.\frac{m-2}{2}z^3+\&c. \quad (A)$$

is proved to be correct for any value of m, we may consider the formula

$$(x+a)^m=x^m+max^{m-1}+m.\frac{m-1}{2}a^2x^{m-2}$$
$$+m.\frac{m-1}{2}.\frac{m-2}{3}a^3x^{m-3}+\&c. (B)$$

exact for any value of m. For by substituting $\frac{a}{x}$ for z in the formula (A), and multiplying by x^m, we obtain

$$(x+a)^m=x^m\left(1+m\frac{a}{x}+m.\frac{m-1}{2}.\frac{a^2}{x^2}+\&c.\right),$$

from which, by performing the operations indicated, we obtain the formula (B).

Now, when m is a whole number, we have

$$(1+z)^m=1+mz+m.\frac{m-1}{2}z^2+m.\frac{m-1}{2}.\frac{m-2}{3}z^3+\&c.$$

but m being a fraction $\frac{p}{q}$, we do not know from what algebraic expression the developement

$1+mz+m\frac{m-1}{2}z^2+m.\frac{m-1}{2}.\frac{m-2}{3}z^3+$ &c. . . . is derived.

Denoting this unknown expression by y; we have the equation

$$y=1+mz+m.\frac{m-1}{2}z^2+m.\frac{m-1}{2}.\frac{m-2}{3}z^3+\ \&c. \ldots . (1).$$

If m' is another fractional exponent, we will have in like manner,

$$y'=1+m'z+m'.\frac{m'+1}{2}.z^2+m'.\frac{m'-1}{2}\times\frac{m'-2}{3}z^3+\&c. \ldots . (2).$$

Multiplying the equalities (1) and (2) member by member, we will have for the first member of the result yy'. As to the second, it would be very difficult to obtain the true form of it, *by the common* rule for the multiplication of polynomials; but by observing that *the form of a product does not depend upon the particular values of the letters which enter in the two factors of this product*, (No. 20) we see that the above product will be of the same form as in the case where m and m' are positive whole numbers. Now in this case we have

$$1+mz+m.\frac{m-1}{2}z^2+\ldots=(1+z)^m,$$

$$1+m'z+m'.\frac{m'-1}{2}z^2+\ldots=(1+z)^{m'}.$$

whence

$$\left(1+mz+m.\frac{m-1}{2}z^2+\ldots\right)\left(1+m'z+m'.\frac{m'-1}{2}z^2+\ldots.\right)$$

$$=(1+z)^{m+m'}=1+(m+m')z+(m-m')\frac{m+m'-1}{2}z^2+\ldots;$$

Therefore this form is true in the case in which m and m' are any quantities whatever, and we have

$$yy'=1+(m+m')z+(m+m')\frac{m+m'-1}{2}.z^2+\ldots (3);$$

Let m'' be a third positive fractional exponent, we will have

$$y''=1+m''z+m''\frac{m''-1}{2}z^2+\ldots$$

Multiplying the two last equations member by member, we will have

$$yy'y''=1+(m+m'+m'')z+(m+m'+m'')\frac{m+m'+m''-1}{2}z^2+\ldots$$

In general, let q denote the number of exponents m, m', m'', m''' ...., q being the denominator of the fraction $\frac{p}{q}$ which we suppose equal to m; we will have, by making r equal to the sum of the exponents $m+m'+m''+m'''+\ldots$

$$yy'y''y'''=1+rz+r.\frac{r-1}{2}z^2+r.\frac{r-1}{2}.\frac{r-2}{3}z^3+\ldots. (4).$$

And by supposing $m=m'=m''=m'''$ .... in which case

$$r=m+m+m+\ldots=mq,$$

the equation (4) becomes

$$y^q=1+mq.z-mq.\frac{mq-1}{2}z^2+mq.\frac{mq-1}{2}.\frac{mq-2}{3}z^3+\ldots$$

Now we have by hypothesis, $m=\frac{p}{q}$, or $mq=p$;

$$\text{hence } y^q=1+pz+p.\frac{p-1}{2}z^2+p.\frac{p-1}{2}.\frac{p-2}{3}z^3+\ldots$$

but p is a whole number, therefore the second member of this equation is the development of $(1+z)^p$, which gives $y^q=(1+z)^p$,

whence $y=(1+z)^{\frac{p}{q}}=(1+z)$;^{m} Consequently

$$(1+z)^m=1+mz+m\frac{m-1}{2}z^2+m\frac{m-1}{2}.\frac{m-2}{3}z^3+\ldots$$

m being any positive fraction.

To demonstrate this formula, for the case in which m is a negative fraction or whole number, it is only necessary to suppose, $m'=-m$, in the equation (3) obtained from the equations (1) and (2), for when $m+m'=0$, the equation (3) reduces to $yy'=1$; whence $y=\frac{1}{y'}$.

But since m is negative by hypothesis, m' or $-m$, must be positive, and we have

$$y'=(1+z)^{m'}, \text{ hence } y=\frac{1}{(1+z)^{m'}}=(1+z)^{-m'}=(1+z)^m,$$

and consequently

$$(1+z)^m=1+mz+m\frac{m-1}{2}.z^2+\ldots$$

183. *Application of the binomial theorem to the extraction of roots by approximation.*

If in the formula $(x+a)^m=$

$$x^m\left(1+m.\frac{a}{x}+m.\frac{m-1}{2}.\frac{a^2}{x^2}+m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{a^3}{x^3}+\ldots\right)$$

we make $m=\frac{1}{n}$, it becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a}=$

$$x^{\frac{1}{n}}\left(1+\frac{1}{n}.\frac{a}{x}+\frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{a^2}{x^2}+\frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{\frac{1}{n}-2}{3}.a^3+\ldots\right)$$

or, reducing, $\sqrt[n]{x+a}=$

$$x^{\frac{1}{n}}\left(1+\frac{1}{n}.\frac{a}{x}-\frac{1}{n}.\frac{n-1}{2n}.\frac{a^2}{x^2}+\frac{1}{n}.\frac{n-1}{2n}.\frac{2n-1}{3n}.\frac{a^3}{x^3}+\ldots\right)$$

The fifth term can be found by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, then changing the sign of the result, and so on.

Extract the cube root of 31. The greatest cube contained in 31 being 27, suppose $n=3$, $x=27$, and $a=4$, and substitute these values in the above formula, we will have

$$\sqrt[3]{31}=\sqrt[3]{27+4}=27^{\frac{1}{3}}\left(1+\frac{4}{27}\right)^{\frac{1}{3}}$$

$$=3\left(1+\frac{1}{3}\cdot\frac{4}{27}-\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{16}{729}+\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{5}{9}\cdot\frac{64}{19683}-\&c.\right)$$

or $$\sqrt[3]{31}=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}\cdots\cdots\cdots$$

From what has been said above, the following term can be obtained by multiplying $\frac{320}{531441}$ by $\frac{3n-1}{4n}\cdot\frac{a}{x}$, or $\frac{2}{3}\cdot\frac{4}{27}$, and changing the sign, which gives $-\frac{2560}{43046721}$.

In the same way, we will find the term which follows this last to be

$$+\frac{250}{43046721}\times\frac{4n-1}{5n}\cdot\frac{a}{x}=\frac{2560}{43046721}\times\frac{11}{15}\times\frac{4}{27}=\frac{112640}{17433922005},$$

and so on, for any number of terms.

By considering only the five first terms, and reducing them to decimals, we obtain for the sum of the additive terms,

$$\left.\begin{array}{r}3=3,00000\\ \frac{4}{27}=0,14815\\ \frac{320}{531441}=0,00060\end{array}\right\}=3,14875,$$

and for the sum of the subtractive terms,

$$\left.\begin{array}{r}-\frac{16}{2187}=-0,00731\\ -\frac{2560}{43046721}=-0,00006\end{array}\right\}=-0,00737.$$

Hence, $\sqrt[3]{31}=3,14138.$

It will be shown presently that this result is exact to within 0,00001.

184. ***Remark.*** When the expression of a number is developed in a series, the terms of which go on decreasing, (in which case it is called a *decreasing* or *converging* series), the greater number of terms we take in the series, the nearer will we approximate to the true value of the proposed number. When the terms of the series are *alternately positive and negative*, we can, by taking a given number of terms, determine the *degree of approximation.*

For, let $a-b+c-d+e-f, +\ldots$, &c. be a decreasing series; $b, c, d \ldots$ being absolute quantities, and let x denote the number represented by this series.

The numerical value of x is contained between any two consecutive sums of the terms of the series. For take any two consecutive sums,

$$a-b+c-d+e-f, \text{ and } a-b+c-d+e-f+g.$$

In the first, the terms which follow $-f$, are $\overline{g-h}, +\overline{k-l}+\ldots$ but since the series is decreasing, the partial differences $g-h$, $k-l, \ldots$ are positive numbers; therefore, in order to obtain the complete value of x, a certain absolute number must be added to the sum $a-b+c-d+e-f$. Hence we have

$$a-b+c-d+e-f<x.$$

In the second series, the terms which follow $+g$ are $-\overline{h+k}$, $-\overline{l+m}\ldots$ Now, the partial differences $-h+k$, $-l+m\ldots$, are negative; therefore, in order to obtain the sum of....... $a-b+c-d+e-f+g$, a negative quantity must be added to it, or, in other words, it is necessary to diminish it. Consequently

$$a-b+c-d+e-f+g>x.$$

Therefore x *is comprehended between these two sums.*

Since the numerical value of the difference between these two sums is evidently less than g, it follows that the *error committed by taking a certain number of terms,* a—b+c—d+e—f, *for the value of* x, *is numerically less than the following term.*

Therefore, in the example of the preceding number, all of the terms after the first being positive and negative alternately, we may conclude that the sum of the first five terms,

$$3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\frac{2560}{43046721},$$

differ from $\sqrt[3]{31}$ by a quantity less than the 6th term, which has been found (No. 183) to be $\frac{112640}{17433922005}$; but we see by inspection that this fraction is less than $\frac{1}{100000}$; hence . . . . .

$\sqrt[3]{31}=3{,}14138$ to within 0,00001.

185. The process for approximating to the n^{th} root of a number N by series, *consists in decomposing it into two parts,* p^n+q, (p *being the root of* N *to within unity*), *and making* $x=p^n$, $a=q$, *in the development of* $\sqrt[n]{x+a}$. (No. 183.) *Then performing the operations indicated upon all the terms which precede that which is less than a unit of the decimal, within which it is required to approximate; then convert these terms into decimals, and reduce the additive and subtractive terms.*

This method is not very advantageous, except when $\frac{q}{p^n}$ is a very small fraction, for if it was not, the terms of the series would not diminish very rapidly, and a great many of them would have to be taken, in order to obtain the required degree of approximation, which would render the calculation very laborious.

There is a case in which it is necessary to modify the above rule, viz. when $p^n<q$; for then $\frac{a}{x}$ or $\frac{q}{p^n}$ is greater than unity, as well as all the powers of $\frac{a}{x}$ which go on increasing, numerically, in proportion as the degree of the power increases.

For example, suppose it is required to extract the cube root of 56; the greatest cube contained in 56, being 27; we have,

$$x=27,\ a=29;\ \text{whence}\ \frac{a}{x}=\frac{29}{27},$$

and the terms of the series will go on increasing instead of diminishing. But $56=64-8=4^3-8$; now $\frac{8}{64}$ or $\frac{1}{8}$ is a small fraction; and if in the expression for $\sqrt[n]{x+a}$ (No. 183), we put $-a$ in the place of a, it becomes

$$\sqrt[n]{x+a}=x^{\frac{1}{n}}\left(1-\frac{1}{n}\cdot\frac{a}{x}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{a^2}{x^2}-\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3}-\dots\right)$$

then making $x=64$, $a=8$, we will obtain a *converging series.*

In this series all the terms excepting the first are negative, and we cannot apply to it what has been said (184) upon the manner of fixing the degree of approximation for the sum of a certain number of terms.

Examples.

$\sqrt[5]{39} = \sqrt[5]{32+7} = 2{,}0807$ to within 0,0001;

$\sqrt[3]{65} = \sqrt[3]{64+1} = 4{,}02073$ to within, 0,00001;

$\sqrt[4]{260} = \sqrt[4]{256+4} = 4{,}01553$ to within 0,00001;

$\sqrt[7]{108} = \sqrt[7]{128-20} = 1{,}95204$ to within 0,00001.

186. The binomial formula also serves to develope algebraic expressions into series.

Take for example, the expression $\frac{1}{1-z}$, we have

$$\frac{1}{1-z}=(1-z)^{-1}.$$

In the binomial formula, make $m=-1$, $x=1$, and $a=-z$, it becomes

$$(1-z)^{-1}=1-1.(-z)-1.\frac{-1-1}{2}.(-z)$$
$$-1.\frac{-1-1}{2}.\frac{-1-2}{3}.(-z)^3-\dots$$

or, performing the operations, and observing that each term is composed of an even number of factors affected with the sign $-$,

$$(1-z)^{-1}=\frac{1}{1-z}=1+z+z^2+z^3+z^4+z^5+\dots.$$

The same result will be obtained by applying the rule for division (No. 26.)

	1	$1-z$
1st. remainder .	$+z$	$1+z+z^2+z^3+z^4+\ldots$
2d.	$+z^2$	
3d.	$+z^3$	
4th.	$+z^4$	
	$+\ldots$	

Again, take the expression $\frac{2}{(1-z)^3}$, or $2(1-z)^{-3}$.

We have $2(1-z)^{-3}=$

$$2[1-3.(-z)-3.\frac{-3-1}{2}.(-z)^2-3.\frac{-3-1}{2}.\frac{-3-2}{3}.(-z)^3-.]$$

or $\quad 2(1-z)^{-3}=2(1+3z+6z^2+10z^3+15z^4+\ldots.)$

To develope the expression $\sqrt[3]{2z-z^2}$ which reduces to . . . . $\sqrt[3]{2z}\left(1-\frac{z}{2}\right)^{\frac{1}{3}}$, we first find

$$\left(1-\frac{z}{2}\right)^{\frac{1}{3}}=1+\frac{1}{3}\left(-\frac{z}{2}\right)+\frac{1}{3}.\frac{\frac{1}{3}-1}{2}.\left(-\frac{z}{2}\right)^2+\ldots\ldots\ldots=$$

$$1-\frac{1}{6}z-\frac{1}{36}z^2-\frac{5}{648}z^3-\ldots;$$

hence $\sqrt[3]{2z-z^2}=\sqrt[3]{2z}\left(1-\frac{1}{6}z-\frac{1}{36}z^2-\frac{5}{648}z^3-\text{\&c.}\right)$

§ V. *Method of Indeterminate Coefficients. Recurring Series.*

187. Algebraists have invented another method of developing algebraic expressions into series, which is in general, more simple than those we have just considered, and more extensive in its applications, as it can be applied to algebraic expressions of any nature whatever.

In order to give some idea of this method, we will suppose it is required to develope the expression $\frac{a}{a'+b'x}$ into a series arranged according to the ascending powers of x. It is visible that the expression can be developed; for $\frac{a}{a'+b'x}$ reduces to $a(a'+b'x)^{-1}$; and by applying the binomial formula to it, we would evidently obtain a series of terms arranged according *to the ascending powers of x.* We may therefore assume

$$\frac{a}{a'+b'x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ \ldots\ldots (1)$$

the coefficients $A, B, C, D, \ldots$ being functions of a, a', b', but independent of x, it is required to determine these coefficients, which are called *indeterminate coefficients.*

For this purpose, multiply both members of the equation (1) by $a'+b'x$; arranging the result with reference to the powers of x, and transposing a, it becomes

$$0=\left\{\begin{array}{c|c|c|c|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ldots (2). \\ -a+Ab' & +Bb' & +Cb' & +Db' & \end{array}\right.$$

Now if the values of $A, B, C, D, \ldots$ were determined, the equation (1) would be verified by any value given to x; this must therefore be the case also in the equation (2).

But by supposing $x=0$, this equation becomes, $\quad 0=Aa'-a$;

Whence $\quad A=\frac{a}{a'}$;

A being equal to $\frac{a}{a'}$, when $x=0$, this must be the value of it when x is any quantity whatever, since A is independent of x by hypothesis; therefore whatever may be the value of x, the equation (2) reduces to

$$0=\left\{\begin{array}{c|c|c|l} Ba' & x+Ca' & x^2+Da' & x^3+\ldots; \\ +Ab' & +Bb' & +Cb' & \end{array}\right. \text{ or, dividing by } x,$$

$$0=\left\{\begin{array}{cc|c|l} Ba' & x+Ca' & x^2+Da' & x^2+\ \ldots\ldots (3) \\ +Ab' & +Bb' & +Cb' & \end{array}\right.$$

This equation being also satisfied by any value for x, by making $x=0$, it becomes $\quad Ba'+Ab'=0.$

Whence $B=-\frac{Ab'}{a'}$, or $B=\frac{a}{a'}\times-\frac{b'}{a'}=\frac{ab'}{a'^2}$.

As this must be the value of B whatever may be that of x, we will suppress the first term $Ba'+Ab'$ [of the equation (3),] which this value of B makes equal to zero, and divide by x; it thus becomes

$$0=\left\{\begin{array}{c|c|l} Ca'+Da' & x+Ea' & x^2+\ \ldots\ldots \\ +Bb'+Cb' & +Db' & \end{array}\right.$$

Making $x=0$, there results $\quad Ca'+Bb'=0.$

Whence $C=-\frac{Bb'}{a'}$, or $C=-\frac{ab'}{a^2}\times-\frac{b'}{a}=\frac{ab'^2}{a'^3}$.

In the same way we would find $\quad Da'+Cb'=0,$

Whence $D=-\frac{Cb'}{a'}$ or $D=\frac{ab'^2}{a'^3}\times-\frac{b'}{a'}=-\frac{ab'^3}{a'^4}$; and so on.

It is easily perceived that any coefficient is formed from that which precedes it, by multiplying by $-\frac{b'}{a'}$; therefore we have,

$$\frac{a}{a'+b'x}=\frac{a}{a'}-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2-\frac{ab'^3}{a'^4}x^3+\frac{ab'^4}{a'^5}-\dots$$

188. By reflecting upon the preceding reasoning, we perceive, that the fundamental principle of the method of indeterminate coefficients, depends upon this, viz., *when an equation of the form* $0=M+Nx+Px^2+Qx^3+\dots(M, N, P, Q,\dots$ being independent of x), *is verified by any value of* x *whatever, each of the coefficients must necessarily be equal to* 0.

For since these coefficients are independent of x, when they are determined by any particular hypothesis made with respect to x, the values must answer for any value of x whatever. Now, making $x=0$, we find $M=0$, and dividing the equation by x, it reduces to

$$0=N+Px+Qx^2+\dots$$

making $x=0$ in this equation, it becomes $N=0$, and dividing the equation by x, it reduces to $0=P+Qx+\dots$ and so on. Hence we have

$$M=0,\ N=0,\ P=0,\ Q=0\ \dots;$$

in this manner we obtain as many equations as there are coefficients to be determined.

This principle may be enunciated in another manner, viz.

When an equation of the form

$$a+bx+cx^2+dx^3+\dots=a'+b'x+c'x^2+d'x^3+\dots$$

is satisfied by whatever value we give to x, the terms involving the same powers in the two members are respectively equal; for, by transposing all the terms into the second member, the equation will take the form $0=M+Px+Qx^2+\dots$, whence

$$a'-a=0,\ b'-b=0,\ c'-c=0\ \dots\dots,$$

and consequently,

$$a'=a,\ b'=b,\ c'=c,\ d'=d\ \dots\dots,$$

Every *equation in which* the terms are arranged with reference *to a certain letter,* and which is satisfied by any value which can be given to this letter, is called an *identical equation*, in order to distinguish it from a *common equation*, that is, an equation which can only be satisfied by giving particular values to this letter. (42.)

189. The method of *indeterminate coefficients* requires that we should know, *à priori*, the form of the development with reference to the exponents of x. The development is generally supposed to be arranged according to the ascending powers of x, commencing with the power x^0; sometimes, however, this form is not exact; in this case, the calculus detects the error in the supposition.

For example, develope the expression $\frac{1}{3x-x^2}$.

Suppose that $\frac{1}{3x-x^2}=A+Bx+Cx^2+Dx^3+\ldots$,

whence, by clearing the fraction, and arranging the terms,

$$0=-1+3Ax+3B \left| x^2+3C \right| x^3+3D \left| x^4+\ldots, \right.$$
$$\quad\quad\quad\quad\; -A \quad\;\; -B \quad\;\; -C$$

whence, (No. 188),

$$-1=0,\ 3A=0,\ 3B-A=0\ldots.$$

Now the first equation, $-1=0$, is absurd, and indicates that the above form is not a suitable one for the expression $\frac{1}{3x-x^2}$; but if we put this expression under the form $\frac{1}{x}\times\frac{1}{3-x}$, and suppose that

$$\frac{1}{x}\times\frac{1}{3-x}=\frac{1}{x}\,(A+Bx+Cx^2+Dx^3+\ldots),$$

it will become, after the reductions are made,

$$0=\left\{\begin{matrix}3A+3B\\-1\ -A\end{matrix}\right. \left| \begin{matrix}x+3C\\-B\end{matrix} \right| \begin{matrix}x^2+3D\\-C\end{matrix} \left| x^3+\ldots, \right.$$

which gives the equations

$$3A-1=0,\ 3B-A=0,\ 3C-B=0\ldots,$$

whence

$$A=\frac{1}{3},\ B=\frac{1}{9},\ C=\frac{1}{27},\ D=\frac{1}{81}\ldots$$

Therefore, $\dfrac{1}{3x-x^2}=\dfrac{1}{x}\left(\dfrac{1}{3}+\dfrac{1}{9}x+\dfrac{1}{27}x^2+\dfrac{1}{81}x^3+\ldots\right),$

or $\qquad=\dfrac{1}{3}x^{-1}+\dfrac{1}{9}x^0+\dfrac{1}{27}x+\dfrac{1}{81}x^2+\ldots;$

that is, the development contains a term affected with a negative exponent.

190. *Demonstration of the Binomial Formula, by the method of indeterminate coefficients.*

Since $(x+a)^m$ can be put under the form $x^m\left(1+\dfrac{a}{x}\right)^m$, or $x^m(1+y)^m$, by making $y=\dfrac{a}{x}$, it is only necessary to develope $(1+y)^m$, m being any quantity whatever.

First, let m be equal to a positive number $\dfrac{p}{q}$. Assume

$$(1+y)^{\frac{p}{q}}=1+Ay+By^2+Cy^3+Dy^4+\text{----}\ (1).$$

[We are led to assume this form for the development, from the formation of the first entire powers, and by observing that when $y=0$, the first member reduces to 1, from which it follows that the part of the second member which is independent of y, must be equal to 1.]

To determine the coefficients A. B, C, D ...., substitute z for y in the equation (1); it becomes

$$(1+z)^{\frac{p}{q}}=1+Az+Bz^2+Cz^3+Dz^4+\text{-----}(2).$$

It is evident that the values of A, B, C .... are the same in this equation as in the equation (1), since they are independent of any value given to y.

Subtracting these two equations, member from member, we obtain

$$(1+y)^{\frac{p}{q}}-(1+z)^{\frac{p}{q}}=A(y-z)+B(y^2-z^2)+C(y^3-z^3)$$
$$+D(y^4-z^4)+\text{-----}(3).$$

Making $(1+y)^{\frac{1}{q}}=u$, and $(1+z)^{\frac{1}{q}}=v$, there will result $1+y=u^q$, $1+z=v^q$; whence $y-z=u^q-v^q$, and the equation (3) becomes

$$u^p-v^p=A(y-z)+B(y^2-z^2)+C(y^3-z^3)+D(y^4-z^4)+\ldots(4).$$

or, dividing the first member by u^q-v^q, and the second by its equal $y-z$, we have

$$\frac{u^p-v^p}{u^q-v^q}=\frac{A(y-z)+B(y^2-z^2)+C(y^3-z^3)+D(y^4-z^4)+\ldots}{y-z}$$

Now u^p-v^p is divisible by $u-v$, (No. 31), and gives the quotient,

$$u^{p-1}+vu^{p-2}+v^2u^{p-3}+\ldots\ldots+v^{p-1}.$$

In like manner, $(u^q-v^q):(u-v)$ gives

$$u^{q-1}+vu^{q-2}+v^2u^{q-3}+\ldots\ldots+v^{q-1}$$

Moreover, $y-z$, y^2-z^2, y^3-z^3, $y^4-z^4\ldots\ldots$ divided by $y-z$, give the quotients

$$1, y+z, y^2+yz+z^2, y^3+yz^2+y^2z+z^3\ldots\ldots;$$

therefore, the equation (4) becomes

$$\frac{u^{p-1}+vu^{p-2}+v^2u^{p-3}+\ldots\ldots+v^{p-1}}{u^{q-1}+vu^{q-2}+v^2u^{q-3}+\ldots\ldots+v^{q-1}}=A+B(y+z)$$
$$+C(y^2+yz+z^2)+D(y^3+yz^2+y^2z+z^3)+\ldots$$

Making $y=z$ in this last equation, whence $u=v$ (since $(1+y)^{\frac{1}{q}}=u$, and $(1-z)^{\frac{1}{q}}=v$), the first member reduces to $\frac{p.u^{p-1}}{q.u^{q-1}}$, or $\frac{p}{q}.\frac{u^p}{u^q}$; and by putting in the place of u^p its value $(1+y)^{\frac{p}{q}}$, or $1+Ay+By^2+Cy^3+\ldots\ldots$, and in place of u^q, its value $1+y$, this first member becomes

$$\frac{p}{q}.\frac{1+Ay+By^2+Cy^3+\ldots\ldots}{1+y}.$$

Moreover, the second member reduces to

$$A+2By+3Cy^2+4Dy^3+\ldots\ldots;$$

hence we have the equation

$$\frac{p}{q}.\frac{1+Ay+By^2+Cy^3+Dy^4+\ldots\ldots}{1+y}=A+2By+3Cy^2$$
$$+4Dy^3+\ldots$$

Clearing the fraction, and performing the operations indicated,

$$\frac{p}{q}+\frac{p}{q}\cdot Ay+\frac{p}{q}\cdot By^2+\frac{p}{q}\cdot Cy^3+\frac{p}{q}\cdot Dy^4+\ldots.$$

$$\begin{array}{r|r|r|r|l} A+2B & y+3C & y^2+4D & y^3+5E & y^4+\ldots. \\ +\ A & +2B & +3C & +4D & \end{array}$$

Comparing the terms of the two members of this *identical equation*, we obtain the following equations:

$$\frac{p}{q}=A, \text{ whence } A=\frac{p}{q}.$$

$$\frac{p}{q}.A=2B+A,\ 2B=A\left(\frac{p}{q}-1\right);\ \text{hence}\ B=\frac{A\left(\frac{p}{q}-1\right)}{2},$$

$$\frac{p}{q}.B=3C+2B,\ 3C=B\left(\frac{p}{q}-2\right);\ \text{hence}\ C=\frac{B\left(\frac{p}{q}-2\right)}{3},$$

$$\frac{p}{q}.C=4D+3C,\ 4D=C\left(\frac{p}{q}-3\right);\ \text{hence}\ D=\frac{C\left(\frac{p}{q}-3\right)}{4},$$

and so on for the rest of the coefficients.

The law for the formation of the coefficients from each other, is manifest. Let N be that coefficient which has n coefficients before it, and M that which immediately precedes it. We evidently have

$$\frac{p}{q}M=nN+(n-1)M;\ \text{whence}\ N=\frac{M\left(\frac{p}{q}-n+1\right)}{n},$$

By examining the preceding demonstration, it will be perceived that it applies to the case in which $q=1$; that is, in which the exponent is a whole number.

As to the case in which m is equal to a negative fraction, $-\frac{p}{q}$, we proceed exactly in the same manner as above until we obtain the equation which corresponds to equation (4), viz.

$$u^{-p}-v^{-p}=A(y-z)+B(y^2-z^2)+C(y^3-z^3)+\ldots.$$

We then observe, that

$$u^{-p}-v^{-p}=\frac{1}{u^p}-\frac{1}{v^p}=\frac{v^p-u^p}{u^pv^p}=-\frac{u^p-v^p}{u^pv^p}.$$

Therefore, by dividing the first member by u^q-v^q, and the second by its equal $y-z$, we have

$$-\frac{1}{u^pv^p}\cdot\frac{u^p-v^p}{u^q-v^q}=\frac{A(y-z)+B(y^2-z^2)+\ldots.}{y-z};$$

or suppressing the factors $u-v$, and $y-z$,

$$-\frac{1}{u^pv^p}\cdot\frac{u^{p-1}+vu^{p-2}+\ldots.v^{p-1}}{u^{q-1}+vu^{q-2}+\ldots.v^{q-1}}=A+B(y+z)+\ldots..$$

then making $y=z$, whence $u=v$, it becomes

$$-\frac{1}{u^{2p}}\cdot\frac{pu^{p-1}}{qu^{q-1}}=-\frac{p}{q}\cdot\frac{u^{-p}}{u^q}=A+2By+3Cy^2+\ldots$$

The remainder of the calculus is exactly similar to that of the preceding case.

191. *Recurring Series.* The development of algebraic fractions by the method of indeterminate coefficients, gives rise to certain series, called *recurring series.*

It has been shown in No. 187, that the development of the expression $\frac{a}{a'+b'x}$ is the series $\frac{a}{a'}$, $-\frac{ab'}{a'^2}x+\frac{ab'^2}{a'^3}x^2+\frac{ab'^3}{a'^4}+\ldots$, in which each term is formed by multiplying that which precedes it by $-\frac{b'}{a}$.

This property is not peculiar to the proposed fraction; it belongs to all rational algebraic fractions, and it consists in this, viz.: *Every rational fraction involving* x, *when developed, gives a series of terms, each of which is equal to the algebraic sum of a certain number of preceding terms, multiplied respectively by certain constant quantities, which are the same for any term of the series.*

The collection of constant quantities, by which a certain number of the preceding terms should be multiplied, is called the *scale* of the series.

In the preceding series, the *scale* is $-\frac{b'}{a}x$, and the series is *called a recurring series of the first order.*

Let it be required to develope the expression . . . .
$\frac{a+bx+cx^2}{a'+b'x+c'x^2+d'x^3}$, into a series.

Assume $\frac{a+bx+cx^2}{a'+b'x+c'x^2+d'x^3}=A+Bx+Cx^2+Dx^3+Ex^4+..$

Clearing the fraction, and transposing, we have,

$$0=\left\{\begin{array}{c|c|c|c|l} Aa'+Ba' & x+Ca' & x^2+Da' & x^3+Ea' & x^4+\ldots \\ -a+Ab' & +Bb' & +Cb' & +Db' & \\ -b & +Ac' & +Bc' & +Cc' & \\ & -c & +Ad' & +Bd' & \end{array}\right.$$

which gives the equations

$$Aa'-a=0, \text{ or } A=\frac{a}{a'}.$$

$$Ba'+Ab'-b=0, \qquad B=-\frac{b'}{a'}A+\frac{1}{a'}b=\frac{-ab'+ba'}{a'^2}.$$

$$Ca'+Bb'+Ac'-c=0, \; C=-\frac{b'}{a'}B-\frac{c'}{a'}A+\frac{1}{a'}c.$$

or $$C=\frac{ab'^2-ba'b'-aa'c'-ca'^2}{a'^3}.$$

$$Da'+Cb'+Bc'+Ad'=0, \; D=-\frac{b'}{a'}C-\frac{c'}{a'}B-\frac{d'}{a'}A,$$

$$Ea'+Db'+Cc'+Bd'=0, \; E=-\frac{b'}{a'}D-\frac{c'}{a'}C-\frac{d'}{a'}B,$$

. . . .

. . . .

Whence we perceive that the three first coefficients, are not obtained by any law; but commencing at the fourth, each coefficient is formed from the sum of the three which precede it, by multiplying them respectively by $-\frac{b'}{a'}$, $-\frac{c'}{a'}$, $-\frac{d'}{a'}$, viz. that which immediately precedes the required coefficient by $-\frac{b'}{a'}$, that which precedes it two terms by $-\frac{c'}{a'}$, and that which precedes it three terms by $-\frac{d'}{a'}$; thus the coefficients A, B, C, D... form a recurring series, the *scale* of which is composed of

$$\left(-\frac{b'}{a'}, -\frac{c'}{a'}, -\frac{d'}{a'}\right).$$

From this law for the formation of the coefficients it follows that the fourth term of the series, Dx^3 is equal to

$$-\frac{b'}{a'}Cx^3 - \frac{c'}{a'}Bx^3 - \frac{d'}{a'}Ax^3,$$

$$\text{or } -\frac{b'}{a'}x.Cx^2 - \frac{c'}{a'}x^2.Bx - \frac{d'}{a'}x^3.A.$$

In like manner, we have for the term Ex^4,

$$-\frac{b'}{a'}Dx^4 - \frac{c'}{a'}Cx^4 - \frac{d'}{a'}Bx^4,$$

$$\text{or } -\frac{b'}{a'}x.Dx^3 - \frac{c'}{a'}x^2.Cx^2 - \frac{d'}{a'}x^3.Bx.$$

Hence each term of the required series, commencing at the fourth, is equal to the sum of the three preceding terms multiplied respectively by $\left(-\frac{b'}{a'}x, -\frac{c'}{a'}x^2 - \frac{d'}{a'}x^3.\right)$

The three first terms $A+Bx+Cx^2$, are obtained by substituting for A, B, C, their values obtained above.

192. Recurring series are divided into orders, and the order is estimated by the number of terms contained in the *scale*. Thus, the expression $\frac{a}{a'+b'x}$ gives a recurring series of *the first order*, the scale of which is $-\frac{b'}{a'}x$.

The expression $\frac{a+bx}{a'+b'x+c'x^2}$ will give a recurring series *of the second order*, of which the scale will be

$$\left(-\frac{b'}{a'}x, -\frac{c'}{a'}x^2\right).$$

The series obtained in the preceding No. is of the third order.

In general, an expression of the form $\frac{a+bx+cx^2+\dots kx^{n-1}}{a'+b'x+c'x^2+\dots+k'x^n}$, gives a recurring series *of the* n[th] *order*, the scale of which is

$$\left(-\frac{b'}{a'}x, -\frac{c'}{a'}x^2 \dots -\frac{k'}{a'}x^n\right).$$

N. B. It is here supposed that the degree of x in the numerator is less than it is in the denominator. If it was not, it would first be necessary to perform the division, arranging the quantities with reference to x, which would give an entire quotient, plus a fraction similar to the above.

Thus, in the expression $\frac{1-x-3x^2+4x^3+x^4}{2-5x+3x^2-x^3}$.

$$\begin{array}{l|l} x^4+4x^3-3x^2-x+1 & -x^3+3x^2-5x+2 \\ \quad +7x^3-8x^2+x & -x-7. \\ \qquad\qquad +13x^2-34x+15. & \end{array}$$

Performing the division, we find the quotient to be $-x-7$, plus the fraction.

$$\frac{13x^2-34x+15}{-x^3+3x^2-5x+2}, \text{ or } \frac{15-34x+13x^2}{2-5x+3x^2-x^3}.$$

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EXAMPLES FOR PRACTICE.

The following Examples are from Bonnycastle's Algebra.

MULTIPLICATION.

1. Multiply $12ax$ by $3a$. *Ans.* $36a^2x$.
2. Multiply $4x^2-2y$ by $2y$. *Ans.* $8x^2y-4y^2$.
3. Multiply $2x+4y$ by $2x-4y$. *Ans.* $4x^2-16y^2$.
4. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.
5. Multiply x^2+xy+y^2 by x^2-xy+y^2.
6. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
7. Multiply $3x^2-2xy+5$ by $x^2+2xy-3$.
8. Multiply $3x^3+2x^2y^2+3y^3$ by $2x^3-3x^2y^2+5y^3$.

DIVISION.

1. Divide $18x^2$ by $9x$. *Ans.* $2x$.
2. Divide $10x^2y^2$ by $-5x^2y$. *Ans.* $-2y$.
3. Divide $-9ax^2y^2$ by $9x^2y$. *Ans.* $-ay$.
4. Divide $-8x^2$ by $-2x$. *Ans.* $+4x$.
5. Divide $10ab+15ac$ by $5a$. *Ans.* $2b+3c$.
6. Divide $30ax-54x$ by $6x$. *Ans.* $5a-9$.
7. Divide $10x^2y-15y^2-5y$ by $5y$. *Ans.* $2x^2-3y-1$.
8. Divide $13a+3ax-17x^2$ by $21a$.
9. Divide $3a^2-15+6a+3b$ by $3a$.
10. Divide $a^2+2ax+x^2$ by $a+x$. *Ans.* $a+x$.

11. Divide $a^3-3a^2y+3ay^2-y^3$ by $a-y$.
Ans. $a^2-2ay+y^2$.

12. Divide 1 by $1-x$. *Ans.* $1+x+x^2+x^3$, &c.

13. Divide $6x^4-96$ by $3x-6$. *Ans.* $2x^3+4x^2+8x+16$.

14. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$. *Ans.* $a^3-3a^2x+3ax^2-x^3$.

15. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.

16. Divide $y^6-3y^4x^2-3y^2x^4-x^6$ by $y^3-3y^2x+3yx^2-x^3$.

FRACTIONS.

1. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

2. Let $10+\frac{2x-8}{3x}$ be reduced to an improper fraction.

3. Let $a+\frac{1-x-b}{b}$ be reduced to an improper fraction.

4. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

5. Let $\frac{35}{8}$ and $\frac{3ab-b^2}{a}$ be reduced to whole or mixed quantities. *Ans.* $4\frac{3}{8}$ and $3b-\frac{b^2}{a}$.

6. Let $\frac{2x^2y}{2x}$ and $\frac{a^2+x^2}{a-x}$ be reduced to whole or mixed quantities.

7. Let $\frac{x^2-y^2}{x+y}$ and $\frac{x^3-y^3}{x-y}$ be reduced to whole or mixed quantities.

8. Let $\frac{10x^2-x+3}{5x}$ be reduced to a whole or mixed quantity.

9. Let $\frac{12x^3+3x^2}{4x^3+x^2-4x-1}$ be reduced to a whole or mixed quantity.

10. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions, having a common denominator. *Ans.* $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

11. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to fractions, having a common denominator. *Ans.* $\frac{ac}{bc}$ *and* $\frac{ab+b^2}{bc}$.

12. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to fractions having a common denominator. *Ans.* $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ *and* $\frac{6acd}{6ac}$.

13. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{a}$, to fractions having a common denominator. *Ans.* $\frac{9a}{12a}$, $\frac{8ax}{12a}$, *and* $\frac{12a^2+24x}{12a}$.

14. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$ and $\frac{a^2+x^2}{a+x}$, to fractions having a common denominator.

15. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$, to equivalent fractions having a common denominator.

16. To find the greatest common divisor of $\frac{x^2-1}{xy+y}$.

Ans. $x+1$.

17. To find the greatest common divisor of $\frac{x^4-b^4}{x^5+b^2x^3}$.

Ans. x^2+b^2.

18. To find the greatest common measure of

$$\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}.$$

19. Reduce $\frac{x^4-b^4}{x^5+b^2x^3}$, to its lowest terms. *Ans.* $\frac{x^2+b^2}{x^3}$.

20. Reduce $\frac{x^2-y^2}{x^4-y^4}$, to its lowest terms. *Ans.* $\frac{1}{x^2+y^2}$.

21. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax+x^3}$, to its lowest terms.

Ans. $\frac{a^2+x^2}{a-x}$.

22. Reduce $\frac{5a^5-10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$, to its lowest terms.

23. Add $\frac{3x}{2b}$ and $\frac{x}{5}$ together. *Ans.* $\frac{15x+2bx}{10b}$.

24\. Add $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$ together. *Ans.* $x+\frac{x}{12}$.

25\. Add $\frac{x-2}{3}$ and $\frac{4x}{7}$ together. *Ans.* $\frac{19x-14}{21}$.

26\. Add $x+\frac{x-2}{3}$ to $3x+\frac{2x-3}{4}$ *Ans.* $4x+\frac{10x-17}{12}$.

27\. It is required to add $4x$, $\frac{5x^2}{2a}$, and $\frac{x+a}{2x}$ together.

Ans. $4x+\frac{5x^3+ax+a^2}{2ax}$.

28\. It is required to add $\frac{2x}{3}$, $\frac{7x}{4}$, and $\frac{2x+1}{5}$ together.

Ans. $2x+\frac{49x+12}{60}$.

29\. It is required to add $4x$, $\frac{7x}{9}$, and $2+\frac{x}{5}$ together.

Ans. $\frac{44x+90}{45}$.

30\. It is required to add $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$ together.

Ans. $3x+\frac{23x}{45}$.

31\. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$. *Ans.* $\frac{39x}{35}$.

32\. Required the difference of $5y$ and $\frac{3y}{8}$. *Ans.* $\frac{37y}{8}$.

33\. Required the difference of $\frac{3x}{7}$ and $\frac{2x}{9}$ *Ans.* $\frac{13x}{63}$.

34\. Required the difference between $\frac{x+a}{b}$ and $\frac{c}{d}$.

Ans. $\frac{dx+ad-bc}{bd}$.

35\. Required the difference of $\frac{3x+a}{5b}$ and $\frac{2x+7}{8}$.

Ans. $\frac{24x+8a-10bx-35b}{40b}$.

36\. Required the difference of $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$.

Ans. $2x+\frac{cx+bx+ab}{bc}$.

37. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

38. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

39. Find the continued product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$.
Ans. $9ax$.

40. It is required to find the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.
Ans. $\frac{ab+bx}{x}$.

41. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

42. Required the product of $x+\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

43. Let $\frac{7x}{5}$ be divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

44. Let $\frac{4x^2}{7}$ be divided by $5x$. *Ans.* $\frac{4x}{35}$.

45. Let $\frac{x+1}{6}$ be divided by $\frac{2x}{3}$. *Ans.* $\frac{x+1}{4x}$.

46. Let $\frac{x}{x-1}$ be divided by $\frac{x}{2}$. *Ans.* $\frac{2}{x-1}$.

47. Let $\frac{5x}{3}$ be divided by $\frac{2a}{3b}$. *Ans.* $\frac{5bx}{2a}$.

48. Let $\frac{x-b}{8cd}$ be divided by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

49. Let $\frac{x^4-b^4}{x^2-2bx+b^2}$ be divided by $\frac{x^2+bx}{x-b}$. *Ans.* $x+\frac{b^2}{x}$.

SIMPLE EQUATIONS.

1. Given $3y-2+24=31$ to find y. *Ans.* $y=3$.
2. Given $x+18=3x-5$ to find x. *Ans.* $x=11\frac{1}{2}$.
3. Given $6-2x+10=20-3x-2$ to find x. *Ans.* $x=2$.
4. Given $x+\frac{1}{2}x+\frac{1}{3}x=11$ to find x. *Ans.* $x=6$.
5. Given $2x-\frac{1}{2}x+1=5x-2$ to find x. *Ans.* $x=\frac{6}{7}$.

6. Given $3ax+\frac{a}{2}-3=bx-a$ to find x. *Ans.* $x=\frac{6-3a}{6a-2b}$.

7. Given $\left\{\begin{matrix}2x+3y=23\\5x-2y=10\end{matrix}\right\}$ to find x and y. $\left\{\text{Ans. } \begin{matrix}y=5\\x=4\end{matrix}\right.$.

8. Given $\left\{\begin{matrix}\frac{1}{2}x+\frac{1}{3}y=7\\\frac{1}{3}x+\frac{1}{2}y=8\end{matrix}\right\}$ to find x and y.

9. Given $4x+y=34$, and $4y+x=16$, to find x and y.
Ans. $x=8$, and $y=2$.

10. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$, to find x and y.
Ans. $x=\frac{1}{2}$, and $y=\frac{1}{3}$.

11. Given $x+y=s$, and $x^2-y^2=d$, to find x and y.
Ans. $x=\frac{s^2+d}{2s}$, and $y=\frac{s^2-d}{2s}$.

12. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find, x, y, and z. *Ans.* $x=24$, $y=6$, and $z=23$.

13. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find x, y, and z.

14. Given $\left\{\begin{matrix}ax+by+cz=m\\dx+ey+fz=n\\gx+hy+kz=p\end{matrix}\right\}$, to find x, y, and z.

15. What two numbers are those, whose difference is 7, and sum 33? *Ans.* 13 *and* 20.

16. To divide the number 75 into two such parts, that three times the greater may exceed seven times the less by 15.
Ans. 54 *and* 21.

17. In a mixture of wine and cider, $\frac{1}{2}$ of the whole *plus* 25 gallons was wine, and $\frac{1}{3}$ part *minus* 5 gallons was cider; how many gallons were there of each?
Ans. 85 *of wine, and* 35 *of cider.*

18. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; how many were there of each? *Ans.* 50 *of each.*

19. Two travellers set out at the same time from London and York, whose distance is 150 miles; one of them goes 8 miles a day, and the other 7; in what time will they meet?
Ans. in 10 *days.*

20. At a certain election, 375 persons voted, and the candidate chosen had a majority of 91; how many voted for each?
Ans. 233 *for one, and* 142 *for the other.*

21. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

22. A post is $\frac{1}{4}$ in the mud, $\frac{1}{3}$ in the water, and 10 feet above the water; what is its whole length? *Ans.* 24 *feet.*

23. There is a fish whose tail weighs 9lb.; his head weighs as much as his tail and half his body; and his body weighs as much as his head and his tail; what is the whole weight of the fish? *Ans.* 72*lb.*

24. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse; what was in it at first?

Ans. 120 *guineas.*

25. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's=84, B's=42, *and* C's=14.

26. Two persons A and B, lay out equal sums of money in trade; A gains 126l. and B loses 87l. and A's money is now double of B's; what did each lay out? *Ans.* 300l.

27. A person bought a chaise, horse, and harness, for 60l. the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. 13l. 6s. 8d. *for the horse,* 6l. 13s. 4d. *for the harness, and* 40l. *for the chaise.*

28. Two persons, A and B, have both the same income: A saves $\frac{1}{5}$ of his yearly, but B, by spending 50l. *per annum* more than A, at the end of 4 years finds himself 100l. in debt; what is their income? *Ans.* 125l.

29. A person has two horses, and a saddle worth 50l. now if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first; what is the value of each horse? *Ans. one* 30l. *and the other* 40l.

30. To divide the number 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other. *The parts are* 8, 12, *and* 16.

31. A footman agreed to serve his master for 8l. a year and a livery, but was turned away at the end of 7 months, and received only 2l. 13s. 4d. and his livery; what was its value?

Ans. 4l. 16s.

32. A person was desirous of giving 3d. a piece to some beg-

gars, but found he had not money enough in his pocket by 8d. he therefore gave them each 2d. and had then 3d. remaining; required the number of beggars. *Ans.* 11.

33. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and then won 2 shillings; lastly he lost $\frac{1}{7}$ of what he then had; and this done, found he had but 12s. remaining; what had he at first?

Ans. 20s.

34. To divide the number 90 into four such parts, that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient shall be all equal to each other.

Ans. The parts are 18, 22, 10, *and* 40, *respectively.*

35. The hour and minute hand of a clock are exactly together at 12 o'clock; when are they next together?

Ans. 1 *h.* $5\frac{5}{11}$ *min.*

36. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how many days would the man alone be in drinking it? *Ans.* 20 *days.*

37. If A and B together can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days; how many days would it take each person to perform the same work alone? *Ans.* A $14\frac{34}{49}$ *days*, B $17\frac{23}{41}$, *and* C $23\frac{7}{31}$.

38. If three agents, A, B, C, can produce the effects a, b, c, in the times e, f, g, respectively; in what time would they jointly produce the effect d?

$$Ans.\ d \div \left(\frac{a}{e}+\frac{b}{f}+\frac{c}{g}\right) time.$$

QUADRATIC EQUATIONS.

1. Given $x^2-8x+10=19$, to find x. *Ans.* $x=9$.
2. Given $x^2-x-40=170$, to find x. *Ans.* $x=15$.
3. Given $3x^2+2x-9=76$, to find x. *Ans.* $x=5$.
4. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find x. *Ans.* $x=1\frac{1}{2}$.
5. Given $2x^4-x^2=496$, to find x. Ans. $x=4$.
6. *Given* $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$ to find x. Ans. $x=49$.

7. Given $\frac{2}{3}x^2+\frac{3}{4}x=\frac{4}{5}$ to find x. *Ans.* .6689.

8. Given $x^6+6x^3=2$, to find x. *Ans.* $x=\sqrt[3]{-3+\sqrt{11}}$.

9. Given $x^2+x=a$, to find x. *Ans.* $x=\sqrt{(a+\frac{1}{4})}-\frac{1}{2}$.

10. Given $x-\sqrt{x}=a$, to find x. *Ans.* $x=(\frac{1}{2}+\sqrt{a+\frac{1}{4}})^2$.

11. Given $3x^{2n}-2x^n=25$, to find x. *Ans.* $x=(\frac{1}{3}\sqrt{76}+\frac{1}{3})^{\frac{1}{n}}$.

12. Given $\sqrt{1+x}-2\sqrt[4]{1+x}=4$, to find x.

Ans. $x=(1+\sqrt{5})^4-1$.

13. A grazier bought as many sheep as cost him 60l., and after reserving 15 out of the number, he sold the remainder for 54l., and gained 2s. a head by them. How many sheep did he buy? *Ans.* 75.

14. There are two numbers whose difference is 15, and half their product is equal to the cube of the lesser number. What are those numbers? *Ans.* 3 *and* 18.

15. A person bought cloth for 33l. 15s., which he sold again at 2l. 8s. per piece, and gained by the bargain as much as one piece cost him. Required the number of pieces. *Ans.* 15.

16. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Ans.* 24.

17. What two numbers are those whose sum, multiplied by the greater, is equal to 77; and whose difference, multiplied by the lesser, is equal to 12?

Ans. 4 *and* 7, or $\frac{3}{2}\sqrt{2}$ *and* $\frac{11}{2}\sqrt{2}$.

18. To find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product shall be 21. *Ans.* 7 *or* 3.

19. To divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 *and* 36.

20. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference.

Ans. 10 *and* 14.

21. The sum of two numbers is 8, and the sum of their cubes is 152. What are the numbers? *Ans.* 3 *and* 5.

22. The sum of two numbers is 7, and the sum of their 4th powers is 641. What are the numbers? *Ans.* 2 *and* 5.

23. The sum of two numbers is 6, and the sum of their 5th powers is 1056. What are the numbers? Ans. 2 and 4.

EVOLUTION.

1. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

2. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

3. Required the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$. *Ans.* $a^2+2ax+x^2$.

4. Required the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{x}{2}+\frac{1}{16}$. *Ans.* $x^2-x+\frac{1}{4}$.

5. It is required to find the square root of a^2+x^2.

$$Ans.\ a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-, \text{ \&c.}$$

6. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

7. Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

8. Required the square root of $a^2+2ab+2ac+b^2+2bc+c^2$. *Ans.* $a+b+c$.

9. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. *Ans.* x^2-2x+1.

10. Required the biquadrate root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. *Ans.* $2a-3x$.

11. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$. *Ans.* $2x-1$.

SURDS.

1. Reduce $\sqrt{125}$ to its most simple terms. *Ans.* $5\sqrt{5}$.

2. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms. *Ans.* $\frac{5}{21}\sqrt{6}$.

3. Reduce $\sqrt[3]{243}$ to its most simple terms. *Ans.* $3\sqrt[3]{9}$.

4. Reduce $\sqrt[3]{\frac{16}{81}}$ to its most simple terms. *Ans.* $\frac{2}{9}\sqrt[3]{18}$.

5. Reduce $\sqrt{98a^2x}$ to its most simple terms. *Ans.* $7a\sqrt{2x}$.

6. Reduce $\sqrt{(x^3-a^2x^2)}$ to its most simple terms.

7. Reduce $(a^3x+3a^3x^2)^{\frac{1}{3}}$ to its most simple terms.

8. Reduce $(32a^6-96a^5x)^{\frac{1}{5}}$ to its most simple terms.

9. Required the sum of $\sqrt{72}$ and $\sqrt{128}$. *Ans.* $14\sqrt{2}$.
10. Required the sum of $\sqrt{27}$ and $\sqrt{147}$. *Ans.* $10\sqrt{3}$.
11. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$. *Ans.* $\frac{19}{30}\sqrt{6}$.
12. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.
13. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.
14. Required the difference of $320^{\frac{1}{3}}$ and $40^{\frac{1}{3}}$. *Ans.* $2\sqrt[3]{5}$.
15. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$. *Ans.* $\frac{4}{45}\sqrt{15}$.
16. Required the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$. *Ans.* $\frac{1}{12}\sqrt[3]{18}$.
17. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$. *Ans.* $30\sqrt{10}$.
18. Required the product of $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$. *Ans.* $\sqrt[3]{4}$.
19. Required the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$. *Ans.* $\frac{1}{40}\sqrt{35}$.
20. It is required to find the product of $a^{\frac{1}{3}}$ and $a^{\frac{2}{3}}$.

Ans. $(a^3)^{\frac{1}{3}}$ *or* a.

21. Required the product of $(x+y)^{\frac{1}{2}}$ and $(x+y)^{\frac{1}{3}}$.
22. Let $6\sqrt{10}$ be divided by $3\sqrt{5}$. *Ans.* $2\sqrt{2}$.

INFINITE SERIES.

1. To find the value of $\frac{1}{(a+b)^2}$, or its equal $(a+b)^{-2}$ in an infinite series.

2. To find the value of $\frac{r^2}{r+x}$, in an infinite series.

Ans. $r-x+\frac{x^2}{r}-\frac{x^3}{r^2}+\frac{x^4}{r^3}$, &c.

3. Required the square root of $\frac{a^2+x^2}{a^2-x^2}$ in an infinite series.

Ans. $1+\frac{x^2}{a^2}+\frac{x^4}{2a^4}+\frac{x^6}{2a^6}$ &c.

4. Required the cube root of $\frac{a^2}{(a^2+x^2)^2}$ in an infinite series.

Ans. $\frac{1}{a^{\frac{2}{3}}}\times : 1-\frac{2x^2}{3a^2}+\frac{5x^4}{9a^4}-\frac{40x^6}{81a^6}$, &c.

CHAPTER V.

Of Progressions and Logarithms.

THIS chapter is naturally connected with the last, as the object of the first paragraph is the investigation of the properties of two kinds of series, and as it also presents an application of the theory of exponents; it moreover completes that part of algebra which is absolutely necessary for the study of *Trigonometry*, and the *Application of Algebra to Geometry*.

§ I. *Of Progressions by Differences, and Quotients.*

Progressions by Differences.

193. A *progression by differences*, or an *Arithmetical progression*, is a series in which the successive terms continually increase or decrease by a constant quantity, which is called the *ratio* or *difference* of the progression.

Thus, in the two series

1, 4, 7, 10, 13, 16, 19, 22, 25. . . .
60, 56, 52, 48, 44, 40, 36, 32, 28. . . .

The first is called an *increasing progression*, of which the ratio is 3, and the second a *decreasing progression*, of which the ratio is 4.

In general, let a, b, c, d, e, f, . . . designate the terms of a progression by differences; it has been agreed to write them thus:

$$\div a.b.c.d.e.f.g.h.i.k. \ldots$$

This series is read, a is to b as b is to c, as c is to d, as d is to e, . . . or a is to b is to c is to d is to e. . . This is a series of *continued equidifferences*, in which each term is at the same time a consequent and antecedent, with the exception of the first term, which is only an *antecedent*, and the last, which is only a *consequent*.

194. Let r represent the ratio of the progression, which we will consider as increasing. In the case of a decreasing progression, it will only be necessary to change r into $-r$, in the results.

From the definition of the progression, it evidently follows that

$$b=a+r,\ c=b+r=a+2r,\ d=c+r=a+3r\ ;$$

and in general, any term of the series is equal to *the first, plus as many times the ratio as there are terms before this term.*

Thus let l be this term, and n the number which marks the place of it, the expression for this *general term*, is

$$l=a+(n-1)r.$$

In fact, if we suppose n successively equal to 1, 2, 3, 4 .... we will obtain the first, second, third, ... term of the progression.

If the progression was decreasing, we would have

$$l=a-(n-1)r.$$

The formula $l=a+(n-1)r$, serves to find any term whatever, without our being obliged to determine all those which precede it.

Thus, by making $n=50$, we find the 50^{th} term of the progression,

$$\div 1.4.7.10.13.16.19. \ldots \qquad l=1+49.3=148.$$

195. A progression by differences being given, it is proposed to *determine a certain number of terms.*

Let $\div\ a.b.c.d.e.f. \ldots i.\ k.\ l$, be the proposed progression, and n the number of terms.

We will first observe that, if x denotes a term which has p terms before it, and y a term which has p terms after it, we have, from what has been said, $x=a+p\times r$,

and $y=l-p\times r$;

whence, by addition, - - - - - - - - - - - - - - $x+y=a+l$.

which demonstrates that, in any progression, *the sum of any two terms, taken at equal distances from the two extremes, is equal to the sum of these extremes*, or *the two extremes and any two terms taken at equal distances from them form an equidifference,* in the order in which they are written.

This being the case, conceive the progression to be written below itself, but in an inverse order, viz.

$$\div a.\ b.\ c.\ d.\ e.\ f\ .\ .\ .\ .\ i.\ k.\ l.$$
$$\div l.\ k.\ i\ .\ .\ .\ .\ .\ .\ .\ .\ .\ c.\ b.\ a.$$

Calling S the sum of the terms of the first progression, 2 S will be the sum of the terms in both progressions, and we will have

$$2S=(a+l)+(b+k)+(c+i)\ldots\ldots+(i+c)+(k+b)+(l+a);$$

or, since the number of the parts $a+l$, $b+k$, $c+i$ ..... is equal to n,

$$2S=(a+l)n, \text{ or } S=\frac{(a+l)n}{2}.$$

That is, *the sum of a progression by differences, is equal to half the sum of the two extremes, multiplied by the number of terms.*

If in this formula we substitute for l its value, $a+(n-1)r$, obtain

$$S=\frac{[2a+(n-1)r]\,n.}{2};$$

but the first expression is the most useful.

Applications.

Find the sum of the fifty first terms of the series 2. 9. 16. 23 ...

For the 50th term we have $l=2+49\times7=345$,

hence $$S=\frac{(2+345).50}{2}=347\times25=8675.$$

In like manner, we find for the 100th term,

$$l=2+99\times7=695,$$

and for the sum of the 100 first terms,

$$S=\frac{(2+695).100}{2}=34850.$$

196. The formulas $l=a+(n-1)r$, $S=\dfrac{(a+l)n}{2}$, contain five quantities, a, r, n, l and S, and consequently give rise to the following general problem, viz.: *Any three of these five quantities being given, determine the other two.* This problem is

subdivided into as many particular problems as there can be formed *different combinations* of five letters, taken *three and three*, or two and two. Now we have found the number of these combinations 2 and 2, and 3 and 3, to be (No. 150)

$$\frac{m(m-1)}{2} \text{ and } \frac{m(m-1)(m-2)}{2.3}.$$

Making $m=5$ in these formulas, they become $\frac{5\times4}{2}$, or 10, and $\frac{5\times4\times3}{2.3}$, or 10. Whence 5 letters combined 3 and 3, give the same number of combinations as 5 letters combined 2 and 2. This result agrees with the consequence of No. 153, from which the number of combinations of m letters taken p and p is found to be equal to the number of combinations taken $m-p$ and $m-p$.

Hence the above problem is subdivided into 10 particular problems, which are ennunciated as follows.

Having given,	1st.	a,	r,	n,	to find	l and S;
	2d.	a,	r,	l,	- - - -	n and S;
	3d.	a,	r,	S,	- - - -	n and l;
	4th.	a,	n,	l,	- - - -	r and S;
	5th.	a,	n,	S,	- - - -	r and l;
	6th.	a,	l,	S,	- - - -	r and n;
	7th.	r,	n,	l,	- - - -	a and S;
	8th.	r,	n,	S,	- - - -	a and l;
	9th.	r,	l,	S,	- - - -	a and n;
	10th.	n,	l,	S,	- - - -	a and r;

The first problem is already resolved, for the two formulas give at once l and S in functions of a, r, n. As to the other problems, their resolution does not present any difficulties, but we recommend the student to resolve them, as they will serve to render him familiar with the resolution of equations of the first and second degree. It will be well to remark, that, although the first powers of the quantities a, r, n, l and S only are involved in the two formulas, they nevertheless lead to the resolution of an equation of the second degree, when a and n, or l and n, are unknown quantities; for a and n, or l and n, both enter in each of the two equations, and are multiplied into each other in the second.

197. We will limit ourselves to the resolution of the fourth problem, viz. knowing a, n, and l, it is required to find r and S.

The formula $l=a+(n-1)r$ gives $r=\frac{l-a}{n-1}$,

and the formula $S=\frac{(a+l)n}{2}$, gives the value of S.

From the first expression $r=\frac{l-a}{n-1}$, we deduce the solution of the following question, viz.: Find a number m of *arithmetical means* between two given numbers a and b.

To resolve this last question, it is only necessary to determine the ratio. Now by substituting in the above formula, b for l, and $m+2$ for n, which expresses the whole number of terms, it becomes $r=\frac{b-a}{m+2-1}$, or $r=\frac{b-a}{m+1}$; that is, *the ratio* of the required progression is obtained by dividing the difference between the given numbers a and b, by one more than the required number of means.

Having obtained the ratio, form the second term of the progression, or the *first arithmetical mean*, by adding r, or $\frac{b-a}{m+1}$, to the first term a. The *second mean* is obtained by augmenting the first by r, &c.

For example, let it be required to find 12 arithmetical means between 12 and 77. We have $r=\frac{77-12}{13}=\frac{65}{13}=5$, which gives the progression $\div 12.\ 17.\ 22.\ 27 \ldots .\ 72.\ 77.$

Consequence. If the same number of arithmetical means are inserted between all of the terms, taken two and two, these terms, and the arithmetical means united, will form but one and the same progression.

For, let $\div a.\ b.\ c.\ d.\ e.\ f. \ldots .$ be the proposed progression, and m the number of means to be inserted between a and b, b and c, c and $d \ldots .$

From what has just been said, the ratio of each partial progression will be expressed by $\frac{b-a}{m+1}$, $\frac{c-b}{m+1}$, $\frac{d-c}{m+1} \ldots$, which are equal to each other, since a, b, $c \ldots .$ are in progression, therefore the ratio is the same in each of the partial progressions and since the *last term* of the first forms the *first term* of the second, &c. we may conclude that all of these partial progressions form a single progression.

Problems.

198. 1st. In a progression by difference, having given the ratio 6, the last term 185, and the sum of the terms 2945, find the first term, and the number of terms.

(*Ans.* First term $=5$, number of terms 31.)

2d. Find 9 arithmetical means between each antecedent and consequent of the progression $\div 2.\ 5.\ 8.\ 11.\ 14\ldots.$

(*Ans.* Ratio, or $r=0.3$.)

3d. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2, the third 3, and so on to the n^{th}, which contains n. In other words, find the expression for the sum of the natural numbers $1, 2, 3\ldots$, from 1 to n, inclusively.

$$\left(Ans.\ S=\frac{n(n+1)}{2}\right).$$

4th. Find the sum of the n first terms of the progression of uneven numbers $1, 3, 5, 7, 9\ldots.$

(*Ans.* $S=n^2$.)

Progressions by Quotients.

199. A *Geometrical progression* or *progression by quotients* is a series of terms, each of which is equal to the product of that which precedes it, by a *constant number*, which is called the ratio of the progression; thus in the two series:

$$3,\ 6,\ 12,\ 24,\ 48,\ 96\ \ldots$$

$$64,\ 16,\ 4,\ 1,\ \frac{1}{4},\ \frac{1}{16}\ \ldots$$

each term of the first contains that which precedes it *twice*, or is equal to double that which precedes it; and each term of the second is contained in that which precedes it four times, or is a *fourth* of that which precedes it; they are then progressions by quotients, of which the ratio is 2 for the first, and $\frac{1}{4}$ for the second.

Let a, b, c, d, e, f... be numbers in a progression by quotients, it is written thus: $\div\!\!\div\, a : b : c : d : e : f : g$ . . . . , and it is enunciated in the same manner as a progression by differences; however it is necessary to make the distinction that one is a series of equal differences, and the other a series of equal quotients or ratios, in which each term is at the same time an antecedent and a consequent, except the first, which is only an antecedent, and the last, which is only a consequent.

200. Let q denote the ratio of the progression . . . . $\div\!\!\div\, a : b : c : d$ . . . , q being >1 when the progression is *increasing*, and <1 when it is *decreasing:* we deduce from the definition, the following equalities.

$$b=aq,\ c=bq=aq^2.\ d=cq=aq^3,\ e=dq=aq^4 \ldots$$

and in general, any term u, that is, one which has $n-1$ terms before it, is expressed by aq^{n-1}.

Let l be this term; we have the formula $l=aq^{n-1}$, by means of which we can obtain the value of any term without being obliged to find the values of all those which precede it. For example, the 8th term of the progression $\div\!\!\div\, 2 : 6 : 18 : 54$..., is equal to $2\times3^7=2\times2187=4374$.

In like manner, the 12th term of the progression . . . . . . $\div\!\!\div\, 64 : 16 : 4 : 1 : \frac{1}{4}$ . . . is equal to

$$64\left(\frac{1}{4}\right)^{11}=\frac{4^3}{4^{11}}=\frac{1}{4^8}=\frac{1}{65536}.$$

201. We will now proceed to determine the sum of n terms of the progression $\div\!\!\div\, a : b : c : d : e : f : \ldots : i : k : l$, l denoting the n^{th} term.

We have (No. 199) the equations

$$b=aq,\ c=bq,\ d=cq,\ e=dq,\ \ldots\ k=iq,\ l=kq;$$

and by adding them all together, member to member, we deduce

$$b+c+d+e+\ldots+k+l=(a+b+c+d+\ldots+i+k)q;$$

or, representing the required sum by S,

$$S-a=(S-l)q=Sq-lq, \text{ or } Sq-S=lq-a;$$

whence $$S=\frac{lq-a}{q-1};$$

That is, in order to obtain the sum of a certain number of terms of a progression by quotients, *multiply the last term by the ratio, subtract the first term from this product, and divide the remainder by the ratio diminished by unity.*

When the progression is decreasing, we have $q<1, l<a$; and the above formula is then written under the form $S=\frac{a-lq}{1-q}$, in order that the two terms of the fraction may be positive.

By substituting aq^{n-1} for l in the two expressions for S, they become, $S=\frac{aq^n-a}{q-1}$, and $S=\frac{a-aq^n}{1-q}$.

From the preceding formulas we will find

1st. For the sum of the 8 first terms of the progression

$$\div 2:6:18:54\ldots:2\times3^7 \text{ or } 4374.$$

$$S=\frac{lq-a}{q-1}=\frac{13122-2}{2}=6560.$$

2d. For the sum of the 12 first terms of the progression

$$\div 64:16:4:1:\frac{1}{4}:\ldots:64\left(\frac{1}{4}\right)^{11}, \text{ or } \frac{1}{65536},$$

$$S=\frac{a-lq}{1-q}=\frac{64-\frac{1}{65536}\times\frac{1}{4}}{\frac{3}{4}}=\frac{256-\frac{1}{65536}}{3}=85+\frac{65535}{196608}.$$

We perceive that the principal difficulty consists in obtaining the numerical value of the last term, a tedious operation, even when the number of terms is not very great.

202. *Remark.* If, in the formula $S=\frac{a(q^n-1)}{q-1}$, we suppose $q=1$, it becomes $S=\frac{0}{0}$.

This result, which is sometimes a symbol of indetermination, is also often a consequence of the existence of a common factor, (No. 72), which becomes nothing by making a particular hypothesis respecting the given question. This, in fact, is the case in the present question; for the expression q^n-1 is divisible by $q-1$, (No. 31), and gives the quotient

$$q^{n-1}+q^{n-2}+q^{n-3}+\ldots\ldots+q+1;$$

hence the value of S takes the form

$$S=aq^{n-1}+aq^{n-2}+aq^{n-3}+\ldots\ldots+aq+a.$$

Now, making $q=1$, we have $S=a+a+a+\ldots.+a=na$. We can obtain the same result by going back to the proposed progression, $\because a : b : c : \ldots : l$, which, in the particular case of $q=1$, reduces to $\div\!\div a : a : a : \ldots. : a$, the sum of which series is equal to na.

The result $\frac{0}{0}$, given by the formula, may be regarded as indicating the *insufficiency* of this formula to give the expression for the sum in this particular case. In fact, the progression, being entirely composed of equal terms, is no more a progression by quotients than it is a progression by differences. Therefore, in seeking for the sum of a certain number of the terms, there is no reason for using the formula $S=\frac{a(q^n-1)}{q-1}$, in preference to the formula $S=\frac{(a+l)n}{2}$, relative to the progression by differences.

203. *Of Infinite Progressions by Quotients.* Let there be the decreasing progression $\div\!\div a : b : c : d : e : f : \ldots..$, containing an indefinite number of terms. The formula $S=\frac{a-aq^n}{1-q}$, which represents the sum of n of its terms, can be put under the form $S=\frac{a}{1-q}-\frac{aq^n}{1-q}$.

Now since the progression is decreasing, q is a fraction; q^n is also a fraction, which diminishes as n increases. Therefore the greater the number of terms we take, the more will $\frac{a}{1-q}\times q^n$ diminish, and consequently, the more will the partial sum of these terms approximate to an equality with the first part of S, that is, to $\frac{a}{1-q}$. Finally, when n is taken greater than any given magnitude, or $n=\infty$, then $\frac{a}{1-q}\times q^n$ will be less than any given magnitude, or will become equal to 0; and the expression $\frac{a}{1-q}$ will represent the true value of the sum of all the terms of the series.

Whence we may conclude, that the expression for *the sum of the terms of a decreasing infinite progression, is*

$$S=\frac{a}{1-q}.$$

This is, properly speaking, the *limit* to which the *partial sums* (obtained by taking a greater number of terms) of the progression continually approximates. The difference between these sums and $\frac{a}{1-q}$ can become as small as we please, and will only become *nothing* when the number of terms taken is infinite.

Applications.

$$1:\frac{1}{3}:\frac{1}{9}:\frac{1}{27}:\frac{1}{81}.$$

We have for the expression of the sum of the terms

$$S=\frac{a}{1-q}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}.$$

The error committed by taking this expression for the value of the sum of the n first terms, is expressed by

$$\frac{a}{1-q}\cdot q^n=\frac{3}{2}\left(\frac{1}{3}\right)^n.$$

First take $n=5$; it becomes $\frac{3}{2}\left(\frac{1}{3}\right)^5=\frac{1}{2.3^4}=\frac{1}{162}.$

When $n=6$, we find $\frac{3}{2}\left(\frac{1}{3}\right)^6=\frac{1}{162}\cdot\frac{1}{3}=\frac{1}{486}.$

Whence we see that the *error committed*, when $\frac{3}{2}$ is taken for the sum of a certain number of terms, is less in proportion as this number is greater.

Again take the progression

$$1:\frac{1}{2}:\frac{1}{4}:\frac{1}{8}:\frac{1}{16}:\frac{1}{32}:\ldots$$

We have $\quad S=\frac{a}{1-q}=\frac{1}{1-\frac{1}{2}}=2.$

The expression $S=\frac{a}{1-q}$, can be obtained directly from the progression $\div a : b : c : d : e : f : g : \ldots$

For take the equations $b=aq$, $c=bq$, $d=cq$, $e=dq$ . . . . . of which the number is indefinite, and add them together, member to member; we have

$$b+c+d+e+ \cdots =(a+b+c+d+ \cdots)q.$$

Now, the first member is evidently the proposed series, diminished by the first term a; it is therefore expressed by $S-a$; the second member is q multiplied by the entire series, since there is no last term, or rather this last term is nothing; hence the expression for this member is qS, and the above equality becomes $S-a=q\,S$, whence $S=\frac{a}{1-q}$.

In fact, by developing $\frac{a}{1-q}$ into a series by the rule for division, we will find the result to be $a+aq+aq^2+aq^3+ \cdots$, which is nothing more than the proposed series, having $b, c, d\ldots$ replaced by their values in functions of a.

204. When the series is increasing, the expression $\frac{a}{1-q}$ cannot be considered as a *limit of the partial sums;* because, the sum of a determined number of terms being (201) $S=\frac{a}{1-q}-\frac{aq^n}{1-q}$, the second part $\frac{aq^n}{1-q}$ augments numerically in proportion to the increase of n; hence the greater the number of terms taken, the more the expression of their sum will differ numerically from $\frac{a}{1-q}$.

The formula $S=\frac{a}{1-q}$ is, in this case, merely the algebraic expression which, by its development, gives the series - - - - - -

$$a+aq+aq^2+aq^3 \cdots$$

There is another circumstance presents itself here, which appears very singular at first sight. Since $\frac{a}{1-q}$ is the fraction which generates the above series, we should have

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+aq^4+\cdots$$

Now, by making, $a=1, q=2$, this equality becomes

$$\frac{1}{1-2} \text{ or } -1=1+2+4+8+16+32+\cdots$$

an equation of which the first member is negative whilst the second is positive and greater in proportion to the magnitude of q.

To interpret this result, we will observe that, when in the equation $\frac{a}{1-q}=a+aq+aq^2+aq^3+\cdots$, we stop at a certain term of the series, it is necessary to complete the quotient in order that the equality may subsist. Thus, in stopping, for example, at the fourth term, aq^3.

	a	$1-q$
1st remainder	$+aq$	$a+aq+aq^2+aq^3+\frac{aq^4}{1-q}$.
2d.	$+aq^2$	
3d.	$+aq^3$	
4th.	$+aq^4$	

It is necessary to add the fractional expression $\frac{aq^4}{1-q}$ to the quotient, which gives rigorously,

$$\frac{a}{1-q}=a+aq+aq^2+aq^3+\frac{aq^4}{1-q}.$$

If in this exact equation we make $a=1, q=2$, it becomes - - -

$$-1=1+2+4+8+\frac{16}{-1}=1+2+4+8-16,$$

which verifies itself.

In general, when an expression involving x, designated by $f(x)$, (which is called *a function* of x), is developed into a series of the form $a+bx+cx^2+dx^3+\ldots$, we have not rigorously $f(x)=a+bx+cx^2+dx^3+\ldots$, unless we conceive that (in stopping at a certain term in the second member) the series is completed by a certain expression involving x.

When, in particular applications, the series is *decreasing*, the expression which serves to complete it may be conceived as *small as we* please, by prolonging the series; but the contrary is the case when the series is *increasing*, and it must not be neglected. This is the reason why increasing series cannot be used for approximating to the value of numbers. It is for this reason, also, that algebraists have called those series which go on diminishing from term to term, *converging series*, and those in which the terms go on increasing, *diverging series*. In the first, the greater the number of terms taken, the nearer the sum approximates numerically to the expression of which this series is the development; whilst in the others, the more terms we take, the more their sum differs from the numerical value of this expression.

206. The consideration of the five quantities a, q, n, l and S, which enter in the two formulas $l=bq^{n-1}$, $S=\frac{lq-a}{q-1}$, (Nos. 200 and 201), gives rise to ten particular problems, the enunciations of which do not differ from those relative to progressions by differences, (196), except that the letter r is replaced by q. We will then, as in progressions by differences, determine q and S, knowing a, l and n.

The first formula gives $q^{n-1}=\frac{l}{a}$, whence $q=\sqrt[n-1]{\frac{l}{a}}$. Substituting this value in the second formula, the value of S will be obtained.

The expression $q=\sqrt[n-1]{\frac{l}{a}}$ furnishes the means for resolving the following question, viz.

To find m *mean proportionals between two given numbers* a *and* b; that is, *to find a number* m *of quantities, which will form with* a *and* b, *considered as extremes, a progression by quotients.*

For this purpose, it is only necessary to know the *ratio;* now the required number of terms being m, the total number of terms r is equal to $m+2$. Moreover, we have $l=b$, therefore the value of q becomes $q=\sqrt[m+1]{\frac{b}{a}}$; that is, we must *divide one of the given numbers* (b) *by the other* (a), *then extract that*

root of the quotient which is denoted by one more than the required number of terms.

Hence, the progression is

$$\div\, a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m+1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} : \ldots\ldots b.$$

Thus, to insert six mean proportionals between the numbers 3 and 384, we make $m=6$, whence $q=\sqrt[7]{\frac{384}{3}}=\sqrt[7]{128}=2$, whence we deduce the progression

$$\div 3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

We will hereafter explain the most expeditious means of calculating numerically the number expressed by $q=\sqrt[m+1]{\frac{b}{a}}$.

When the same number of mean proportionals are inserted between all the terms of a progression by quotients, taken two and two, all the progressions thus formed will constitute a single progression. The demonstration is analogous to that of No. 197.

207. Of the ten principal problems that may be proposed upon progressions, *four* are susceptible of being easily resolved. The following are the enunciations, with the formulas relating to them.

1st. a, q, n, being given, to find l and S.

$$l=aq^{n-1};\quad S=\frac{lq-a}{q-1}=\frac{a(q^n-1)}{q-1}.$$

2d. a, n, l, being given, to find q and S.

$$q=\sqrt[n-1]{\frac{l}{a}};\quad S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}.$$

3d. q, n, l, being given, to find a and S.

$$a=\frac{l}{q^{n-1}};\quad S=\frac{l(q^n-1)}{q^{n-1}(q-1)}.$$

4th. q, n, S, being given to find a and l.

$$a=\frac{S(q-1)}{q^n-1},\quad l=\frac{Sq^{n-1}(q-1)}{q^n-1}.$$

Two other problems depend upon the resolution of equations of a degree superior to the second; they are those in which the unknown quantities are supposed to be a and q, or l and q.

For, from the second formula we deduce $a=lq-Sq+S$;

Whence, by substituting this value of a in the first $l=aq^{n-1}$,

$$l=(lq-Sq+S)q^{n-1},$$

or,
$$(S-l)q^{n}-Sq^{n-1}+l=0.$$

an equation of the n^{th} degree.

In like manner, in determining l and q, we would obtain the equation $aq^{n}-Sq+S-a=0$.

208. Finally, the other *four* problems lead to the resolution of equations of a peculiar nature; they are those in which n and one of the other four quantities are unknown.

From the second formula it is easy to obtain the value of one of the quantities a, q, l, and S, *in functions* of the other three; hence the problem is reduced to finding n by means of the formula $l=aq^{n-1}$.

Now this equality becomes $q^{n}=\frac{lq}{a}$, an equation of the form $a^{x}=b$, a and b being known quantities. Equations of this kind are called *exponential equations*, to distinguish them from those previously considered, in which the unknown quantity is raised to a power denoted by a known number.

§ II. *Of Exponential Quantities and Logarithms.*

209. *Resolution of the equation* $a^{x}=b$. The object of the question is, to find the exponent of the power to which it is necessary to raise a given number a, in order to produce another given number b.

We will first consider some particular cases.

Suppose it is required to resolve the equation $2^{x}=64$. By raising 2 to its different powers, we find that $2^{6}=64$; hence $x=6$ will satisfy the conditions of the question.

Again, let there be the equation $3^{x}=243$. The solution is $x=5$. In fact, so long as the second member b is a *perfect power* of the given number a, x will be an entire number which may be obtained by raising a to its successive powers, commencing at the first.

Suppose it is required to resolve the equation $2^x=6$. By making $x=2$, and $x=3$, we find $2^2=4$ and $2^3=8$; from which we perceive that x has a value comprised between 2 and 3.

Suppose then, that $x=2+\frac{1}{x'}$ - - - (x' is then >1).

Substituting this value in the proposed equation, it becomes

$$2^{2+\frac{1}{x'}}=6 \text{ or } (180)\, 2^2\times 2^{\frac{1}{x'}}=6; \text{ hence } 2^{\frac{1}{x'}}=\frac{3}{2},$$

or raising both members to the x' power, $\left(\frac{3}{2}\right)^{x'}=2$.

To determine x', make successively $x'=1$, $x'=2$; we find $\left(\frac{3}{2}\right)^1=\frac{3}{2}$, less than 2, and $\left(\frac{3}{2}\right)^2=\frac{9}{4}$, which is greater than 2; therefore x' is comprised between 1 and 2.

$$\text{Suppose } x'=1+\frac{1}{x''} \text{ - - - } (x''>1).$$

By substituting this value in the equation $\left(\frac{3}{2}\right)^{x'}=2$.

$$\left(\frac{3}{2}\right)^{1+\frac{1}{x''}}=2 \text{ or } \frac{3}{2}\times\left(\frac{3}{2}\right)^{\frac{1}{x''}}=2,$$

or reducing $\left(\frac{4}{3}\right)^{x''}=\frac{3}{2}$.

The two hypotheses $x''=1$ and $x''=2$, give $\left(\frac{4}{3}\right)^1=\frac{4}{3}$ which is less than $\frac{3}{2}$, and $\left(\frac{4}{3}\right)^2=\frac{16}{9}=1+\frac{7}{9}$, which is greater than $\frac{3}{2}$, therefore x'' is comprised between 1 and 2.

Let $x''=1+\frac{1}{x'''}$, there will result

$$\left(\frac{4}{3}\right)^{1+\frac{1}{x'''}}=\frac{3}{2}, \text{ or } \frac{4}{3}\times\left(\frac{4}{3}\right)^{\frac{1}{x'''}}=\frac{3}{2};$$

whence, reducing, $\left(\frac{9}{8}\right)^{x'''}=\frac{4}{3}$.

Making successively $x'''=1, 2, 3$, we find for the two last hypotheses $\left(\frac{9}{8}\right)^2=\frac{81}{64}=1+\frac{17}{64}$, which is $<1+\frac{1}{3}$, and . . . . . $\left(\frac{9}{8}\right)^3=\frac{729}{512}=1+\frac{217}{512}$, which is $>1+\frac{1}{3}$: therefore x''' is comprised between 2 and 3.

Let $x'''=2+\frac{1}{x^{\text{iv}}}$, the equation involving x''' becomes

$$\left(\frac{9}{8}\right)^{2+\frac{1}{x^{\text{iv}}}}=\frac{4}{3}, \text{ or } \frac{81}{64}\left(\frac{9}{8}\right)^{\frac{1}{x^{\text{iv}}}}=\frac{4}{3};$$

and consequently $\left(\frac{256}{243}\right)^{x^{\text{iv}}}=\frac{9}{8}$.

Operating upon this exponential equation in the same manner as upon the preceding equations, we will find two entire numbers k and $k+1$, between which x^{iv} will be comprised. Making $x^{\text{iv}}=k+\frac{1}{x^{\text{v}}}$, x^{v} can be determined in the same manner as x^{iv}, and so on.

Making the necessary substitutions in the equations

$$x=2+\frac{1}{x'},\quad x'=2+\frac{1}{x''},\quad x''=1+\frac{1}{x'''},\quad x'''=2+\frac{1}{x^{\text{iv}}}\ \ldots,$$

we obtain the value of x under the form of a continued fraction

$$x=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{x^{\text{iv}}}}}}.$$

Now we know (Arith. No. 175) that in any continued fraction, the greater the number of integral fractions we take, the nearer we will approximate to the value of the number represented by this continued fraction; therefore we may by this means find the value of x in the equation $2^x=6$, if not exactly, we may at least approximate to it as near as we please.

For example, forming the four first reductions by the rule (Arith. No. 169), we find

$$\frac{2}{1},\ \frac{3}{1},\ \frac{5}{2},\ \frac{13}{5},$$

and $\frac{13}{5}$ only differs from the true value of x by a quantity less than $\frac{1}{(5)^2}$ or $\frac{1}{25}$. But the approximation is still nearer; for if we calculate the value of x^{v} from the equation . . . . $\left(\frac{256}{243}\right)^{x^{iv}}=\frac{9}{8}$, we will find that x^{v} is between 2 and 3; therefore $x^{v}=2+\frac{1}{x^{vi}}$; hence the 5th reduction is $\frac{13\times2+5}{5\times2+2}$, or $\frac{31}{12}$.

Therefore $\frac{13}{5}$ differs from the true value of x by a quantity less than $\frac{1}{12\times5}$, or $\frac{1}{60}$. The reduction $\frac{31}{12}$ differs from it by a quantity less than $\frac{1}{(12)^2}$, or $\frac{1}{144}$.

General Method.

Let the equation be $a^{x}=b$.

In forming the successive powers of a, we find that b is comprised between a^{n} and a^{n+1}; we then make $x=n+\frac{1}{x'}$. Substituting this value in the equation, we obtain $a^{n+\frac{1}{x'}}=b$, which can be put under the form $a^{n}\times a^{\frac{1}{x'}}=b$, whence $\left(\frac{b}{a^{n}}\right)^{x'}=a$, or supposing, for greater simplicity, $\frac{b}{a^{n}}=c$, . . . $c^{x'}=a$.

Operating upon this as upon the proposed equation, we will find that x' is comprised between n' and $n'+1$; this will give $x'=n'+\frac{1}{x''}$. Substituting this value in the equation involving x', we will again be led to the resolution of an equation of the form $d^{x''}=c$, (the value of d being $\frac{a}{c^{n'}}$), and so on. Consequently, we will obtain for the value of x, an expression of the form

$$x=n+\cfrac{1}{n'+\cfrac{1}{n''+\cfrac{1}{n'''+\ldots\ldots}}}$$

By continuing the operation, we may approximate as near as we please to the value of x; and the degree of the approximation may always be estimated (Arith. No. 174) *by unity divided by the square of the denominator of the last reduction.*

210. *Remarks.* 1st. When, in the equation $a^x=b$, we suppose $b<a$, as we have $a^0=1$, (No. 24), and $a^1=a$, it follows that x is comprised between 0 and 1. We must then suppose $x=\frac{1}{x'}$.

2d. When b is a fraction, and a greater than unity, we must suppose $x=-y$ in the equation $a^x=b$, which gives $a^{-y}=b$, whence (No. 179) $a^y=\frac{1}{b}$; and as $\frac{1}{b}$ is greater than 1, we determine y by the above method, and the corresponding value of x will be equal to that of y taken *negatively*.

Examples.

$3^x = 15$ - - - - - $x =$ 2,46 to within 0,01.

$10^x = 3$ - - - - - $x =$ 0,477 - - - - - 0,001.

$5^x = \frac{2}{3}$ - - - - - $x =-$ 0,25 - - - - - - 0,01.

$\left(\frac{7}{12}\right)^x = \frac{3}{4}$ - - - - - $x =$ 0,53 - - - - - - 0,01.

Theory of Logarithms.

213. *Introduction.* If in the equation $a^x=y$, (a preserving always the same value), we suppose y to be replaced by all possible absolute numbers, we may, for each value of y determine the corresponding value of x, either exactly, or by approximation.

First suppose, $a>1$.

Making successively $x=0, 1, 2, 3, 4, 5,$ - - - - - - - - - -,
there will result $y=a^0$, or $1, a, a^2, a^3, a^4, a^5,$ - - - - -,
hence, *every value of* y *greater than unity is produced by the powers of* a, *the exponents of which are* positive, *entire, or fractional; and the value of* x *increases as* y becomes greater.

Now make $x=0, -1, -2, -3, -4, -5, \ldots,$
there will result $y=a^0$, or $1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5}, \ldots,$
hence *every value of* y *less than unity, is produced by powers of* a, *of which exponents are* negative, *and the value of* x *is greater, negatively, as the value of* y *approximates to* 0.

When, $a<1$ and equal to a fraction $\frac{1}{a'}$; by making

$$x=0, \ 1, \ 2, \ 3, \ 4, \ 5, \ldots\ldots,$$

we find $y=\left(\frac{1}{a'}\right)^x$, or $1, \frac{1}{a'}, \frac{1}{a'^2}, \frac{1}{a'^3}, \frac{1}{a'^4}, \frac{1}{a'^5}, \ldots\ldots,$
and when we make $x=0, -1, -2, -3, -4, -5, \ldots$
we obtain $y=\left(\frac{1}{a'}\right)^x$ or $1, \ a', \ a'^2, \ a'^3 \ a'^4, \ a'^5. \ldots;$

That is, in the hypothesis $a<1$, all numbers are generated with the different powers of a, in the inverse order of that in which they are generated when we suppose $a>1$.

Hence, *every possible absolute number can be generated with any constant absolute number whatever, by raising it to suitable powers.*

N. B. a must always be supposed to be *different from unity*, because all the powers of 1 are equal to 1.

214. By conceiving that a table has been formed, containing in one column, every entire number, and in another, the *exponents of the powers* to which it is necessary to raise *an invariable number*, to form all these numbers, an idea will be had of a table of *logarithms*.

The logarithm of a number, is the exponent of the power, to which it is necessary to raise a certain invariable number, in order to produce the first number.

The invariable number may in the first place be taken arbitrarily (provided it be > or < 1); but once chosen, it must remain the same for the formation of all numbers, and it is called the *base* of the system of logarithms.

Whatever the base of the system may be, *the logarithm of the base is unity, and the logarithm of* 1 *is* 0.

For, 1st, we have $a^1=a$, whence $\log a=1$,

2d, $a^0=1$, whence $\log 1=0$.

(The word logarithm is commonly denoted by the three first letters *log*, or simply by the first letter *l.*, followed by *a point*, and the number under consideration.)

We will now show some of the advantages of tables of logarithms in making numerical calculations.

215. *Multiplication and Division.* Let it be required to multiply together a series of numbers y, y', y'', y''' .... Denote the base of a system of logarithms by a, (supposed to be calculated), and the logarithms of y, y', y'', y''' .... by x, x', x'', x''' .....

From the definition, (No. 214), we have the equations

$$y=a^x,\quad y'=a^{x'},\quad y''=a^{x''},\quad y'''=a^{x'''}\ \ldots.$$

Multiplying these equations member by member, and applying the rule for the exponents (No. 180), we find

$$yy'y''y'''\ \ldots\ =a^{x+x'+x''+x'''+}\ \ldots$$

Hence

$$\log yy'y''\ldots=x+x'+x''+\ldots=\log y+\log y'+\log y''+\ldots;$$

that is, *the logarithm of a product is equal to the sum of the logarithms of the factors of this product.*

Secondly. Suppose it is required to divide y by y', and let x and x' represent their logarithms; we have the equations $y=a^x$, $y'=a^{x'}$, from which we deduce (No. 180) $\frac{y}{y'}=a^{x-x'}$.

Hence,

$$\log\frac{y}{y'}=x-x'=\log y-\log y';$$

that is, *the logarithm of the quotient is equal to the difference between the logarithms of the dividend and divisor.*

Consequences of these properties. A multiplication can be performed by taking the logarithms of the two factors from the tables, and *adding* them together; this will give the logarithm of the product. Then finding this new logarithm in the tables, and taking the number which corresponds to it, we will obtain the required product. Therefore, *by a simple addition, we find the result of a multiplication.*

In like manner, when one number is to be divided by another, subtract the logarithm of the divisor from that of the dividend, then *find the number* corresponding to this difference; this will be the required quotient. Therefore, *by a simple subtraction we obtain the quotient of a division.*

216. *Formation of Powers, and Extraction of Roots.* Let it be required to raise a number y to a power designed by $\frac{m}{n}$; a denoting the base, and x the logarithm of y, we have the equation

$$y=a^x;$$

whence, raising both members to the power $\frac{m}{n}$

$$y^{\frac{m}{n}}=a^{\frac{m}{n}.x}.$$

Therefore,

$$\log y^{\frac{m}{n}}=\frac{m}{n}.\,x=\frac{m}{n}.\log y;$$

that is, *the logarithm of any power of a number is equal to the product of the logarithm of the number, by the exponent of the power.*

Take $n=1$, as a particular case; there will result. . . . . $\log y^m=m.\log y$, an equation which is susceptible of the above enunciation.

Let $m=1$; there will result

$$\log y^{\frac{1}{n}} \text{ or } \log \sqrt[n]{y}=\frac{1}{n}.\log y;$$

that is, *the logarithm of any root of a number is equal to the logarithm of this number, divided by the index of the root.*

Consequence. To form any power of a number, take the logarithm of this number from the tables, multiply it by the exponent of the number, then find the number corresponding to this product, it will be the required power.

In like manner, to extract the root of a number, divide the logarithm of the proposed number by the index of the root, then find the number corresponding to the quotient, it will be the required root. Therefore, *by a simple multiplication, we can raise a quantity to a power, and extract its root by a simple division.*

217. The properties just demonstrated are independent of any system of logarithms; but the consequences which have been deduced from them, that is, the use that may be made of them in numerical calculations, supposes the construction of a

table, containing all the numbers in one column, and the *logarithms* of these numbers in another, calculated from a given *base*. Now, in calculating this table, it is necessary, in considering the equation $a^x=y$, to make y pass through all possible states of magnitude, and determine the value of x corresponding to each of the values of y, by the method of No. 209.

The tables in common use, are those of which the base is 10, and their construction is reduced to the resolution of the equation $10^x=y$. Making in this equation, y successively equal to the series of natural numbers, 1, 2, 3, 4, 5, 6, 7,- - - -, we have to resolve the equations

$$10^x=1,\ 10^x=2,\ 10^x=3,\ 10^x=4\ ---$$

We will moreover observe, that it is only necessary to calculate directly (by the method of No. 209) the logarithms of the prime numbers 1, 2, 3, 5, 7, 11, 13, 17, - - - -; for as all the other entire numbers result from the multiplication of these factors, their logarithms may be obtained (No, 215) by the addition of the logarithms of the prime numbers.

Thus, since 6 can be decomposed into 2×3, we have

$$\log 6=\log 2+\log 3,$$

in like manner. $24=2^3\times3$; hence $\log 24=3\log 2+\log 3$.

Again $360=2^3\times3^2\times5$; hence

$$\log 360=3\log 2+2\log 3+\log 5.$$

It is only necessary to place the logarithms of the entire numbers in the tables; for, by the property of division (No. 215) we obtain the logarithm of a fraction by subtracting the logarithm of the divisor from that of the dividend.

218. If we had a table of logarithms constructed, it would be easy to construct from this as many as we wished.

For let a be the base of a system already formed, and b be the base of a system which it is required to construct; let N represent any number whatever, and log N and X, its two logarithms calculated from the bases a and b; we have the equation $b^X=N$.

Whence taking the logarithms of both members, in the system of which the base is a, $X.\log b=\log N$.

Hence, $$X=\frac{\log N}{\log b}.$$

This proves that, *knowing the logarithm of a number in one system, in order to have the logarithm of the same number in another system, we must divide the logarithm of the number calculated in the first system, by the logarithm of the base of the second system, also calculated in the first system.*

Thus the logarithm of 4, in the system of which the base is 3, is $\frac{\log 4}{\log 3}$, log 4 and log 3 being calculated in the known system of which the base is 10.

Let N, N′, N″ - - - - be a series of numbers, a the base of a system already formed, b that of a system to be constructed, we have the equations

$$X=\frac{\log N}{\log b}=\frac{1}{\log b}.\log N\,;\ X'=\frac{1}{\log b}.\log N'\,;\ X''=\frac{1}{\log b}.\log N''$$

- - - ; from which we see that, a table being already formed, in order to construct a new one from it, we must *multiply the logarithms of the first system by the constant quantity* $\frac{1}{\log b}$. This constant quantity which serves to pass from one table to another, is called the *modulus* of the new table, with reference to the old.

§ III. *Logarithmic and Exponential Series.*

The method of resolving the equation $a^x=b$, exposed in No. 209, is sufficient to give an idea of the construction of logarithmic tables; but this method is very laborious when we wish to approximate very near the value of x. Analysts have discovered much more expeditious methods for constructing new tables, or for verifying those already calculated. These methods consist in the development of logarithms into series.

231. Let it be required to develope a number, represented by y, into a series, and apply the method of indeterminate coefficients (No. 189).

It is immediately visible that we cannot suppose

$$ly=A+By+Cy^2+Dy^3+\&c.\,;$$

for making $y=0$, the first member reduces (219) to an *infinite negative*, or an *infinite* positive quantity, according as the base is greater or less than 1; whilst the second member would be reduced to A.

Neither can we suppose $l.\,y=Ay+By^2+\&c.,$
since $y=0$ gives $l.\,0$, or $-\infty=0$;
but if we put y under the form $1+x$, and suppose

$$l.\,(1+x)=Ax+Bx^2+Cx^3+Dx^4+\text{----}\ (1),$$

making $x=0$, the equation is reduced to $l.\,1=0$, which does not present any absurdity.

In order to determine the coefficients A, B, C...., we will follow the process of No. 190. Substituting z for x, the equation becomes

$$l.\,(1+z)=Az+Bz^2+Cz^3+Dz^4+\text{----}\ (2).$$

Subtracting the equation (2) from (1), we obtain

$$l.\,(1+x)-l.(1+z)=A(x-z)+B(x^2-z^2)+C(x^3-z^3)+\ldots(3).$$

The second member of this equation is divisible by $x-z$; we will see if we can by any artifice put the first under such a form that it shall also be divisible by $x-z$.

We have $l.\,(1+x)-l.\,(1+z)=l.\dfrac{1+x}{1+z}=l.\left(1+\dfrac{x-z}{1+z}\right)$;

but since $\dfrac{x-z}{1+z}$ can be regarded as a single number u, we can develope $l.(1+u)$, or $l.\left(1+\dfrac{x-z}{1+z}\right)$, in the same manner as $l.\,(1+x)$, which gives

$$l.\left(1+\frac{x-z}{1+z}\right)=A\frac{x-z}{1+z}+B\left(\frac{x-z}{1+z}\right)^2+C\left(\frac{x-z}{1+z}\right)^3+\ldots.$$

Substituting this development for $l.(1+x)-l.(1+z)$ in the equation (3), and dividing both members by $x-z$, it becomes

$$A.\frac{1}{1+z}+B\frac{x-z}{(1+z)^2}+C\frac{(x-z)^2}{(1+z)^3}+\ldots..$$
$$=A+B(x+z)+C(x^2+xz+z^2)+\ldots.$$

Since this equation, like the preceding, must be verified by any values of x and z, make $x=z$, and there will result

$$\frac{A}{1+x}=A+2Bx+3Cx^2+4Dx^3+5Ex^4+\ldots.$$

Whence, clearing the fraction, and transposing

$$0=\begin{array}{l|l|l|l|l} A+2B & x+3C & x^2+4D & x^3+5E & x^4+\ldots\ldots \\ -A\ +A & \ \ +2B & \ \ +3C & \ \ +4D & \end{array}$$

Putting the coefficients of the different powers of x equal to zero, we obtain the series of equations

$$A-A=0,\ 2B+A=0,\ 3C+2B=0,\ 4D+3C=0\ ---;$$

whence

$$A=A,\ B=-\frac{A}{2},\ C=-\frac{2B}{3}=+\frac{A}{3},\ D=-\frac{3C}{4}=-\frac{A}{4}\ldots\ldots$$

The law of the series is evident; the coefficient of the n^{th} term is equal to $\mp\frac{A}{n}$, according as n is even or odd; hence we will obtain for the development of $l.(1+x)$,

$$l.(1+x)=Ax-\frac{A}{2}x^2+\frac{A}{3}x^3-\ldots.$$

$$=A\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\ldots\ldots\right)$$

N. B. By the above method, the coefficients B, C, D, E.... have all been determined in functions of A; but A remains entirely undetermined. Now this should be the case from the nature of the expression which it has been proposed to develope; for since an infinite number of systems of logarithms can be formed, the general development of l. $(1+x)$ must necessarily involve an indeterminate quantity, which serves to distinguish the systems from each other. Moreover, we have seen (218) that the logarithms of the same number, taken in two systems, only differ by a factor, *which is the same for all numbers;* therefore the indeterminate quantity ought to be a factor of the series.

The number A is (218), the *modulus*, the particular value of which characterizes the system of logarithms.

232. The most simple hypothesis that can be made is $A=1$, which gives

$$l'.(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\ ---\ (5)$$

denoting this particular system of logarithms by $l'.(1+x)$.

By giving all possible values to x, we will form all the logarithms of this system, which is called the *natural or Naperien System*. We will now proceed with the formation of this system, since it will be easy to form all the other systems from it, either by giving different values to A, or by the formula of No. 218.

In the series (5) make $x=0$; it becomes $l'.1=0$.

Again, let $x=1$, there will result $l'.2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$; a series which does not converge rapidly, and in which it would be necessary to take a great number of terms for a near approximation; for example, it would be necessary to take the 100 first terms to have the value to within 0,01, (No. 184). In general, this series will not serve for determining the logarithms of entire numbers, since for every number greater than 2 we would obtain a series in which the terms would go on increasing continually.

The following are the principal transformations for converting the above series into converging series for the purpose of obtaining the logarithms of entire numbers, which are the only logarithms placed in the tables.

First Transformation. In the series (5) by making $x=\frac{1}{y}$; and observing that $l'\left(1+\frac{1}{y}\right)=l'(1+y)-l'y$, it becomes

$$l'(1+y)-l'y=\frac{1}{y}-\frac{1}{2y^2}+\frac{1}{3y^3}-\frac{1}{4y^4}+\cdots \quad (6)$$

This series becomes more converging as y increases; besides the first member of this equation expresses the difference between two consecutive logarithms. Making successively $y=2$, 3, 4, 5 - - - - we have

$$l'3-l'2=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}+\cdots$$

$$l'4-l'3=\frac{1}{3}-\frac{1}{18}+\frac{1}{81}-\frac{1}{324}+\cdots$$

$$l'5-l'4=\frac{1}{4}-\frac{1}{32}+\frac{1}{192}-\frac{1}{1024}+\cdots$$

The first series will give the logarithm of 3 by means of the logarithm of 2; the second, the logarithm of 4, in functions of the logarithm of 3 - - - &c. The degree of approximation can be estimated (No. 184), since the series are composed of terms alternately positive and negative.

Second Transformation. A much more converging series is obtained in the following manner.

In the series $l'(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+ - - -$ substitute $-x$

for x; and it becomes $l'(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}- - - -$

Subtracting the second series from the first, observing that $l'(1+x)-l'(1-x)=l'\frac{1+x}{1-x}$, we obtain

$$l'\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+ - - -\right)$$

This series will not converge very rapidly unless x is a very small fraction, in which case, $\frac{1+x}{1-x}$ will be greater than unity, but will differ very little from it.

Take $\frac{1+x}{1-x}=1+\frac{1}{z}$ (z being an entire number);

we have $(1+x)z=(1-x)(z+1)$: whence $x=\frac{1}{2z+1}$.

Hence the preceding series becomes $l'\left(1+\frac{1}{z}\right)$ or

$$l'(z+1)-l'z=2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+ - - - -\right)$$

This series also gives the difference between two consecutive logarithms, but it converges much more rapidly than the series (6).

Making successively $z=1, 2, 3, 4, 5 \ldots$, we find

$$l'2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\ldots\right),$$

$$l'3-l'2=2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\frac{1}{7.5^7}+\ldots\ldots\right),$$

$$l'4-l'3=2\left(\frac{1}{7}+\frac{1}{3.7^3}+\frac{1}{5.7^5}+\frac{1}{7.7^7}+\ldots\ldots\right).$$

- -

- -

Let $z=100$; there will result

$$l'101=l'100+2\left(\frac{1}{201}+\frac{1}{3(201)^3}+\frac{1}{5(201)^5}+\ldots\right);$$

whence we see, that knowing the logarithm of 100, the first term of the series is sufficient for obtaining that of 101 to seven places of decimals.

There are formulas more converging than the above, which serve to obtain logarithms in functions of others already known, but the preceding are sufficient to give an idea of the facility with which tables may be constructed.

233. The Naperien logarithms being calculated, those of any other system can easily be obtained.

For example, in order to form the common system, multiply (No. 218) each Naperien logarithm by the *modulus* $\frac{1}{l'10}$. This number has been calculated to as great a degree of approximation as can be desired, and its value in decimals is .... 0,4342944819.... ; it is the *modulus for passing from the natural system to that of which the base is* 10.

The modulus also expresses the *common logarithm of the base of the Naperien system*; for, calling this base e, we have the equation $e^{l'10}=10$; whence, taking the logarithms in the common system,

$$l'10\times l.e=l.10=1\text{; therefore, } l.e=\frac{1}{l'10}=0,43429\ldots\ldots$$

As the common tables are supposed to be constructed, we can make use of them to determine the number corresponding to the above logarithm, and we find

$$0.4342944819\ldots=l.e=l.2.718281828\ldots$$

Therefore,

$$e=2.718281828.$$

234. *Development of the exponential* a^x *into series.* The connexion between exponential quantities and logarithms leads us to investigate the possibility of developing a^x according to the powers of x, which would give the development of a number in functions of its logarithm, the inverse of the preceding question.

Suppose this development found, and let

$$a^x=1+Ax+Bx^2+Cx^3+Dx^4+\text{- - - -}(1);$$

by making $x=0$, the equation is reduced to $a^0=1$, an exact result; therefore this form for the development is admissible.

To determine A, B, C, D . . . , substitute z for x in (1); it will become

$$a^z=1+Az+Bz^2+Cz^3+Dz^4+\text{- - - - - -}(2);$$

subtracting the equation (2) from the equation (1), we have the equation

$$a^x-a^z=A(x-z)+B(x^2-z^2)+C(x^3-z^3)+D(x^4-z^4)+\ldots(3);$$

the second member of which is divisible by $x-z$; we will therefore try to convert the first member into an expression involving $x-z$ as a factor. Now a^x-a^z can be put under the form $a^z(a^{x-z}-1)$; hence, by substituting $x-z$ for x in the series (1), it becomes

$$a^z(a^{x-z}-1)=a^z[A(x-z)+B(x-z)^2+C(x-z)^3+\ldots.]:$$

then substituting for a^x-a^z, in the equation (3), the value just obtained, and dividing both members by $x-z$, we find

$$a^z[A+B(x-z)+C(x-z)^2+\ldots.]$$
$$=A+B(x+z)+C(x^2+xz+z^2)+\ldots.$$

Making $x=z$ in this last equation, it is reduced to

$$a^x.A=A+2Bx+3Cx^2+4Dx^3+5Ex^4+\text{- - - -};$$

or, replacing a^x by its development (1),

$$A+A^2x+ABx^2+ACx^3+\ldots=A+2Bx+3Cx^2+4Dx^3\ldots.$$

Equating the coefficients of the same powers, we obtain the equations

$$A=A,\quad A^2=2B,\quad AB=3C.\quad AC=4D, \; - - -$$

whence we deduce

$$A=A,\ B=\frac{A^2}{2},\ C=\frac{A^3}{2.3},\ D=\frac{A^4}{2.3.4}.$$

The law of the series is manifest; $\frac{A^n}{2.3.4\ldots n}$ being the term of the series (1), which has n terms before it.

We perceive that the coefficients B, C, D - - - are expressed in functions of the coefficient A, which still remains *indeterminate;* nevertheless it has a fixed value, which can be found by the following artifice; a^x can be put under the form $(1+\overline{a-1})^x$ $=(1+b)^x$ by supposing $a-1=b$.

Now, developing $(1+b)^x$ by the binomial theorem, we have

$$(1+b)^x=1+x.b+x\,\frac{x-1}{2}\,b^2+x.\frac{x-1}{3}.\frac{x-2}{3}\,b^3+ - - -$$

The sum of those terms of this development which involve the first power of x is equal to $\left(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\ldots\right)x$; moreover the coefficient of this part of the development is represented by A, in the series (1); hence, replacing b by $a-1$, we have

$$A=a-1-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+ - - - =k.$$

Substituting k for A in the values obtained for B, C, D - - -, and then substituting these values in the series (1), we will finally obtain for the development of a^x

$$a^x=1+kx+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}+\frac{k^4x^4}{1.2.3.4}+\frac{k^5x^5}{1.2.3.4.5}+ - - - \quad (4).$$

235. *Consequences.* If we suppose $x=1$ in this series it becomes

$$a=1+k+\frac{k^2}{1.2}+\frac{k^3}{1.2.3}+\frac{k^4}{1.2.3.4}+ - - -$$

Whence we see that a is expressed in functions of k, as k was already in functions of a, viz. $k=a-1-\frac{(a-1)^2}{2}+ - - -$

The particular value (e) of a, corresponding to $k=1$, we find to be,

$$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{ - - -}$$

The twelve first terms of this series, which converges pretty rapidly, gives $e=2.7182818$, exact to within 0.0000001.

This number, which corresponds to $k=1$, being very remarkable, we will seek for the development of e^x. For this purpose it is only necessary to make $a=e$, and $k=1$ in the formula (4), which gives

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\text{ - - -}$$

The two quantities a and k have another relation with each other, which can be obtained by means of the particular number e.

For, suppose $x=\frac{1}{k}$ in the equation (4); we find

$a^{\frac{1}{k}}=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+$ - - ; but we have already

$e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ - - - - ; hence $a^{\frac{1}{k}}=e$, or $a=e^k$.

Taking the logarithm of both members of this equation in the system of which the base is a, it becomes, since $\text{l}.e=1$,

$$\text{l}.\ a=k;$$

or, substituting for k its value $a-1-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\ldots$,

$$\text{l}.\ a=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\frac{(a-1)^4}{4}+\text{ - - - - };$$

and supposing $a=1+x$, whence $a-1=x$, we obtain

$$\text{l}.\ (1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\text{- - - -},$$

which series is identical with that which gives the natural logarithms (No. 232).

Whence we see that e or 2.7182818..., is the base of the Naperien system. It also follows from this, that the development of *exponentials* into series leads to logarithmic series.

CHAPTER VI.

General Theory of Equations.

THE most celebrated analysts have tried to resolve equations of any degree whatever, but hitherto their efforts have been unsuccessful with respect to equations of a higher degree than the fourth. However, their investigations on this subject have conducted them to some properties common to equations of every degree, which they have since used, either to resolve certain classes of equations, or to reduce the resolution of a given equation to that of one more simple. In this chapter it is proposed to make known these properties, and their use in facilitating the resolution of equations.

§ I. *Divisibility of Entire Functions. General properties of Equations. Complete Theory of the Greatest Common Divisor.*

DIVISIBILITY OF ENTIRE FUNCTIONS.

239. The development of the properties relating to equations of every degree, leads to the consideration of polynomials of a particular nature, and entirely different from those considered in the first chapter. These are, expressions of the form

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Tx + U,$$

in which m is a positive whole number; but the coefficients A, B, C, ... T, U, denote any quantities whatever, that is, entire or fractional quantities, commensurable or incommensurable. Now, in algebraic division, as exposed in chapter 1st, the object was this, viz.: *given two polynomials* entire, with reference to all the letters and particular numbers involved in it, *to find a third polynomial of the same kind, which, multiplied by the second, would reproduce the first.*

But when we have two polynomials,

$$Ax^m+Bx^{m-1}+Cx^{m-2}+\ .\ .\ .\ .+Tx+U,$$
$$A'x^n+B'x^{n-1}+C'x^{n-2}+\ .\ .\ .\ .+T'x+U',$$

which are *necessarily* entire only with respect to x, and in which the coefficients A, B, C . . . ., A′, B′, C′ . . . ., may be any quantities whatever, it may be proposed *to find a third polynomial*, of the same form and nature as the two preceding, *which, multiplied by the second, will reproduce the first.*

The process for effecting this division is analogous to that for common division; but there is this difference, viz.: In this last, *the first term of each partial dividend must be exactly divisible by the first term of the divisor;* whereas, in the new kind of division, we divide the first term of each partial dividend (that is, the part affected with the highest power of the principal letter,) by the first term of the divisor, whether the coefficient of the corresponding partial quotient is entire or fractional; and the operation is continued *until a quotient is obtained, which, multiplied by the divisor, will cancel the last partial dividend*, in which case the division is said to be exact; or, *until a remainder is obtained, of a degree less than that of the divisor*, with reference to the principal letter, in which case the division is considered impossible, since by continuing the operation, quotients would be obtained containing the principal letter affected with *negative exponents*, or this same letter in the denominator of them, which would be contrary to the nature of the question, which requires that the quotient should be of the same form as the proposed polynomials.

240. To distinguish polynomials which are entire with reference to a letter, x for example, but the coefficients of which are any quantities whatever, from ordinary polynomials, that is, from polynomials which are entire with reference to all the letters and particular numbers involved in them, it has been agreed to call the first *entire functions* of x, and those polynomials which

divide them exactly, in the sense above mentioned, ***relative divisors***.

241. It follows from these definitions, that when an entire function of x has another polynomial of the same kind for a relative divisor, *the product of this divisor by any factor*, independent of the principal letter, ***is also a relative divisor of the first polynomial***.

For suppose we have

$$\frac{Ax^m+Bx^{m-1}+\ldots+U}{A'x^n+B'x^{n-1}+\ldots+U'}=A''x^{m-n}+B''x^{m-n-1}+\ldots+U''.$$

Let K be any factor independent of x; we necessarily have

$$\frac{Ax^m+Bx^{m-1}+\ldots+U}{K(A'x^n+B'x^{n-1}+\ldots+U')}=\frac{A''}{K}x^{m-n}+\frac{B''}{K}x^{m-n-1}+\ldots+\frac{U''}{K};$$

and the second member of this identity is an entire function of x; hence $K(A'x^n+B'x^{n-1}+\ldots)$ is a relative divisor of $Ax^m+Bx^{m-1}+\ldots$

242. First. *Every entire function* D *which will exactly divide the product* A×B *of two other entire functions, and which is prime with one of them,* A *for example, will necessarily divide the other* B.

(Two functions are said to be prime with each other, when they do not contain a common factor involving x).

Demonstration. First, when A is independent of x, the proposition is evident, for let

$$\frac{A\times B}{D}=Px^n+Qx^{n-1}+\ldots+Tx+U;$$

by dividing both members by A, we have

$$\frac{B}{D}=\frac{P}{A}x^n+\frac{Q}{A}x^{n-1}+\ldots+\frac{T}{A}x+\frac{U}{A}.$$

Now, A being, by hypothesis, independent of x, the second member of this equation is an entire function. Hence, &c.

We will now suppose that A is a function of x, and of a higher degree than D.

Dividing A by D, we have the equation

$$A=DQ+R \text{ - - - - } (1),$$

(the remainder R is not 0, since A and D are supposed to be prime with each other.)

Multiplying both members of the equation (1) by B, and then dividing by D, we have

$$\frac{AB}{D}=BQ+\frac{BR}{D}.$$

Now, $\frac{AB}{D}$ and BQ being entire functions, $\frac{BR}{D}$ must also be one, that is D a relative divisor of AB, is also a relative divisor of BR; and if R was independent of x, which might be the case, the proposition would be demonstrated.

But again suppose R is a *function* of x, and divide D by R; we obtain the new equation

$$D=RQ'+R' \quad - - - - \quad (2).$$

(R is not equal to 0, for in this case D would be divisible by R, and so would A, in virtue of equation (1); therefore A and D would have a common factor involving x, which would be contrary to the hypothesis.)

From equation (2) we deduce

$$B=\frac{BRQ'}{D}+\frac{BR'}{D};$$

Now, the first member B, and the first term $\frac{BRQ'}{D}$ of the second member, are entire functions; hence $\frac{BR'}{D}$ must also be an entire function, that is, D being a relative divisor of AB and of BR, it must necessarily be a relative divisor of BR'; and if R' was independent of x, the theorem would be demonstrated. But by again supposing R' a *function* of x, we may divide R by R', which will give a new remainder R'', different from 0, without which R', a relative divisor R, would also be a relative divisor of D, in virtue of the equality (2), and consequently of A, from the equality (1); hence A and D would have a common factor involving x, which would be contrary to the hypothesis. Hence, D a relative divisor of AB, BR, BR', is necessarily a relative divisor of BR''.

We will now observe that the degrees (with respect to x) of the remainders R, R', R'' - - - go on diminishing; therefore, as *we cannot* obtain a remainder equal to zero, immediately after a *remainder* involving x (for this would be to suppose this last remainder a relative divisor of D and A), we must obtain a remainder $R^{(n)}$ independent of x, and such that the product $B \times R^{(n)}$ will be divisible by D, which requires that B itself should be divisible by D.

N. B. The degree of A has been supposed higher than that of D. If it were not, we would divide D by A; and so on.

243. *No prime function* D, *prime with two or more entire functions* A, B, C...., *can divide their product.*

For, in order that D may divide ABC . . . ., or A×BC . . ., as it is prime with A, it must divide BC . . . .; in like manner, in order that it may divide BC . . . . ., or B×CE . . . . ., it must divide CE . . . . . In this way we would prove that D would divide the last entire function, which would be contrary to the hypothesis.

244. Second. *Every entire function* D, *of the first degree with reference to* x, *which divides the product* AB *of two entire functions, will divide one of these functions.*

For, suppose that D would not divide A, it could only be prime with A; hence, from the first principle of No. 242, D must divide B.

245. Consequences. 1st. *Every entire function* D, *of the first degree with reference to* x, *which will divide* A^2, *and, in general, any power* A^n *of an entire function* A, *will divide* A; for A^2 is the same as A×A. Now every polynomial *of the first degree with reference to* x, which divides this product, will, by the second principle, divide one of the factors.

In like manner, A^3 being equal to $A \times A^2$, every polynomial *of the first degree with reference to* x, which divides A^3, will divide A or A^2, and so on.

2d. *When* A *and* B *are entire functions, prime with each other,* A^n *and* $B^{n'}$ *will also be prime with each other.*

For every polynomial *of the first degree with reference to* x, which divides A^n and $B^{n'}$ at the same time, will also divide A and B, which would be contrary to the hypothesis.

246. Third. *Every entire function* A, *which is divisible by two or more entire functions* D, D′, D″ . . . ., *prime with each other, is divisible by their product* DD′D″ . . . .

For we have by hypothesis, A=DQ, Q being an entire function. Now since D′ will divide A, it will also divide its value DQ; and as D′ is prime with D, it must divide Q, and we have Q=D′Q′; whence, substituting in the expression for A, we have A=DD′Q′, or $\frac{A}{DD'}=Q'$, an entire function. In like manner, D″ dividing A, will divide DD′×Q′; but it is prime with each of the polynomials D, D′, and consequently with DD′; hence D″ will divide Q′, and we have

$$Q'=D''Q'', \text{ whence } A=DD'D''Q'',$$

or

$$\frac{A}{DD'D''}=Q'',$$

and so on.

247. *Consequence.* If d, d', d'' . . . ., being entire functions, and prime with each other, another entire function A, has for particular relative divisors certain powers of d, d', d''. . . ., viz. d^n, $d'^{n'}$, $d''^{n''}$ . . . ., the same polynomial A is divisible by the product $d^n.\ d'^{n'}.\ d''^{n''}$. . . . ., and by all the polynomials that can be formed by multiplying together two and two, three and three, ..., the different powers of d, d', d'' . . . ., comprehended between the first and that which is denoted by n for d, by n' for d' . . . . .

For the polynomials d, d', d'' . . . . being prime with each other. d^2, d^3, ...; d'^2, d'^3. . . . ., will also be prime with each other, (No. 246); hence the products two and two, three and three, ..., are so many relative divisors of A.

248. General scholium. *When an entire function* P *has been formed by the multiplication of several entire functions* . . . . . P′, P″, P‴, P^{iv} . . . ., *in such a manner that we have*

$$P=P'P''P'''P^{iv}\ldots.,$$

this polynomial can have no relative divisors of the first degree with reference to x, *except those which enter into the different polynomials* PP′P″ . . . ., *or the products of these relative divisors by factors independent of* x.

General Properties of Equations.

249. Every complete equation of the m^{th} degree (m being a *positive* whole number) may, by the transposition of terms, and

by the division of both members by the coefficient of x^m, be put under the form

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots+Tx+U=0;$$

P, Q, R . . . . T, U, being coefficients taken in the most general algebraic sense.

Any expression, whatever the nature of it may be, that is, *numerical* or *algebraic*, *real* or *imaginary*, *which, substituted in place of* x *in the equation, renders its first member equal to* 0, is called a *root* of this equation.

250. As every equation may be considered as the algebraic translation of the relations which exist between the given and unknown quantities of a problem, we are naturally led to this principle; viz. EVERY EQUATION *has at least one root.* Indeed, the conditions of the enunciation may be incompatible, but then we must suppose that we will be warned of it by some *symbol of absurdity*, such as a formula, containing as a necessary operation, the extraction of an even root of a negative quantity; yet there will still exist an expression which, substituted for x in the equation, will satisfy it. We will admit this principle, which we will have occasion to verify hereafter for most equations.

The following proposition may be regarded as the fundamental property of the theory of equations.

251. First Property. *If* a *is a root of the equation* - - - - $x^m+Px^{m-1}+Qx^{m-2}+$ - - - $Tx+U=0$, *the first member of it is divisible by* $x-a$; and reciprocally, *if a factor of the form* $x-a$ *will divide the first member of the proposed equation*, a *is a root of it*.

For, perform the division, and see what takes place when the operation is continued until the exponent of x becomes 0 in the first term of the dividend.

(This operation is of the nature of that spoken of in No. 239, since a, P, Q, - - - - are any quantities whatever.)

$$+Px^{m-1}+Qx^{m-2}+\ldots+Tx+U \quad\Big|\; x-a$$

$$\begin{array}{l|l} +a & x^{m-1} \\ +P & \end{array} \qquad\qquad \Big|\; x^{m-1} \begin{array}{r|l} +a & x^{m-2} \\ +P & \end{array} \begin{array}{r|l} +a^2 & x^{m-3} \\ +Pa & \\ +Q & \end{array} +\ldots \begin{array}{l} +a^{m-1} \\ +Pa^{m-2} \\ +Qa^{m-3} \\ +\ldots \\ +T \end{array}$$

$$\begin{array}{r|l} +a^2 & x^{m-2}+\ldots \\ +Pa & \\ +Q & \end{array}$$

- -
- -
- -

By reflecting a little upon the manner in which the partial quotients are obtained, we will first discover from analogy, and afterwards by a method employed several times (No. 31 and 86), *a law of formation* for the coefficients of these quotients ; and we may conclude, 1st. that there will be m partial quotients, 2d. that the coefficient of the m^{th} quotient, that is of x, must be

$$a^{m-1}+Pa^{m-2}+Qa^{m-3}+ \text{----} +T,$$

T being the coefficient of the last term but one of the proposed equation.

Hence, by multiplying the divisor by this quotient, and reducing it with the dividend, we obtain for a remainder

$$a^m+Pa^{m-1}+Qa^{m-2}+ \text{---} +Ta+U.$$

Now, by hypothesis a is a root of the equation ; hence, ***this remainder is nothing***, since it is nothing more than the result of the substitution of a for x in the equation; *therefore the division is exact.*

Reciprocally, if $x-a$ is an exact divisor of $x^m+Px^{m-1}+...$, the remainder $a^m+Pa^{m-1}+...$ will be *nothing ;* therefore, (No. 249), a is a root of the equation.

252. From this it results that, in order to discover whether a binomial of the form $x-a$ is an exact divisor of a polynomial involving x, it will be sufficient to see if the result of the substitution of a for x, is equal to 0.

To ascertain whether a is a root of a polynomial involving x and placed equal to 0, it will be sufficient to try the division of it by $x-a$. If it is exact, we may be certain that a is a root of the equation.

253. *Remark.* By inspecting the quotient of the division of No. 251, we perceive the following law for the coefficients: *Each coefficient is obtained by multiplying that which precedes it by the root* a, *and adding to the product that coefficient of the proposed equation, which occupies the same rank as that we wish to obtain in the quotient.*

Thus, the coefficient of the 3d term, a^2+Pa+Q, is equal to $(a+P)a+Q$, or to the product of the preceding coefficient $a+P$, by the root a, augmented by the coefficient Q of the 3d *term of the* proposed equation.

The coefficient of the 4th term is

$$(a^2+Pa+Q)a+R, \text{ or } a^3+Pa^2+Qa+R.$$

This law should be remembered.

254. Second property. *Every equation involving but one unknown quantity, has as many roots as there are units in the exponent of its degree, and no more.*

Let the proposed equation be

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots\ldots+Tx+U=0.$$

Since (No. 250) every equation has at least one root, if we denote the root that the above equation must necessarily have by a, its first member will be divisible by $x-a$, and we will have the identity

$$x^m+Px^{m-1}+\ldots\ldots=(x-a)\,(x^{m-1}+P'x^{m-2}+\ldots.)\ldots(1).$$

But by supposing $x^{m-1}+P'x^{m-2}+\ldots.=0$,
we obtain an equation which has at least one root.

Denote this root by b, we have (No. 251)

$$x^{m-1}+P'x^{m-2}+\ldots.=(x-b)\,(x^{m-2}+P''x^{m-3}+\ldots),$$

an equality which, multiplied member by member, with the equality (1), gives

$$x^m+Px^{m-1}+\ldots=(x-a)\,(x-b)\,(x^{m-2}+P''x^{m-3}+\ldots.)\ldots(2).$$

Reasoning upon the polynomial $x^{m-2}+P''x^{m-3}+\ldots\ldots$ as upon the preceding polynomial, we have

$$x^{m-2}+P''x^{m-3}+\ldots.=(x-c)\,(x^{m-3}+P'''x^{m-4}+\ldots.),$$

which, multiplied by the equality (2), gives

$$x^m+Px^{m-1}+\ldots.=(x-a)\,(x-b)\,(x-c)\,(x^{m-3}+\ldots.)\ldots(3).$$

Observe that for each indicated factor of the first degree with reference to x, the degree of x in the polynomial is diminished by unity; therefore, after $m-2$ factors of the first degree have been divided out, the exponent of x will be reduced to $m-(m-2)$, or 2; that is, we will obtain a polynomial of the second degree with reference to x, which (No. 98) can be decomposed into the product of two factors of the first degree, $(x-k)$ $(x-l)$. Now, as the $m-2$ factors of the first degree have already been indicated, it follows that we have the identity

$$x^m+Px^{m-1}+\ldots.=(x-a)\,(x-b)\,(x-c)\ldots(x-k)\,(x-l).$$

From which we see, that the *first member of the proposed equation is decomposed into the product of* m *factors of the first degree.*

As there is *a root* (No. 251) corresponding to each divisor of the first degree, it follows that the *m* factors of the first degree $x-a$, $x-b$, $x-c$ . . . ., give the *m* roots *a*, *b*, *c* . . . . for the proposed equation.

Moreover, it evidently follows from the proposition of No. 248, that the polynomial $x^m + Px^{m-1}$. . . . cannot have any relative divisors of the first degree except $x-a$, $x-b$, $x-c$, ... $x-k$, $x-l$, or the product of one of these relative divisors by a factor independent of x. Hence the equation can have no other roots than *a*, *b*, *c* . . . *k*, *l*, since if it had a root α, different from *a*, *b*, *c* . . . . *l*, it would follow that it would have a relative divisor $x-\alpha$, different from $x-a$, $x-b$, $x-c$ . . . . $x-l$, which is impossible.

Finally, *every equation of the* m$^{\text{th}}$ *degree has* m *roots, and can have no more.*

255. There are some equations in which the number of roots is apparently less than the number of units in the exponent of their degree. They are those in which the first member is the product of equal factors, such as the equation

$$(x-a)^4(x-b)^3(x-c)^2(x-d)=0,$$

which has but *four* different roots, although it is of the 10th degree.

It is evident that no quantity α, different from *a*, *b*, *c*, *d*, can verify it; for if it had this root α, the first member would be divisible by $x-\alpha$, which is impossible, (No. 248), since $x-\alpha$ is prime with $x-a$, $x-b$, $x-c$, $x-d$.

But this is no reason why the proposed equation should not have ten roots, *four* of which are equal to *a*, *three* equal to *b*, *two* equal to *c*, and *one* equal to *d*.

256. Consequence of the second property.

The first member of every equation of the m^{th} degree, having *m* divisors of the first degree, of the form

$$x-a,\ x-b,\ x-c, \ldots .\ x-k,\ x-l,$$

if we multiply these divisors together, *two and two*, *three and three*, ..., we will obtain as many relative divisors of the second, third, ... degree with reference to x, as we can form different combinations of m quantities, taken two and two, three and three, ... Now the number of these combinations is (No. 150) expressed by $m.\frac{m-1}{2}$, $m.\frac{m-2}{3}$. . . . Besides, these products are (No. 248) the only divisors of the same degree that the first member of the proposed equation can have, unless we consider the products of these relative divisors by factors independent of x.

Thus the proposed equation contains $m.\frac{m-1}{2}$ divisors of the second degree, $m.\frac{m-1}{2}.\ \frac{m-2}{3}$ divisors of the third degree, and so on.

257. *Composition of equations.* If in the identical equation

$$x^m+Px^{m-1}+\ -\ -\ -\ =(x-a)(x-b)(x-c)\ -\ -\ -\ (x-l),$$

we perform the multiplication of the m factors of the second member, (see No. 151), and compare the terms of the two members, we will find the following relations between the coefficients P, Q, R, - - - T, U, and the roots a, b, c, - - - k, l, of the proposed equation, viz.

$$-a-b-c\ -\ -\ -\ -\ -k-l=\text{P, or } a+b+c+\ -\ -\ -\ +k+l=-\text{P};$$
$$ab+ac+\ -\ -\ -\ -\ -\ +kl=\text{Q}\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -$$
$$-abc-abd\ -\ -\ -\ -\ -ikl=\text{R, or } abc+abd\ -\ -\ -\ -\ +ikl=-\text{R};$$

- -

- -

$$\pm abcd\ -\ -\ -\ -\ kl=\text{U, or } abcd\ -\ -\ -\ -\ kl=\pm\text{U}.$$

(The double sign has been placed in the last relation, because the product $-a\times-b\times-c$ - - - $\times-l$ will be *plus* or *minus* according as the degree of the equation is *even* or *odd*.)

Hence, 1st. The algebraic sum of the roots, taken with contrary signs, is equal to the coefficient of the second term; or, the algebraic sum of the roots themselves, is equal to the coefficient of the second term taken with a contrary sign.

2d. The sum of the products of the roots taken two and two with their respective signs, is equal to the coefficient of the *third term.*

3d. The sum of the products of the roots three and three taken with contrary signs, is equal to the coefficient of the fourth term; or the coefficient of the fourth term, taken with a contrary sign, is equal to the sum of the product of the roots three and three taken with their signs. And so on.

Finally, the product of all the roots, taken with contrary signs, is equal to the last term; or, the product of all the roots, taken with their respective signs, is equal to the last term of the equation, taken with its sign, *when the equation is of an even degree*, and with a contrary sign, *when the equation is of an odd degree.*

The properties demonstrated (No. 98) with respect to equations of the second degree, are only particular cases of the above. The last term, taken with its sign, is equal to the product of the roots themselves, because the equation is of an even degree.

Complete Theory of the greatest Common Divisor.

258. By reflecting upon the preceding properties, we will perceive a certain analogy between the research of the relative divisors, of the different degrees of an entire function of x, and the resolution of an equation. In fact, it follows from the property of No. 254, that every polynomial - - - - - $Ax^m+Bx^{m-1}+Cx^{m-2}+\ldots+Tx+U$ can be decomposed into m factors of the first degree; and to do this, it will be sufficient to place the polynomial equal to 0, and resolve the equation with reference to the principal letter x.

We see, moreover, from the same property, that polynomials of the first degree are the only ones that cannot be decomposed.

Thus, $a^2-ab+bc$ being a polynomial of the second degree with reference to x, can be decomposed into

$$\left(a-\frac{b+\sqrt{b^2-4bc}}{2}\right)\left(a-\frac{b-\sqrt{b^2-4bc}}{2}\right)$$

by solving the equation $a^2-ba+bc=0$.

The polynomial $a^3-3ab^2+2b^2c+3bc^2$, can be decomposed into two factors involving b, which are obtained by solving the equation

$$(2c-3a)b^2+3c^2b+a^3=0.$$

with reference to b.

It can also be decomposed into two factors with respect to c; but to obtain the three factors of the first degree with reference to a, it is necessary to resolve an equation of the 3d degree.

If it is necessary to know how to resolve an equation in order to obtain the relative divisors of a polynomial, it is not necessary to know this in order to obtain what is called the *greatest relative common divisor of two polynomials.* Analysts have even referred to this last question in the resolution of certain classes of equations. Therefore, before proceeding any farther with the theory of equations, we will discuss the theory of the the greatest common divisor.

We will first treat of the case of two entire functions of x, and then we will consider that in which the two polynomials are entire with respect to all the letters and coefficients.

Of the Greatest Relative Common Divisor.

259. The greatest relative common divisor of two entire functions of x, is the polynomial of the highest degree with respect to x, that will divide the proposed polynomials.

It evidently follows from this definition, that when the two polynomials have been divided by their greatest common divisor, the quotients will not contain any common factor involving x; for if they did, the product of this factor by the divisor already considered would be of a higher degree with reference to x than this divisor, and would be a relative divisor of the two polynomials.

Let d, d', d'', be the only factors of the first degree with respect to x, which are common to the two entire functions, and suppose that n, n', n'', are the exponents of the powers of these two factors, which are common to the two polynomials; as d^n, $d'^{n'}$, $d''^{n''}$, are prime relative divisors, it follows (No. 247) that the product, $d^n.d'^{n'}.d''^{n''}$, is the relative divisor of the highest degree, common to the two polynomials, since all the others can only be combinations of the different powers of d, d', d'', comprehended between 1 and n, n', n'', taken 2 and 2, 3 and 3 .... Therefore,

First principle. 1st. *The greatest relative common divisor of two entire functions, is the product of the highest powers of all the divisors of the first degree with reference to x, common to* *the two polynomials. 2d. Every relative common divisor of two entire functions, will divide their greatest relative common divisor.*

N. B. An infinite number of relative common divisors may be formed, of the same degree as $d^n.d'^{n'}.d''^{n''}$, by multiplying this last by factors independent of x. (No. 241.)

260. Second principle. *The greatest relative common divisor of two entire functions, is the same as that which exists between the polynomial of the lowest degree and the remainder after their division; or, at least, it can only differ from it by a factor independent of* x.

For, let A and B be the two polynomials, D their greatest relative common divisor, Q and R the quotient and remainder after their division, D′ the greatest relative common divisor of B and R; we have the equation

$$A = B \times Q + R,$$

from which we deduce

$$\frac{A}{D} = \frac{BQ}{D} + \frac{R}{D}, \text{ and } \frac{A}{D'} = \frac{BQ}{D'} + \frac{R}{D'}.$$

Now D being a relative divisor of A and B, it follows that $\frac{A}{D}$ and $\frac{BQ}{D}$ are entire functions of x; therefore $\frac{R}{D}$ must also be an entire function of x; that is, D is a relative divisor of B and R. Hence, from the first principle, D must divide D′, which is the greatest common divisor of B and R.

In like manner, D′, a relative divisor of B and R, is also a relative divisor of BQ and of R, and consequently of BQ + R or A. Therefore D′, a relative divisor of A and B, must divide D, which is the greatest common divisor of A and B.

Hence the two polynomials are reciprocally divisible by each other; they must therefore be of the same degree, and consequently (No. 159) equal to each other, or only differ from one another by a factor which is independent of x.

261. From these two principles, we deduce the following rule for finding the greatest relative common divisor of two entire functions.

Divide the polynomial of the highest degree by that of the lowest; if there is no remainder, the second polynomial is the *required g. c. d. If a remainder is obtained, divide the second polynomial by it; if the division can be performed exactly, this remainder is the g. c. d. of itself, and the second polynomial, and consequently the g. c. d. of the proposed polynomials. If a second remainder is obtained, divide the first remainder by the second, and continue the operation in this manner until a remainder is obtained which will exactly divide the preceding remainder; this will be the required greatest common divisor.*

In applying the above rule, when a remainder is obtained which is independent of x, we may conclude that the two polynomials are *prime with each other*, in this sense, viz. they will not admit of a common divisor involving x; for the greatest common divisor being (259) a relative divisor of the remainder after each division, it would also divide the remainder independent of x, which is impossible.

262. To determine the greatest relative common divisor of several entire functions A, B, C, E....

Let D, represent the greatest common divisor of A and B, D′ the g. c. d. of D and C; then will D′ be the g. c. d. of A, B, C.

For since the g. c. d of A, B, C must divide A and B, it will divide their g. c. d. D; it must also divide C, therefore it will divide D′, which is the g. c. d. of D and C; hence it cannot be of a higher degree than D′. But D′ is evidently common to the three polynomials, A, B, C; therefore D′ is their g. c. d.

It might be proved in an analogous manner, that D″, the g. c. d. of D′ and E, is the g. c. d. of A, B, C, E; and so on.

N. B. In the application to particular cases, we begin with finding the g. c. d. of the two polynomials of the lowest degree, then of this g. c. d. and the next most simple polynomial, &c.

263. By referring to the preceding rules, it will be perceived that, by a series of algebraic divisions, we can always obtain the greatest relative common divisor of two or more polynomials involving x, whatever may be the nature of the coefficients of the different powers of this principal letter.

Of the ordinary Greatest Common Divisor.

264. The polynomials that we are now going to consider are of the same nature as those in the first chapter, and are called

rational and entire polynomials, because, being composed of a limited number of terms, which is also the case with ***entire functions***, they do not contain any signs for division or the extraction of roots; that is, the numerical or algebraic coefficients are entire, and the exponents of all the letters are positive whole numbers.

One rational polynomial is called a *factor* or ***divisor*** of another of the same nature, when there is a third rational and entire polynomial which, multiplied by the first, will produce the second.

A rational and entire polynomial is said to be ***absolutely prime***, when it has no rational and entire divisor except itself and unity; and two rational and entire polynomials are ***prime with each other***, when they do not admit of any rational and entire common factor except unity.

265. *Every prime rational and entire polynomial* P, *which will exactly divide the product* (A×B) *of two other rational and entire polynomials, must necessarily divide one of these polynomials.*

This general theorem depends upon several propositions, which are only particular cases of it; and which we will demonstrate successively.

First Case. Let P be a prime number, A any entire number, B a rational and entire polynomial but depending upon the single letter α, that is to say such that

$$B=a\alpha^n+b\alpha^{n-1}+c\alpha^{n-2}+ \text{- - - -} s\alpha+t;$$

($a, b, c,$ - - - s, t being any entire numbers whatever, positive or negative).

Since the product A×B can be put under the form

$$Aa.\alpha^n+Ab.\alpha^{n-1}+Ac.\alpha^{n-2}+ \text{- - - -} As.\alpha+At,$$

and as P will, by hypothesis, divide this product, it follows (No. 30) that P will divide each of the coefficients - - - - - - Aa, Ab, Ac, - - - - As, At; hence (Arith. 5th edition, No. 133) P must divide A, or each of the numbers $a, b, c,$ - - - - s, t, and consequently B.

Whence we may conclude that *every prime number* P *which, will exactly divide the product* (A×B) *of two quantities, one of which* (A) *is any entire number*, and the other a rational and en-

tire number (B) *depending upon a single letter* (α) *must divide* A *or* B.

Second Case. Let P be a prime number, A and B two rational and entire numbers, depending upon the single letter α, that is such that we may have

$$A=a\alpha^{n}+b\alpha^{n-1}+c\alpha^{n-2}+\ldots\ldots+s\alpha+t,$$
$$B=a'\alpha^{n'-1}+b'\alpha^{n'-2}+c'\alpha^{n'-3}+\ldots\ldots+s'\alpha+t';$$

(a, b, c, ... s, t, a', b', c', ... s', t', being entire numbers).

Denote by A′ the algebraic sum of the terms of A, the coefficients of which contain the factor P, and by A″ the sum of the terms of which the coefficients are not divisible by P, we will have

$$A=A'+A'' \text{ - - - - } (1).$$

In like manner, let B′ and B″ be the two parts, one of which has all its coefficients divisible, and the other none of them divisible by P.

We will have

$$B=B'+B'' \text{ - - - - } (2).$$

Multiplying the equalities (1) and (2) together, we obtain

$$AB=A'B'+A''B'+A'B''+A''B'' \text{ - - - - } (3).$$

This being the case, since P will divide each of the coefficients A′ and B′, it follows that P will divide the three first parts of the equality (3); hence, in order that P may divide the product AB, it must divide the fourth part A″B″. Now, this last division is impossible; for, denote by $k\alpha^{r}$, $k'\alpha^{r'}$, the two terms of A″ and B″, affected with the highest exponent of α; as their product, $kk'.\alpha^{r+r'}$, cannot be reduced with the other partial products involved in A″B″, it must, (No. 30), if it divides A″B″, also divide kk', which is absurd, since the prime number P will neither divide k nor k'.

The only way of avoiding this absurdity, is to suppose A″ or B″ equal to 0, and then all the terms of A, or of B, being divisible by P, it follows that A or B must be divisible by P, in order that A×B may be divisible by it.

Therefore every prime number P, *which will exactly divide the product* A×B *of two rational and entire polynomials, will divide all the coefficients of one of these polynomials, and consequently this polynomial itself.*

Third Case. Let A be any entire number, B a rational and entire polynomial, depending upon the single letter a, P a prime polynomial, of the same nature as B.

Since the product $A\times B$ is, by hypothesis, divisible by P, we have

$$A\times B=P\times Q \text{ - - - - - } (1),$$

(Q being an entire quantity).

Decompose the number A into its prime factors, and let

$$A=f.f'.f''\dots f^{(n)},$$

the equality (1) becomes

$$f.f'.f''\dots f^{(n)}.\ B=P\times Q \text{ - - - - - } (2);$$

whence, dividing both members by f,

$$f'.f''\dots f^{(n)}.\ B=\frac{P\times Q}{f}.$$

Now the first member being a rational and entire quantity, the second must also be rational and entire; but f is a prime number, which cannot divide P, since P is prime. Therefore, from the second case, f must divide Q, and we have $Q=f\times Q'$, (Q' being an entire quantity); whence, by substituting in the equality (2), and dividing by f,

$$f'.f''\dots f^{(n)}=B\times Q' \text{ - - - - - } (3).$$

Reasoning upon this equality in the same way as upon the equality (2), we will find that $Q'=f'\times Q''$, (Q'' being entire); whence, by substituting in (3), and dividing by f',

$$f''.f'''\dots f^{(n)}.\ B=P\times Q'' \text{ - - - - - } (4),$$

and so on. Therefore, having suppressed all the factors .... $f.f'.f''\dots f^{(n)}$, we will obtain an equality of the form

$$B=P\times Q^{(n+1)},$$

$Q^{(n+1)}$ being an entire quantity; which proves that B is divisible by P.

Therefore, *every rational and entire polynomial, depending upon a single letter* a, *which will exactly divide the product* $A\times B$ *of any entire number by a rational and entire polynomial depending upon the same letter* a, *will divide this last polynomial.*

Fourth Case. Let A and B be two rational and entire polynomials depending upon the single letter α, and P a prime polynomial of the same nature.

Suppose that A is not divisible by P, and that it is of a higher degree than P, divide A by P, and continue the operation until a remainder is obtained of a lower degree than P; but in order to obtain quotients affected with entire coefficients, first multiply A by a suitable number m. Denote the quotient of this division by Q, and the remainder by R; we will have the equality

$$m.A=P\times Q+R \text{ - - - - } (1).$$

R cannot be equal to 0, for in this case P would divide $m.A$ and consequently A, by the 3d case, which would be contrary to hypothesis.

This being the case, multiply both members of the equality (1) by B, and divide then by P; it becomes

$$\frac{m.A\times B}{P}=B\times Q+\frac{B\times R}{P}.$$

Now since, by hypothesis, P will divide $A\times B$, and consequently $m.A\times B$, it must divide $B\times R$, and if R is an entire number the proposition is demonstrated, since P divides $B\times R$, it must divide B, by the 3d case.

But suppose that P depends upon α, and divide P by R, after having multiplied P by any number m' that will give entire coefficients in the quotient; we will have

$$m'.P=R\times Q'+R' \text{ - - - - } (2).$$

R' cannot be equal to 0; for if we had $R'=0$, it would follow that R would divide $m.P$, and consequently all prime algebraic factors of R would divide P, which is impossible, since P is prime.

Multiply both members of the equality (2) by B, and divide them by P, it becomes

$$m'.B=\frac{B\times R\times Q'}{P}+\frac{B\times R'}{P},$$

which proves that BR' is divisible by P, since BR is divided by it. If R' is independent of α, the proposition is demonstrated. Since P, dividing $B\times R'$, must divide B, by the 3d case. But suppose that R' depends upon α, and continue to divide P by *R', by R''* - - - -, and so on. We will finally obtain a remainder $R^{(n)}$ independent of α, and such that $B\times R^{(n)}$ will be divisible by P. Therefore B is divisible by P.

(When P is of a higher degree than A, divide P by A, then by R, R′, R″, and the reasoning will be the same.)

Therefore, when *a prime rational and entire polynomial* P, *depending upon the single letter* α, *will exactly divide the product* A×B *of two rational and entire polynomials depending only upon the same letter* α, *we may conclude that* P *will exactly divide* A *or* B.

It is now easy to generalize the proposition; for by supposing that the three polynomials A, B, P may contain the two letters α, c, we can establish the four new cases: viz.

1st. P being a prime number, or a prime polynomial depending upon the single letter α, A any entire number, or a rational and entire polynomial depending upon the single letter α, and B a rational and entire polynomial, containing the two letters α, e;

2d. P being a prime number, or a prime polynomial depending upon the single letter α, and A and B two rational and entire polynomials, containing the two letters α, e;

3d. A being an entire number, or a rational and entire polynomial, depending upon a single letter α, and B and P, two rational and entire polynomials, containing the letters α, e, but P being prime;

4th. Finally, A, B, P, being three polynomials containing the two letters α, e, P being prime.

If we apply to each of these hypotheses, reasoning analogous to that of Nos 1, 2, 3, 4, we will find that, *Every prime rational and entire polynomial* P, *depending upon two letters* α, e, *which will divide the product* A×B *of two polynomials containing the same two letters will divide one of these polynomials.*

The proposition being true in the case of two letters, it may be extended to the cases of *three*, *four*, &c. letters; it is therefore general.

First. Conceive that a rational and entire polynomial A is decomposed into the product of several prime factors, numerical or algebraical, but entire and rational, and we will have

$$A=P.P'.P''.P''' \ldots . P^{(n)}.$$

It follows from the preceding proposition, that no prime factor p, different from P, P′, P″ . . . . $P^{(n)}$, will divide A; for in order that p may divide A, it must divide P×P′P″. . . . $P^{(n)}$. Now if it differs from p it cannot divide it, (since P is prime), and it must consequently divide P′P″ . . . . $P^{(n)}$. For the same reason, if p differs from P′, it must divide P″P‴ . . . . $P^{(n)}$, and so on; whence we would conclude that p is equal to the last factor $P^{(n)}$, which is contrary to the hypothesis.

Therefore, *the only rational and entire factors that can be contained in* A, *are the factors* PP′P″ . . . $P^{(n)}$, *into which* A *is already decomposed, or the products of these factors, taken two and two, three and three, &c.*

266. *Secondly.* Let A and B be two rational and entire polynomials, D their greatest common divisor, that is, (No. 34), *the greatest polynomial with reference to the exponents and coefficients, which will exactly divide the proposed polynomials.* If the quotients of their division by b, be denoted by A′ and B′, we have (No. 34) A=A′D, B=B′D, A′ and B′ being *prime with each other.*

This being the case, since every prime divisor d, common to the two polynomials, will divide A′ and B′, it must, by the fundamental proposition, (No. 265), divide D. It is moreover evident, that no prime divisor p, which divides A and not B, or reciprocally, will divide D.

Hence, *the greatest common divisor of two rational and entire polynomials, contains as factors, all the particular common divisors of the two polynomials, and cannot contain any other factors.*

This is the principle of No. 35 applied to two rational and entire polynomials.

267. *Third.* We may easily discover that D is equal to $d^n . d'^{n'} . d''^{n''} \dots$; $d, d', d'' \dots$ being prime common factors of the two polynomials, and $n, n', n'' \dots .$ the exponents of the powers of these factors.

For conceive A and B to be decomposed into their prime factors, and suppose that (d being a certain number of times a factor of A, and a certain number of times a factor of B) d^n is the highest power of d common to A and B; and that - - - - - $d'^{n'}, d''^{n''}, d'''^{n'''}$, are the highest powers of d′, d″, d‴ common to

A and B. Suppose, moreover, that d, d', d'', d''' are the only common prime factors; we will have

$$A = A'' \times d^n \times d'^{n'} \times d''^{n''} \times d'''^{n'''},$$
$$B = B'' \times d^n \times d'^{n'} \times d''^{n''} \times d'''^{n'''}.$$

A'' and B'' are necessarily prime with each other; for if there is a prime factor which will divide A'' and B'', it must be one of the factors d, d', d'', d''', in which case n, n', n'', n''' would not be the exponents of the common highest powers, or it would be different from d, d', d'', d''', and then these last quantities would not be the only common prime factors, which would be contrary to the hypothesis.

Now it is evident that the only rational and entire common divisors of A and B are (265) the powers

$$d, d^2, d^3 \text{ - - - - } d^n, d', d'^2, d'^3 \text{ - - - - } d'^{n'}, d'', d''^2, d''^3 \text{ - - - - } d''^{n''},$$

or the products of these powers, taken *two and two*, *three and three*, *four and four*. But the greatest common divisor thus obtained, is evidently the product $d^n \times d'^{n'} \times d''^{n''} \times d'''^{n'''}$. Therefore D is equal to this product. This also proves that *two rational and entire polynomials can have but one greatest common divisor*, whilst two *entire functions* have an infinite number of *greatest relative common divisors.*

268. *Fourth.* The greatest common divisor will not be affected by introducing or suppressing a factor in one of the polynomials A or B, provided this factor is not a factor of the other polynomial; for it is evident that the greatest common divisor of the two new polynomials is the same as that of the proposed polynomials, since it must be composed of the same factors.

269. *Fifth.* We will now demonstrate the second principle of No. 35. We will observe, in the first place, that the two polynomials may be supposed such, that after having arranged them with reference to one of the common letters, a, and divided the polynomial of the highest degree, A, for example, by the second, B, we will obtain an *entire* quotient, and an *entire* remainder, in which the highest exponent of a will be less than that of the divisor.

For, in order that the quotient may be fractional for each of the partial operations, it is necessary that the coefficient of the *first term* of each partial dividend should not be divisible by the *coefficient* of the first term of the divisor, and then the denominator of the partial quotient is this last coefficient itself, or one of the factors of it. Now there may be *three cases;* this coefficient may divide the coefficients of the other powers of *a* in the divisor, or it may have factors which are common to all these coefficients, or it may have factors common to some, but not to all of them. (In this case, all the coefficients are said to be *prime* with each other.)

In the two first cases, this coefficient, or one of its factors, would be a factor of B; and since this factor would not be contained in the dividend, it would not be a factor of the greatest common divisor of A and B. Therefore (No. 268) it may be suppressed in B, and the question will be reduced to finding the greatest common divisor of A, and the result, B′, obtained by suppressing this factor.

In the third case, we may multiply the dividend A by the *most simple multiple* of the denominators of the fractional quotients, which multiple will necessarily be *prime with* B. The product of A by this multiple having the same greatest common divisor with B as that which exists between A and B, we may operate with this product A′, and with B, as though they were the primitive polynomials, and then we will be certain of obtaining entire quotients.

The greatest common divisor of A *and* B, *is the same as the* g. c. d *of* B *and* R, R denoting the remainder after the division, which is continued until the remainder is of a lower degree than B, with respect to the principal letter *a*.

For, let D be the greatest common divisor of A and B, D′ the g. c. d. of B and R; we have the equality

$$A = BQ + R,$$

(Q and R being entire polynomials); whence, dividing first by D, and then by D′,

$$\frac{A}{D} = \frac{B \times Q}{D} + \frac{R}{D} \text{ and } \frac{A}{D'} = \frac{B \times Q}{D'} + \frac{R}{D'}.$$

These last equalities prove, 1st. That as D will divide A, B and consequently B × Q, it will also divide R; therefore D, the common divisor of B and R, will (No. 266) divide D′, the g. c. d. of B and R.

2d. Since D′ will divide R, B, and consequently B × Q, it will also divide A; then D′, a common divisor of A and B, will divide D, which is the g. c. d. of A and B.

Since D and D′ reciprocally divide each other, the quotient must be unity, and we have

$$D=D'.$$

270. We will terminate the exposition of these principles by a remark, which will serve as a guide in the solution of this question.

Let A be a rational and entire polynomial, supposed to be arranged with reference to one of the letters involved in it, a, for example.

If this polynomial is not *absolutely prime*, (No. 264), that is, if it can be decomposed into rational and entire factors, it may be regarded as the product of three principal factors, viz.

1st. *Of a monomial factor* A_1, common to all the terms of A. (This factor is composed of the greatest common divisor of all the numerical coefficients, multiplied by the product of the literal factors which are common to all the terms.)

2d. Of a polynomial factor A_2, independent of a, which is (No. 30) common to all the coefficients of the different powers of a, in the arranged polynomial.

3d Of a polynomial factor A_3 depending upon a, and in which the coefficients of the different powers of a are prime with each other (No. 269); so that we will have

$$A=A_1\times A_2\times A_3.$$

Sometimes one or both of the factors A_1, A_2 reduce to unity, but this is the general form of *rational and entire* polynomials. It follows from this, that when there is a greatest common divisor of two polynomials A and B, we will have

$$D=D_1.D_2.D_3;$$

D_1 denoting the greatest monomial common factor, D_2 the greatest polynomial factor independent of a, and D_3 the greatest polynomial factor depending upon this letter.

In order to obtain D_1, *find the monomial factor* A_1 *common to all the terms of* A. This factor is in general composed of literal factors, which are found by inspecting the terms, and of a numerical coefficient, obtained by finding the greatest common divisor of the numerical coefficients in A.

In the same way, find the monomial B_1 *common to all the terms of* B; then *determine the greatest factor* D_1 *common to* A_1 *and* B_1.

This factor D_1, is set aside, as forming the first part of the required common divisor. *The factors* A_1 *and* B_1 *are also suppressed in the proposed polynomials*, and the question is reduced to finding the g. c. d. of two new polynomials A′ and B′ which do not contain a common monomial factor. It is then to be understood that the process developed below, is to be applied to these two polynomials.

271. Several circumstances may occur as regards the number of letters that may be contained in A′ and B′.

1st. *When* A′ *and* B′ *contain but one letter* a.

When A′ and B′ are arranged with reference to *a*, the coefficients will necessarily be *prime with each other;* therefore in this case, we will only have to seek for the greatest common factor depending upon *a*, viz. D_3 (No. 270).

In order to obtain it, we will (No. 269) first prepare the polynomial of the highest degree, so that its first term may be exactly divisible by the first term of the divisor. This preparation consists in *multiplying the whole dividend by the coefficient of the first term of the divisor, or by a factor of this coefficient, or* (No. 36) *by a certain power of it*, in order that we may be able to execute several operations, without any new preparations.

The division is then performed, continuing the operation until a remainder is obtained of a lower degree than the divisor.

If there is a factor common to all the coefficients of the remainder, it must be suppressed, as it cannot form a part of the required g. c. d.; after which, *we operate with the second polynomial, and this remainder, in the same way we did with the polynomials* A′ *and* B′ (No. 260).

Continue this series of operations until a remainder is obtained which will exactly divide the preceding remainder, this remainder will be the g. c. d. D_3 of A′ and B′; and $D_1 \times D_3$ will express the g. c. d. of A and B; *or, continue the operation until a remainder is obtained independent of a*, that is, a *numerical* remainder, in which case, the two polynomials, A′ and *B′, will be prime with each other*.

2d. *When* A′ *and* B′ *contain two letters* a *and* b.

After having arranged the polynomials with reference to *a*, we first find the polynomial factor which is *independent of a*, if there is one.

To do this, we determine the greatest common divisor A_2 of all the coefficients of the different powers of *a* in the polynomial A′. This common divisor is obtained by applying the rule (No. 262) for finding the greatest common divisor of several polynomials, as well as the rule for the last case, since these coefficients contain only one letter *b*. *In the same way we determine the greatest common divisor* B_2 *of all the coefficients of* B′. Then comparing A_2 and B_2, *we set aside their greatest common divisor* D_2, as forming a part of the required g. c. d; and we also suppress the factors A_2 and B_2 in A′ and B′; which produces two new polynomials A″ and B″, the coefficients of which are *prime with each other*, and to which we may consequently apply the rule for the first case.

Care must always be taken to ascertain, in each remainder, *whether the coefficients of the different powers of the letter* a, *do not contain a common factor, which must be suppressed*, as not forming a part of the common divisor. We have already seen (No. 38) that the suppression of these factors is absolutely necessary.

We will in this way obtain the common divisor D_3 of A′ and B″, and $D_1 \times D_2 \times D_3$, for the g. c. d. of the polynomials A and B.

N. B. In applying the rule for the first case to A″ and B″, we would ascertain when these two polynomials *were prime with each other*, from this circumstance, viz. : *a remainder would be obtained which would be numerical, or a function of* b, *but independent of* a. The greatest common divisor of A and B would be $D_1 \times D_2$.

3d. *When* A *and* B *contain three letters*, a, b, c.

After arranging the two polynomials with reference to *a*, we determine the g. c. d. independent of *a*, which is done by applying to the coefficients of the different powers of a, in both polynomials, the process of No. 262, and the rule for the second case, since these polynomial coefficients contain but two letters, b and c.

The independent polynomial D_2 being thus obtained, and the factor A_2 and B_2, which have given it, being suppressed in A' and B', there will result two polynomials A'' and B'', having their coefficients *prime with each other*, and to which the rules for the preceding cases may be applied, and so on.

272. Let there be the two polynomials

$$a^2d^2-c^2d^2-a^2c^2+c^4, \text{ and } 4a^2d-2ac^2+2c^3-4acd.$$

The second contains a monomial factor 2. Suppressing it, and arranging the polynomials with reference to d, we have the expressions

$$(a^2-c^2)d^2-a^2c^2+c^4, \text{ and } (2a^2-2ac)d-ac^2+c^3.$$

It is first necessary to ascertain whether there is a common divisor independent of d.

By considering the coefficients a^2-c^2, and $-a^2c^2+c^4$, of the first polynomial, it will be seen that $-a^2c^2+c^4$ can be put under the form $-c^2(a^2-c^2)$; hence a^2-c^2 is a common factor of the coefficients of the first polynomial. In like manner, the coefficients of the second, $2a^2-2ac$, and $-ac^2+c^3$, can be reduced to $2a(a-c)$, and $-c^2(a-c)$; therefore $a-c$ is a common factor of these coefficients.

Comparing the two factors a^2-c^2 and $a-c$, as this last will divide the first, it follows that $a-c$ is a common factor of the proposed polynomials, and it is that part of their greatest common divisor *which is independent of d.*

Suppressing a^2-c^2 in the first polynomial, and $a-c$ in the second, we obtain the two polynomials d^2-c^2 and $2ad-c^2$, to which the ordinary process must be applied.

$$\begin{array}{l|l} d^2-c^2 & 2ad-c^2 \\ \hline 4a^2d^2-4a^2c^2 & 2ad+c^2 \\ \hline +2ac^2d-4a^2c^2 & \\ \hline -4a^2c^2+c^4. & \end{array}$$

Explanation. After having multiplied the dividend by $4a^2$, and *performed* two consecutive divisions, we will obtain a remainder $-4a^2c^2+c^4$, independent of the letter d; hence the two polynomials d^2-c^2, and $2ad-c^2$, are prime with each other. Therefore the greatest common divisor of the proposed polynomials is $a-c$.

Again, taking the same example, and arranging with reference to a, it becomes, after suppressing the factor 2 in the second polynomial,

$$(d^2-c^2)a^2-c^2d^2+c^4, \text{ and } 2da^2-(2cd+c^2)a+c^3.$$

It is easily perceived, that the coefficient of the different powers of a in the second polynomial are prime with each other. In the first polynomial, the coefficient $-c^2d^2+c^4$, of the second term, or of a^0, becomes $-c^2(d^2-c^2)$; whence d^2-c^2 is a common factor of the two coefficients, and since it is not a factor of the second polynomial, it may be suppressed in the first, as not forming a part of the common divisor.

By suppressing this factor, and taking the second polynomial for a dividend and the first for a divisor, (in order to avoid preparation), we have

$$\text{1st.}\quad 2da^2 \begin{array}{r|l} -2cd & a+c^3 \\ -c^2 & \end{array} \Big\} \frac{a^2-c^2}{2d}.$$

$$\text{Rem.} \;\text{- - -}\; \begin{array}{r|l} -2cd & a+2dc^2 \\ -c^2 & +\;\; c^3 \end{array}$$

$$\text{or, - - - - -}\; a-c,$$

by suppressing the common factor $(-2cd-c^2)$;

$$\text{2d.}\quad \begin{array}{r} a^2-c^2 \\ \hline +ac-c^2 \\ \hline 0 \end{array} \Big\} \frac{a-c}{a+c}.$$

Explanation. After having performed the first division, a remainder is obtained which contains $-2cd-c^2$, as a factor of its two coefficients; for $2dc^2+c^3=-c(-2cd-c^2)$. This factor being suppressed, the remainder is reduced to $a-c$, which will exactly divide a^2-c^2.

Hence $a-c$ is the required greatest common divisor.

273. There is a remarkable case, in which, the greatest common divisor may be obtained more easily than by the general method; it is when *one of the two polynomials contains a letter, which is not contained in the other.*

In this case, as it is evident that the greatest common divisor is independent of this letter, it follows that, by arranging the polynomial which contains it, with reference to this letter, *the required common divisor will be the same as that which exists between the coefficients of the different powers of the principal letter and the second polynomial, which, by hypothesis, is independent of it.*

By this method, we will be led to determine the greatest common divisor between three or more polynomials; but they will be more simple than the proposed polynomials. It often happens, that some of the coefficients of the arranged polynomial are monomials, or, that we may discover by simple inspection that they are prime with each other; and, in this case, we are certain that the proposed polynomials are prime with each other.

Thus, in the example of No. 272, treated by the first method, after having suppressed the common factor $a-c$, which gives the results.

$$d^2-c^2 \text{ and } 2ad-c^2,$$

we know immediately that these two polynomials are prime with each other; for, since the letter a is contained in the second and not in the first, it follows from what has just been said, that the common divisor must divide the coefficients $2d$ and $-c^2$, which is evidently impossible; hence, &c.

We will apply this last case to the two polynomials

$$3bcq+30mp+18bc+5mpq,$$

and

$$4adq-42fg+24ad-7fgq,$$

Since q is the only letter common to the two polynomials (which, moreover, do not contain any common monomial factors), we can arrange them with reference to this letter, and follow the ordinary rule. But as b is found in the first polynomial and not in the second, if we arrange the first with reference to b, which gives

$$(3cq+18c)b+30mp+5mpq,$$

the required g. c. d. will be the same as that which exists between the second polynomial and the two coefficients

$$3cq+18c \text{ and } 30mp+5mpq.$$

Now the first of these coefficients can be put under the form $3c(q+6)$, and the other becomes $5mp(q+6)$; hence $q+6$ is a common factor of these coefficients. It will therefore be sufficient to ascertain whether $q+6$ (which is a *prime* divisor) is a factor of the second polynomial.

Arranging this polynomial with reference to q, it becomes

$$(4ad-7fg)q-42fg+24ad;$$

as the second part $24ad-42fg$ is equal to $6(4ad-7fg)$, it follows that this polynomial is divisible by $q+6$, and gives the quotient $4ad-7fg$. Therefore $q+6$ is the greatest common divisor of the proposed polynomials.

274. N. B. It may be ascertained that $q+6$ is an exact divisor of the polynomial $(4ad-7fg)q+24ad-42fg$, by a method derived from the property of No. 151.

Make $q+6=0$ or $q=-6$ in this polynomial; it becomes

$$(4ad-7fg)\times-6+24ad-42fg.$$

which reduces to 0; hence $q+6$ is a divisor of this polynomial.

This method may be advantageously employed in nearly all the applications of the process. It consists in this, viz. after obtaining a remainder of the first degree with reference to a (when a is the principal letter), *make this remainder equal to* 0, *and deduce the value of* a *from this equality.*

If this value, substituted in the remainder of the 2d degree, *destroys it,* then the remainder of the 1st degree, simplified (No. 38), is a common divisor. If the remainder of the 2d degree does not reduce to 0 by this substitution, we may conclude that there is no common divisor depending upon the principal letter.

Farther, having obtained a remainder of the 2d degree with reference to a, it is not necessary to continue the operation any farther.

Decompose this polynomial into two factors of the 1*st degree,* which is done by placing it equal to 0, and resolving the resulting equation of the second degree.

When each of the values of a thus obtained, substituted in the remainder of the 3d degree, *destroy it,* it is a proof that the remainder of the 2d degree, *simplified,* is a common divisor; when only one of the values destroys the remainder of the 3d degree, the common divisor is the factor of the 1st degree with respect to a, which corresponds to this value.

Finally, when neither of these values destroy the remainder of the 3d degree, we may conclude that there is not a common divisor depending upon the letter a.

It is here supposed that the two factors of the 1st degree with reference to a, are rational, otherwise it would be more simple to perform the division of the remainder of the 3d degree by that of the second, and when this last division cannot be performed exactly, we may be certain that there is no rational common divisor, for if there was one, it could only be of the first degree with respect to a, and should be found in the remainder of the second degree, which is contrary to hypothesis.

275. To show the difference between the process for finding the greatest relative common divisor, and that for finding the ordinary common divisor, we will apply the first to an example, in which the two polynomials are not only entire with reference to x, but also with respect to the other quantities involved in it.

Let the proposed polynomials be

$$6x^5-4x^4-11x^3-3x^2-3x-1,$$

and

$$4x^4+2x^3-18x^2+3x-5\,;$$

1st. $$6x^5-4x^4-11x^3-3x^2-3x-1 \;\Big\}\; 4x^4+2x^3-18x^2+3x-5$$

$$-7x^4+16x^3-\frac{15}{2}x^2+\frac{9}{2}x-1 \;\Big\}\; \frac{3}{2}x-\frac{7}{4}$$

$$+\frac{39}{2}x^3-39x^2+\frac{39}{4}x-\frac{39}{4}\,;$$

2d. $$4x^4+2x^3-18x^2+3x-5 \;\Big\}\; \frac{39}{2}x^3-39x^2+\frac{39}{4}x-\frac{39}{4}$$

$$+10x^3-20x^2+5x-5 \;\Big\}\; \frac{8}{39}x+\frac{20}{39}$$

$$0$$

Hence $\frac{39}{2}x^3-39x^2+\frac{39}{4}x-\frac{39}{4}$ is the g. c. d.

By the Common Method.

1st. Multiplying by 16.

$$96x^5-64x^4-176x^3-48x^2-48x-16 \;\Big\}\; 4x^4+2x^3-18x^2+3x-$$

$$-112x^4+256x^3-120x^2+72x-16 \;\Big\}\; 24x-28$$

Rem. - - - $+312x^3-624x^2+156x-156\,;$

or, suppressing the factor 156,

$$2x^3-4x^2+x-1.$$

2d.

$$\begin{array}{r|l} 4x^4+2x^3-18x^2+3x-5 & 2x^3-4x^2+x-1 \\ \hline +10x^3-20x^2+5x-5 & 2x+5 \\ \hline 0. & \end{array}$$

Hence, $2x^3-4x^2+x-1$ is the g. c. d.

By applying the process of No. 261, without making any preparation, we obtain the result

$$\frac{39}{2}x^3-39x^2+\frac{39}{4}x-\frac{39}{4};$$

while, by following the process of No. 271, with all its modifications, we obtain

$$2x^3-4x^2+x-1$$

for the greatest common divisor of the two polynomials.

Now the last result only differs from the first by the factor $\frac{39}{4}$, which is common to all the terms of the first.

Hence we see that the effect produced by the application of the process, *without preparation*, is to give the *ordinary common divisor* of the two polynomials, (supposed to be *rational and entire*), multiplied by *foreign factors*, but which are *independent* of the principal letter.

Examples.

1st. $\begin{cases} x^6+4x^5-3x^4-16x^3+11x^2+12x-9 \\ 6x^5+20x^4-12x^3-48x^2+22x+12; \end{cases}$

g. c. d. *simplified* $= x^3+x^2-5x+3.$

2d. $\begin{cases} 20x^6-12x^5+16x^4-15x^3+14x^2-15x+4 \\ 15x^4-9x^3+47x^2-21x+28; \end{cases}$

g. c. d. *simplified* $= 5x^2-3x+4.$

3d. $\begin{cases} -4b^3ca^6+14b^2ca^5-12bca^4-7b^2c^2a^3+14bc^2a^2 \\ -b^4a^6+b^3a^5+2b^2a^4+b^5a^3-2b^4a^2. \end{cases}$

§ II. *Transformation of Equations. Elimination.*

The object of this paragraph will be, to collect together the *principal* transformations, of which the object is to reduce the resolution of a given equation to that of another, more easily treated of.

276. First transformation. *To make the second term disappear from an equation.*

The difficulty of resolving an equation may be conceived to diminish with the number of powers of the unknown quantity involved in it; thus, the equation $x^2=q$, gives immediately $x=\pm\sqrt{q}$, whilst the complete equation $x^2+px+q=0$, requires preparation before it can be resolved.

Now, any equation being given, it can always be *transformed* into another, in which the second term is wanting.

For, let there be the general equation

$$x^m+Px^{m-1}+Qx^{m-2}+ \text{- - - -} +Tx+U=0.$$

Suppose $x=u+x'$, u being unknown, and x' an *indeterminate quantity;* by substituting $u+x'$ for x, we obtain

$$(u+x')^m+P(u+x')^{m-1}+Q(u+x')^{m-2}+\ldots+T(u+x')+U=0\,;$$

developing by the binomial formula, and arranging according to the decreasing powers of u,

$$\left.\begin{array}{l|l|l} u^m+mx' & u^{m-1}+m.\dfrac{m-1}{2}x'^2 & u^{m-2}+ \text{- - - -} \\ \quad\;\, +P & \qquad +(m-1)Px' & \\ & \qquad +Q & \end{array}\quad \begin{array}{l} +x'^m \\ +Px'^{m-1} \\ +Qx'^{m-2} \\ + \text{- - - - -} \\ \text{- - - - - -} \\ +Tx' \\ +U \end{array}\right\}=0.$$

Since x' is entirely arbitrary, we may dispose of it in such a way that we will have $mx'+P=0$; whence $x'=-\dfrac{P}{m}$. Substituting this value of x' in the last equation, we will obtain an equation of the form

$$u^m+Q'u^{m-2}+R'u^{m-3}+ \text{- - -} +T'u+U'=0,$$

in which the second term is wanting.

If this equation was resolved, we could obtain the values of x corresponding to those of u, by substituting each of the values of u in the relation $x=u+x'$, or $x=u-\dfrac{P}{m}$.

Whence we may deduce the following general rule:

In order to make the second term of an equation disappear, ***substitute for the unknown quantity a new unknown quantity, united with the coefficient of the second term, taken with a contrary sign, and divided by the exponent of the degree of the equation.***

We can show, *a posteriori*, that this substitution will accomplish the proposed object.

For, let $a, b, c, d \ldots$ be the m roots of the given equation, it follows from the relation $x=u-\frac{P}{m}$, which gives $u=x+\frac{P}{m}$, that the values of u are

$$u=a+\frac{P}{m},\ b+\frac{P}{m},\ c+\frac{P}{m},\ d+\frac{P}{m}\ldots;$$

hence the sum of the new roots is

$$a+b+c+d+\cdots+m.\frac{P}{m};$$

but we have (No. 257) $a+b+c+d+\cdots=-P$; hence the preceding sum reduces to $-P+P$, or 0; therefore the coefficient of the second term of the *transformed* equation is *nothing* of itself.

N. B. The coefficient of the first term of the equation has been supposed equal to unity; but if the equation was of the form

$$Ax^m+Px^{m-1}+\cdots+Tx+U+0,$$

by making $x=u+x'$, we would obtain $mAx'+P$ for the coefficient of u^{m-1}, which, placed equal to zero, would give .... $x'=-\frac{P}{mA}$; that is, in this case the denominator of the value of x' would be *the product of the exponent of the degree of the equation by the coefficient* (A) *of the first term.*

Let us apply the preceding rule to the equation $x^2+px=q$. If we take $x=u-\frac{p}{2}$, it becomes $\left(u-\frac{p}{2}\right)^2+p\left(u-\frac{p}{2}\right)=q$, or, by performing the operations, and reducing $u^2-\frac{p^2}{4}=q$, this equation gives $u=\pm\sqrt{\frac{p^2}{4}+q}$, consequently we obtain for the two corresponding values of x,

$$x=\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

277. Instead of making the second term disappear, an equation may be required, which shall be deprived of its third, fourth,......term; this can be obtained by placing thec oefficient of u^{m-2}, u^{m-3}. . . . . equal to 0. For example, to make the third term disappear, we make in the above transformed equation

$$m\frac{m-1}{2}x'^2+(m-1)Px'+Q=0;$$

from which we obtain two values for x', which substituted in the transformed equation reduces it to the form

$$u^m+P'u^{m-1}+R'u^{m-3}+\ldots.T'u+U'=0.$$

Beyond the third term it will be necessary to resolve equations of a degree superior to the second, to obtain the value of x', thus to cause the last term to disappear, it will be necessary to resolve the equation

$$x'^m+Px'^{m-1}+\ldots Tx'+u=0,$$

which is nothing more than what the proposed equation becomes when x' is substituted for x.

It may happen that the value $x'=\frac{P}{m}$ m which makes the second term disappear, (No. 276), causes also the disappearance of the third or some other term. For example, in order that the second and third terms may disappear at the same time, it is necessary that the equation $x'=-\frac{P}{m}$ should agree with

$$m\frac{m-1}{2}x'^2+(m-1)Px'+Q=0$$

Now if in this last equation, we replace x' by $-\frac{P}{m}$ it becomes

$$m\frac{m-1}{2}\cdot\frac{P^2}{m^2}-(m-1)\frac{P^2}{m}+Q=0, \text{ or } (m-1)P^2-2mQ=0;$$

therefore, whenever this relation exists between the coefficients

P and Q, the disappearance of the second term involves that of the third.

278. *Remarks upon the preceding transformation. Formation of derived polynomials.*

The relation $x=u+x'$, of which we have made use in the two preceding numbers, indicates that the roots of the transformed equations are equal to those of the proposed, diminished or increased by a certain quantity. Sometimes this quantity is introduced in the calculus, as an indeterminate quantity, the value of which is afterwards fixed in such a manner as to satisfy a given condition; sometimes it is a particular number given *à priori*, which expresses a *constant difference* between the roots of a primitive equation, and those of another eqvation which we wish to form.

In short, the transformation which consists in substituting $u+x'$ for x in an equation is of very frequent use in the theory of equations. Now there is a very simple method of obtaining, in practice, the transformation which results from this substitution.

To show this we will invert the order of the terms in $u+x'$, that is, for x substitute $x'+u$ in the equation

$$x^m+Px^{m-1}+Qx^{m-2}+Rx^{m-3}+\ldots Tx+U=0;$$

it becomes, by developing and arranging according to the ascending powers of u.

$$\begin{array}{ll|l|l}
x'^m & +mx'^{m-1} & u+m\dfrac{m-1}{2}x'^{m-2} & u^2+\ldots u^m=0 \\
+Px'^{m-1} & +(m-1)Px'^{m-2} & +(m-1)\dfrac{m-2}{2}Px'^{m-3} & \\
+Qx'^{m-2} & +(m-2)Qx'^{m-3} & +(m-2)\dfrac{m-3}{2}Qx'^{m-4} & \\
+\ldots\ldots+\ldots\ldots & & +\ldots\ldots & \\
+Tx' & +T & & \\
+U & & &
\end{array}$$

If we observe how the coefficients of the different powers of u are composed, we will see that the coefficient of u^0 is nothing more than what the first member of the proposed equation becomes when x' is substituted in place of x; we will hereafter denote it by X'.

The coefficient of u^1 is formed by means of the preceding, or X′, by multiplying each of the terms of X′ by the exponent of x' in this term, and then diminishing this exponent by unity; we will call this coefficient Y′.

The coefficient of u^2 is formed from Y′ by multiplying each of the terms of Y′ by the exponent of x in this term, dividing the product by two, and then diminishing the exponent by unity. By calling this coefficient $\frac{Z'}{2}$ it is evident that Z′ is formed from Y′ in the same manner that Y′ is formed from X′.

In general, the coefficient of any term in the above transformed equation, is formed from the preceding one, by multiplying each of its terms by the exponent of x' in this term, dividing the product by the number of coefficients preceding the one required, and then diminishing the exponents of x' by unity.

This law, by which the coefficients X′, Y′, $\frac{Z'}{2}$, $\frac{V'}{2.3}$ are derived from each other, is evidently an immediate consequence of that which regulates the different terms of the formula for the binomial. (See No. 152).

The expressions Y′, Z′, V′, W′... are called derived polynomials of X′, because Z′ is deduced or derived from Y′, as Y′ is derived from X′: V′ is derived from Z′, as Z′ is derived from Y′, and so on. Y′ is called *the first derived polynomial*, Z′ *the second*, *&c.* Recollect that X′ is what the first member of the proposed equation becomes when x' is substituted for x.

The coefficient of the first term of the proposed equation has been supposed equal to unity; if this was not the case, the law of formation for the coefficients of the transformed equations would be absolutely the same, and the coefficient of u^m would be equal to that of x^m.

To show the use of this law in practice, let it be required to make the coefficient of the second term of the following equation disappear.

$$x^4-12x^3+17x^2-9x+7=0.$$

According to the rule of No. 276, take $x=u+\frac{12}{4}$, or . . . . $x=3+u$, which will give a transformed equation of the 4th degree, and of the form

P and Q, the disappearance of the second term involves that of the third.

278. ***Remarks upon the preceding transformation. Formation of derived polynomials.***

The relation $x=u+x'$, of which we have made use in the two preceding numbers, indicates that the roots of the transformed equations are equal to those of the proposed, diminished or increased by a certain quantity. Sometimes this quantity is introduced in the calculus, as an indeterminate quantity, the value of which is afterwards fixed in such a manner as to satisfy a given condition; sometimes it is a particular number given *à priori*, which expresses a *constant difference* between the roots of a primitive equation, and those of another equation which we wish to form.

In short, the transformation which consists in substituting $u+x'$ for x in an equation is of very frequent use in the theory of equations. Now there is a very simple method of obtaining, in practice, the transformation which results from this substitution.

To show this we will invert the order of the terms in $u+x'$, that is, for x substitute $x'+u$ in the equation

$$x^m+Px^{m-1}+Qx^{m-2}+Rx^{m-3}+\ldots Tx+U=0\,;$$

it becomes, by developing and arranging according to the ascending powers of u.

$$\begin{array}{ll|l|l}
x'^m & +mx'^{m-1} & u+m\frac{m-1}{2}x'^{m-2} & u^2+\ldots u^m=0 \\
+Px'^{m-1} & +(m-1)Px'^{m-2} & +(m-1)\frac{m-2}{2}Px'^{m-3} & \\
+Qx'^{m-2} & +(m-2)Qx'^{m-3} & +(m-2)\frac{m-3}{2}Qx'^{m-4} & \\
+\ldots\ldots & +\ldots\ldots & +\ldots\ldots & \\
+Tx' & +T & & \\
+U & & &
\end{array}$$

If we observe how the coefficients of the different powers of u are composed, we will see that the coefficient of u^0 is nothing more than what the first member of the proposed equation becomes when x' is substituted in place of x; we will hereafter denote it by X'.

The coefficient of u^1 is formed by means of the preceding, or X', by multiplying each of the terms of X' by the exponent of x' in this term, and then diminishing this exponent by unity; we will call this coefficient Y'.

The coefficient of u^2 is formed from Y' by multiplying each of the terms of Y' by the exponent of x in this term, dividing the product by two, and then diminishing the exponent by unity. By calling this coefficient $\frac{Z'}{2}$ it is evident that Z' is formed from Y' in the same manner that Y' is formed from X'.

In general, the coefficient of any term in the above transformed equation, is formed from the preceding one, by multiplying each of its terms by the exponent of x' in this term, dividing the product by the number of coefficients preceding the one required, and then diminishing the exponents of x' by unity.

This law, by which the coefficients X', Y', $\frac{Z'}{2}$, $\frac{V'}{2.3}$ are derived from each other, is evidently an immediate consequence of that which regulates the different terms of the formula for the binomial. (See No. 152).

The expressions Y', Z', V', W'... are called derived polynomials of X', because Z' is deduced or derived from Y', as Y' is derived from X': V' is derived from Z', as Z' is derived from Y', and so on. Y' is called *the first derived polynomial*, Z' *the second*, *&c.* Recollect that X' is what the first member of the proposed equation becomes when x' is substituted for x.

The coefficient of the first term of the proposed equation has been supposed equal to unity; if this was not the case, the law of formation for the coefficients of the transformed equations would be absolutely the same, and the coefficient of u^m would be equal to that of x^m.

To show the use of this law in practice, let it be required to make the coefficient of the second term of the following equation disappear.

$$x^4-12x^3+17x^2-9x+7=0.$$

According to the rule of No. 276, take $x=u+\frac{12}{4}$, or .... $x=3+u$, which will give a transformed equation of the 4th degree, and of the form

$$X'+Y'u+\frac{Z'}{2}u^2+\frac{V'}{2\times 3}u^3+u^4=0,$$

and the operation is reduced to finding the values of - - - - -
$X', Y', \frac{Z'}{2}, \frac{V'}{2.3}$.

Now it follows from the preceding law, that

$X'= (3)^4-12.(3)^3+17.(3)^2-9.(3)^1+7$, or $X'=-110$;
$Y'=4.(3)^3-36.(3)^2+34.(3)^1-9$, or - - - - $Y'=-123$;
$\frac{Z'}{2}=6.(3)^2-36.(3)^1+17$, or - - - - - - - - $\frac{Z'}{2}=-37$;
$\frac{V'}{2.3}=4.(3)^1-12$, - - - - - - - - - - - - - - - - $\frac{V'}{2.3}=0$.

Therefore the transformed equation becomes

$$u^4-37u^2-123u-110=0.$$

Again, transform the equation

$$4x^3-5x^2+7x-9=0$$

into another, the roots of which exceed the roots of the proposed equation by unity.

Take $u=x+1$; there will result $x=-1+u$, which gives the transformed equation

$$X'+Y'u+\frac{Z'}{2}u^2+4u^3=0.$$

$X'= 4.(-1)^3- 5.(-1)^2+7.(-1)^1-9$, or $X'=-25$;
$Y'=12.(-1)^2-10.(-1)^1+7$. - - - - - - - - $Y'= 29$;
$\frac{Z'}{2}=12.(-1)^1-5$. - - - - - - - - - - - - - - $\frac{Z'}{2}=-17$;
$\frac{V'}{2.3}=4$. - $\frac{V'}{2.3}= 4$.

Therefore the transformed equation becomes

$$4u^3-17u^2+29u-25=0.$$

The following examples may serve the student for exercises:

Make the second term vanish from the following equations.

1st. $x^5-10x^4+7x^3+4x-9=0$.
(Result, $u^5-33u^3-118u^2-152u-73=0$.)

2d. $3x^3+15x^2+25x-3=0$.

(Result, $3u^3-\frac{152}{9}=0$). See No. 276.

Transform the equation $3x^4-13x^3+7x^2-8x-9=0$ into another, the roots of which shall be less than the roots of the proposed by the fraction $\frac{1}{3}$.

$$\left(\text{Result } 3u^4-9u^3-4u^2-\frac{65}{9}u-\frac{102}{9}=0\right).$$

We will frequently have occasion for the law of formation of *derived polynomials.*

279. These polynomials have the following remarkable property.

Let X or $x^m+Px^{m-1}+Qx^{m-2}+ \text{----} =0$, be the proposed equation, and a, b, c, l, the m roots of it, we will have (No. 257) the identical equation

$$x^m+Px^{m-1}+ \text{----} =(x-a)(x-b)(x-c) \text{----} (x-l).$$

Substituting $x'+u$, or (to avoid the accents) $x+u$, in the place of x; it becomes

$$(x+u)^m+P(x+u)^{m-1}+ \text{----} =(x+u-a)(x+u-b) \text{----};$$

or, changing the order of the terms in the second member, and regarding $x-a$, $x-b$, ---- each as a single quantity.

$$(x+u)^m+P(x+u)^{m-1}+\ldots\ldots=(u+\overline{x-a})(u+\overline{x-b})\ldots\ldots(u+\overline{x-l}).$$

Now, by performing the operations indicated in the two members, we will, by the preceding No. obtain for the first member

$$X+Yu+\frac{Z}{2}u^2+ \text{----} +u^m;$$

X being the first member of the proposed equation, and Y, Z ---- the derived polynomials of this member.

With respect to the second member, it follows from No. 257, 1st. that the part involving u^0, or the last term, is equal to the product $(x-a)(x-b)$ ---- $(x-l)$ of the factors of the proposed equation;

2d. The coefficient of u is equal to the sum of the products of these m factors taken $m-1$ and $m-1$.

3d. The coefficient of u^2 is equal to the sum of the products of these m factors taken $m-2$ and $m-2$; and so on.

Moreover the two members of the last equation are identical; therefore (No. 188) the coefficients of the same powers *are equal.*

Hence $X=(x-a)(x-b)(x-c)$ - - - - $(x-l)$, which was already known.

Y, or the first derived polynomial, ***is equal to the sum of the products of the*** m ***factors of the first degree in the proposed equation, taken*** m−1 ***and*** m−1; or, ***equal to the sum of all the quotients that can be obtained by dividing*** X ***by each of the*** m ***factors of the first degree in the proposed equation;*** that is,

$$Y=\frac{X}{x-a}+\frac{X}{x-b}+\frac{X}{x-c}+ \text{- - - -} \frac{X}{x-l}.$$

$\frac{Z}{2}$ or the second derived polynomial, divided by 2, is equal to the sum of the products of the m factors of the proposed equation taken $m-2$ and $m-2$, or equal to the sum of all the quotients that can be obtained by dividing X by each of the factors of the second degree; that is,

$$\frac{Z}{2}=\frac{X}{(x-a)(x-b)}+\frac{X}{(x-a)(x-c)}+ \text{- - -} \frac{X}{(x-k)(x-l)};$$

and so on.

280. Second Transformation. *To make the denominators disappear from an equation.*

Having given an equation, we can always transform it into another of which the roots will be equal to a given ***multiple*** or *submultiple* of those of the proposed equation.

Take the equation $x^m+Px^{m-1}+Qx^{m-2}+$ - - - - $Tx+U=0$, and denote by y the unknown quantity of a new equation, of which the roots are k times greater than those of the proposed equation. If we take $y=kx$, there will result $x=\frac{y}{k}$; whence, substituting and multiplying every term by k^m, we have

$$y^m+PKy^{m-1}+QK^2y^{m-2}+RKy^{m-3}+\ldots+TKy^{m-1}+UK^m=0.$$

an equation of which the coefficients are equal to those of the proposed equation multiplied respectively by K^0, K^1, K^2, K^3, K^4.

This transformation is principally used *to make the* ***denominators disappear from an equation, without giving to the first term any other coefficient than unity.***

To fix the ideas, take the equation of the 4th degree

$$x^4+\frac{a}{b}x^3+\frac{c}{d}x^2+\frac{e}{f}x+\frac{g}{h}=0,$$

if in this equation we make $x=\frac{y}{K}$, y being a new unknown and K an indeterminate quantity, it becomes

$$y^4+\frac{aK}{b}y^3+\frac{cK^2}{d}y^2+\frac{eK^3}{f}y+\frac{gK^4}{h}=0.$$

Now, there may be two cases,

1st. Where the denominators b, d, f, h, are prime with each other; in this hypothesis, as K is altogether arbitrary, take $K=bdfh$, *the product of the denominators*, it becomes

$$y^4+adfh.y^3+cb^2df^2h^2.y^2+eb^3d^3f^2h^3.y+gb^4d^4f^4h^3=0,$$

an equation the coefficients of which are entire, and that of its first term unity.

We have besides, the equation $x=\frac{y}{bdfh}$, to determine the values of x corresponding to those of y.

2d. When the denominators contain common factors; in which case we will evidently render the coefficients entire by taking for K the smallest multiple of all the denominators. But we can simplify this still more, by observing that it is reduced to determining K in such a manner that K^1, K^2, K^3.... shall contain the prime factors which compose b, d, f, h, raised to powers at least equal to those which are found in the denominators.

Thus, let the equation $x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{7}{150}x-\frac{13}{9000}=0.$

Take $x=\frac{y}{k}$, it becomes $y^4-\frac{5k}{6}.y^3+\frac{5k^2}{12}.y^2-\frac{7k^3}{150}.y-\frac{13k^4}{9000}=0.$

First make $k=9000$, which is a multiple of all the other denominators, it is clear that the coefficients become whole numbers.

But if we decompose 6, 12, 150 and 9000 into their factors, we find

$$6=2\times3,\ 12=2^2\times3,\ 150=2\times3\times5^2,\ 9000=2^3\times3^2\times5^3;$$

and by simply making $k=2\times3\times5$, the product of different simple factors, we obtain

$$k^2=2^2\times3^2\times5^2,\ k^3=2^3\times3^3\times5^3,\ k^4=2^4\times3^4\times5^4,$$

whence we see that the values of k, k^2, k^3, k^4, contain the prime factors of 2, 3, 5, raised to powers at least equal to those which enter in 6, 12, 150 and 9000.

Hence the hypothesis $k=2\times3\times5$ is sufficient to make the denominators disappear. Substituting this value, the equation becomes

$$y^4-\frac{5.2.3.5}{2.3}y^3+\frac{5.2^2.3^2.5^2}{2^2.3}y^2-\frac{7.2^3.3^3.5^3}{2.3.5^2}y-\frac{13.2^4.3^4.5^4}{2^3.3^2.5^3}=0;$$

which reduces to

$$y^4-5.5.y^3+5.3.5^2y^2-7.2^2.3^2.5.y-13.2.3^2.5=0;$$

or $\quad y^4-25y^3+375y^2-1260y-1170=0.$

There are some circumstances which oblige us to augment the exponent of one of the prime factors of k, by one or more units. But we ought to perceive the necessity of taking k as small a number as possible: otherwise, we would obtain a transformed equation, having its coefficient very great, as may be seen by calculating the transformed equation resulting from the supposition $k=9000$ in the preceding equation.

Examples.

1st. $x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0$; $x=\frac{y}{6}$,

whence

$$y^3-14y^2+11y-75=0;$$

2d. $x^5-\frac{13}{12}x^4+\frac{21}{40}x^3-\frac{32}{225}x^2-\frac{43}{600}x-\frac{1}{800}=0$; $x=\frac{y}{2^2.3.5}$,

or $x=\frac{y}{60}$

whence

$$y^5-65y^4+1890y^3-30720y^2-928800y+972000=0.$$

281. The preceding transformations are those most frequently used; there are others very useful, of which we will speak as they present themselves; they are too simple to be treated of separately.

In general, the problem of the transformations of equations should be considered as an application of the problem of *elimination* between two equations of any degree whatever involving two unknown quantities. In fact, an equation being given, suppose *that we* wish to transform it into another, of which the roots have, *with* those of the proposed equation, a determined relation.

Denote the proposed equation by $F(x)=0$, (enunciated function of x equal to zero), and the algebraic expression of the relation which should exist between x and the new unknown quantity y, by $F'(x, y)=0$; the question is reduced to finding, by means of these two equations, a new equation involving y, which will be the required equation. When the unknown quantity x is only of the first degree in $F'(x, y)=0$, the transformed equation is easily obtained, but if it is found therein raised to the second, third - - - power, we must have recourse to the methods of elimination.

Elimination. Part First.

282. To eliminate between two equations of any degree whatever, involving two unknown quantities, is to obtain, by a series of operations, performed on these equations, *a single equation which contains but one of the unknown quantities*, and which gives all the values of this unknown quantity which will satisfy the proposed equation at the same time as the corresponding values of the other unknown quantities.

This equation, *which is a function of one of the unknown quantities*, is called *the final equation*, and the values of the unknown quantity found from this equation, are called *compatible values*.

Of all the known methods of elimination, *the method of the common divisor*, is, in general, the most expeditious; it is the method which we are going to develope.

Let $F(x, y)=0$ and $F'(x, y)=0$ be any two equations whatever, or, more simply,

$$A=0, \quad B=0.$$

Suppose the final equation involving y obtained, and let us try to discover some property of the roots of this equation, which may serve to determine it.

Let $y=\beta$ be one of the compatible values of y; it is clear that since this value satisfies the two equations at the same time as a certain value of x, it is such, that by substituting it in both of the equations, which will then contain only x, *the equation will admit of at least one common value of x*; and to this common value there will necessarily be a corresponding common divisor involving x (No. 252). This common divisor will be of the first, or a higher degree with respect to x, according as the particular value of $y=\beta$ corresponds to one or more values of x.

Reciprocally, every value of y, *which*, substituted in the two equations, gives a common divisor involving x, *is necessarily a compatible value*, because it then evidently satisfies the two equations at the same time as the value or values of x found from this common divisor when put equal to 0.

283. We will remark, that, before *the substitution, the first members of the equations cannot*, in general, *have a common divisor*, which is a function of one or both of the unknown quantities.

In fact, let us suppose for a moment that the equations $A=0$, $B=0$, are of the form

$$A'\times D=0, \quad B'\times D=0.$$

D being a function of x and y.

Making separately $D=y$, we obtain a single equation involving two unknown quantities, which can be satisfied with an *infinite number of systems of values*. Moreover every system which renders D equal to 0, would at the same time cause $A'D$, $B'D$ to vanish, and would consequently satisfy the equations $A=0$, $B=0$.

(Thus, the hypothesis of a common divisor of the two polynomials A and B, containing x and y, would bring with it as a consequence that the proposed equations were indeterminate.) Therefore if there exists a common divisor, involving x and y, of the two polynomials A and B, the proposed equations will be *indeterminate*, that is, they may be satisfied by an infinite number of systems of values of x and y. Then there are no data to determine a *final equation* in y, since the number of values of y is *infinite*.

If the two polynomials A and B were of the form $A'\times D$, $B'\times D$, D being a function of x only, we might conceive the equation $D=0$ resolved with reference to x, which would give one or more values for this unknown. Each of these values substituted in $A'\times D=0$ and $B'\times D=0$, at the same time with any *arbitrary* value of y, would verify these two equations, since D must be nothing, in consequence of the substitution of the value of x. Therefore, in this case, the proposed equations would admit of *a finite number of values* for x, but of an infinite number *of values for y*; then there could not exist a final equation in y.

Hence, when the equations A=0, B=0, are determinate, that is, when they only admit of a *limited number* of systems of values for x and y, their first members cannot have a *function of these unknown quantities for a common divisor*, unless a particular substitution has been made for one of them.

284. From this it is easy to deduce a process for obtaining the *final equation* involving y.

Since the characteristic property of every compatible value of y is, that being substituted in the first members of the two equations, it gives them a common divisor involving x, which they had not before, (unless the equations are indeterminate, which is contrary to the supposition), it follows, that if to the two proposed polynomials, arranged with reference to x, we apply the process for the greatest common divisor, (No. 267), we generally will not find one; but, by continuing the operation properly, we will arrive at a remainder independent of x, and which is a function of y, which, placed equal to 0, will give the required *final equation;* for every value of y found from this equation, reduces to nothing the last remainder of the operation for finding the common divisor; it is, then, such, that substituted in the preceding remainder, it will render this remainder a common divisor of the first members A and B. Therefore, each of the roots of the equation thus formed is a compatible value of y.

285. Admitting that the final equation may be completely resolved, which would give all the compatible values, it would afterwards be necessary to obtain the corresponding values of x. Now it is evident that it would be sufficient for this, to substitute the different values of y in the remainder preceding the last, put the polynomial involving x which results from it equal to 0, and find from it the values of x; for these polynomials are nothing more than the divisors involving x, which become common to A and B.

But as the final equation is generally of a degree superior to the second, we are obliged to defer the second part of the theory of elimination to another chapter, the object of which part is to determine all the systems of values which properly verify two equations of any degree whatever involving two unknown quantities.

We also propose to review the method just exposed, because there are some difficulties which we will endeavour to make disappear. But here our design was principally to show how, *two equations of any degree being given, we can, without supposing the resolution of any equation, arrive at another equation, containing only one of the unknown quantities which enter in the proposed equations.*

286. If we had three equations, (1), (2) and (3), containing the unknown quantities x, y and z, to obtain the final equation in z, that is, the equation containing all the values of the unknown quantity z which satisfy the three equations at the same time as certain values of x and y, it would be necessary to consider y as known, and eliminate x by means of the equations (1) and (2), then by means of (1) and (3), by the method of No. 284; from this there will result two equations involving y and x, to which we must apply the same method to eliminate y.

The reasoning is the same for four equations, with four unknown quantities, &c.

For the present we will limit ourselves to a single application of the method of elimination.

287. Let the following problem be proposed.

An equation of the degree m *with a single unknown quantity being given, it is required to find an equation, having for its roots a certain combination of any two of the roots of the proposed equation.*

Let $x^m+Px^{m-1}+Qx^{m-2}+ - - - - +Tx+U=0$

be the proposed equation; denote its roots by x', x'', x''', and let u be the unknown quantity of the required equation.

If we consider any two roots of the proposed equation, x' and x'', for example, we will have by hypothesis,

$$U=F(x', x'') - - - - - (1).$$

(The letter F expressing here a certain system of operations to be performed upon x' and x'', in order to obtain the value of u).

And since x' and x'' are roots of the given equation, we will have the two equations

$$x'^m+Px'^{m-1}+Qx'^{m-2}+ - - - - - Tx'+U=0 - - - - - (2),$$
$$x''^m+Px''^{m-1}+Qx''^{m-2}+ - - - - - Tx''+U=0 - - - - - (3).$$

The equations (1), (2) and (3) may then be considered as the equations of the problem; and when the nature of the combination or function expressed by the letter F is known and defined, it will be sufficient to eliminate x' and x'' by means of these three equations. The *final equation* involving u will be the equation required. In fact, the result, not containing either of the particular roots x', x'', since we will have eliminated them, will agree to all the roots x', x'', x''', ..., and will consequently have for a root a combination of any two of the roots of the proposed equation, (expressed by the characteristic F).

288. Let it be required, as a particular case of the preceding question, to determine an equation, the roots of which shall be the differences between any two roots of a given equation. This equation we will call *the equation of the differences.*

Solution. Let $x^m + Px^{m-1} + \text{- - - -} = 0$, be the proposed equation, x'. x'', x''' . . . its m roots, and let u denote the value of any one of those differences:

$$x''-x',\ x'''-x',\ x''-x''',\ x''-x^{\text{v}}.$$

From the enunciation, we have

$$u = x'' - x' \text{ - - - - } (1).$$

Moreover, x' and x'' being roots of the proposed equation, will satisfy it, and give

$$x'^m + Px'^{m-1} + \text{ - - - - } = 0 \text{ - - - - } (2),$$

$$x''^m + Px''^{m-1} + \text{ - - - - } = 0 \text{ - - - - } (3).$$

From equation (1) we deduce $x'' = x' + u$; whence, substituting in equation (3),

$$(x' + u)^m + P(x' + u)^{m-1} + \text{ - - - - } = 0 \text{ - - - - } (4).$$

The question is then reduced to eliminating x' by means of the two equations (2) and (4).

Now the equation (4), when developed, takes the form (No. 278)

$$X' + Y'u + \frac{Z'}{2}u^2 + \text{ - - - - } + u^m = 0;$$

and if we observe that X′ is nothing more than

$$x'^m + Px'^{m-1} + \ldots,$$

an expression which is equal to nothing by equation (2), the last equation, freed from the term X′, and afterwards divided by u, (See the N. B.), reduces to

$$Y' + \frac{Z'}{2}u + \frac{V'}{2.3}u^2 + \ldots . u^{m-1} = 0.$$

Hence the equation sought is that which results from the elimination of x' by means of the two equations

$$X'=0,$$

$$Y'+\frac{Z'}{2}u+\frac{V'}{2\times 3}u^2+ - - - - +u^{m-1}=0.$$

Therefore, to form the equation expressing the differences of the roots of the proposed equation, eliminate x' by means of the equation $X'=0$, which is deduced from the proposed equation by replacing x by x', and the equation resulting from the substitution of $x'+u$ in place of x; this equation being first freed from its first term X', and afterwards divided by u.

N. B. 1st. In practice we dispense with placing the accent over the letter x, that is, we eliminate x directly by means of the proposed equation $X=0$, or $x^m+Px^{m-1}+ - - - - =0$, and the equation $Y+\frac{Z}{2}u+\frac{V}{2.3}u^2+\ldots+u^{m-1}=0$, in which x enters the expression $Y+\frac{Z}{2}$, &c., in the same manner that x' enters $Y', \frac{Z'}{2}$, &c.

The result of the elimination is evidently the same.

After having substituted $x+u$ for x in the equation $X=0$, which gives

$$X+Yu+\frac{Z}{2}u^2+ - - - - u^m=0,$$

we omit the first term X, as forming the first member of the first equation, and we obtain a new equation

$$Yu+\frac{Z}{2}u^2+ - - - - u^m=0,$$

all the terms of which are divisible by u, or, which amounts to the same thing, it is satisfied by making $u=0$; and this should be the case, since among the differences between the roots it is necessary to reckon those which exist between each root and itself; but if we suppress this factor u, the equation will then contain only the *differences* between any one root and all the others.

Now these are the only differences which it will be for us to der in the sequel.

Let it be required, for example, to determine the equation containing the differences of the roots of the equation - - - $x^3-6x-7=0$.

It follows from the law of formation, (278), that

$$X=x^3-6x-7,\ Y=3x^2-6,\ \frac{Z}{2}=3x,\ \frac{V}{2.3}=1\ ;$$

which give the two equations

$$x^3-6x-7=0,$$
$$3x^2-6+3x.u+u^2=0,$$

by means of which x must be eliminated.

By applying the process of No. 284 to these two equations, we will obtain for the final equation involving u,

$$u^6-36u^4+324u^2+459=0.$$

This is the equation containing the differences of the roots of the proposed equation.

290. *Composition and form of the equation of the differences.* We can discover, *à priori*, for every equation of the degree m, the form and composition of the equation containing the differences of the roots of this equation.

Let us denote the roots of the proposed equation by x', x'', x'''..., any one of the differences by u, and observe if one of the differences is $x''-x'$, there will necessarily exist another $x'-x''$, which only differs from the first in the sign, that is, if α is a value of u, $-\alpha$ is necessarily another value of it; in like manner β being a root $-\beta$ is another, &c.

Then the first member of the equation involving u may be put under the form

$$(u-\alpha)\ (u+\alpha)\ (u-\beta)\ (u+\beta)\ (u-\gamma)\ (u+\gamma)...=0.$$

or by multiplying the factors two and two

$$(u^2-\alpha^2)\ (u^2-\beta^2)\ (u^2-\gamma^2)\ ---=0.$$

Hence this equation is of an even degree, and moreover contains only even powers of the unknown quantity; that is, it is of the form

$$u^{2n}+P'u^{2n-2}+Q'u^{2n-4}+----T'u^2+U'=0.$$

The degree $2n$ is besides equal to $m(m-1)$ or (No. 149) to the number of arrangements that can be made of m letters taken two and two.

If in the preceding equation we take, to simplify it, $u^2=z$, it becomes

$$z^n+P'z^{n-1}+Q'z^{n-2}+ - - - - Z'z+U'=0,$$

an equation of a degree n or $m\frac{m-1}{2}$, of which the roots are the squares of the differences between the roots x', x'', x''' - - - ; for by putting in the equation $u^2=z$, in place of u, its values $x''-x'$, $x'''-x'$ - - - there will result

$$z=(x''-x')^2,\ z=(x'''-x')^2 \ - - -$$

The equation involving z which we have just obtained, is called for this reason the *equation of the squares of the differences*, and we commonly prefer it to the equation of the differences, as being of a subduple degree.

Thus, in the example of the preceding number, the equation of the differences is of the 6th degree. It contains only the even powers of u, and if we make $u^2=z$, it becomes

$$z^3-36z^2+324z+459=0\ ;$$

an equation having for its roots, the squares of the differences of the roots of the proposed equation.

The equation of the differences, or of the squares of the differences, will be very useful in the sequel.

§ III. *Of Equations susceptible of reduction to those of lower degrees.*

Under this title are comprehended all equations in which two or more roots have particular relations between them, because we may, in general, make the resolution of these equations depend upon others *of an inferior degree.* Such are equations having equal roots, that is, when the first member is (255) the product of several equal factors of different denominations.

Method of Equal Roots.

291. An equation is said to contain equal roots, that is, (No. 255), its first member contains equal factors ; when the first derived polynomial, which (No. 279) is the sum of the products of *its m* factors taken $m-1$ and $m-1$, contains a factor in its different parts, which is two or more times a factor of the proposed

equation. Hence, *there must be a common divisor between the first member of the proposed equation and its first derived polynomial.*

It remains to ascertain the manner in which this common divisor is composed of the equal factors.

292. *Having given an equation, it is required to discover whether it has equal roots, and to determine these roots if possible.*

Let x denote the first member of the equation

$$x^m+Px^{m-1}+Qx^{m-2}+ \text{----} +Tx+U=0,$$

and suppose that it contains n factors equal to $x-a$, n' factors equal to $x-b$, n'' factors equal to $x-c$ ----, and contains also the simple factors $x-p$, $x-q$, $x-r$ ---; so that we may have

$$X=(x-a)^n(x-b)^{n'}(x-c)^{n''} \text{------} (x-p)(x-q)(x-r) \text{---}.$$

With respect to Y, or the derived polynomial of X, we have seen (No. 279) that it is *the sum of the quotients obtained by dividing* X *by each of the* m *factors of the first degree in the proposed equation.* Now, since X contains n factors equal to $x-a$, we will have n partial quotients equal to $\frac{X}{x-a}$; the same reasoning applies to each of the general factors, $x-b$, $x-c$, -----, Moreover we can form but one quotient equal to ------- $\frac{X}{x-p}$, $\frac{X}{x-q}$, $\frac{X}{x-r}$ ---- Therefore Y is necessarily of the form

$$Y=\frac{nX}{x-a}+\frac{n'X}{x-b}+\frac{n''X}{x-c}+ \text{---} +\frac{X}{x-p}+\frac{X}{x-q}+\frac{X}{x-r}+ \text{---}.$$

From this composition of the polynomial Y, it is plain that $(x-a)^{n-1}$, $(x-b)^{n'-1}$, $(x-c)^{n''-1}$ --- are factors common to all its terms; hence the product $(x-a)^{n-1}\times(x-b)^{n'-1}\times(x-c)^{n''-1}$ --- is a *relative divisor* of Y; moreover, it is evident that this product will also divide X, it is therefore a common divisor of X and Y; and it is their greatest common divisor. For; the prime factors of X are $x-a$, $x-b$, $x-c$ --- and $x-p$, $x-q$, $x-r$ ---; now $x-p$, $x-q$, $x-r$, cannot divide Y, since each of them is a common factor of all the parts of Y, except one. Hence, the greatest common divisor of X and Y is

$$D=(x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1} \text{----};$$

that is, *the greatest common divisor is composed of the product of those factors which enter two or more times in the proposed equation, raised to a power less by unity than they are in the given equation.*

293. From the above we deduce the following method:

To discover whether an equation $X=0$ contains any equal roots, *form* Y or *the derived polynomial of* X; *then seek for* (No. 261) *the greatest relative common divisor between* X and Y; if one cannot be obtained, the equation will not have equal roots or equal factors.

If we find a common divisor D, and it is of the first degree, or of the form $x-h$, make $x-h=0$, whence $x=h$; *we may then conclude that the equation has two roots equal to* h, *and has but one species of equal roots*, from which it may be freed by dividing X by $(x-h)^2$.

If D is of the second degree with reference to x, *resolve the equation* $D=0$; there may be two cases; the two roots will be equal, or they will be unequal. 1st. When we find $D=(x-h)^2$, *the equation has three roots equal to* h, *and has but one species of equal roots*, from which it can be freed by dividing X by $(x-h)^3$; 2d, when D is of the form $(x-h)(x-h')$, the proposed equation *has two roots equal to h*, and *two equal to h'*, from which it may be freed by dividing X by $(x-h)^2(x-h')^2$, or by D^2.

Suppose now that D is of any degree whatever; *it is necessary*, in order to know the species of equal roots, and the number of roots of each species, *to resolve completely the equation* $D=0$; *and every simple root of* D *will be twice a root of the proposed equation; every double root of* D *will be three times a root of the proposed equation;* and so on.

Examples.

294. Determine whether the equation

$$2x^4-12x^3+19x^2-6x+9=0$$

contains equal roots.

We have (No. 278) for the derived polynomial

$$8x^3-36x^2+38x-6.$$

Now, seeking (No. 275) for the greatest common divisor of these polynomials we find $D=x-3=0$, whence $x=3$; hence the proposed equation has *two* roots equal to 3.

Dividing its first member by $(x-3)^2$, we obtain

$$2x^2+1=0\ ;\ \text{whence}\ x=\pm\frac{1}{2}\sqrt{-2}.$$

Thus the equation is completely resolved, and its roots are

$$3,\ 3,\ +\frac{1}{2}\sqrt{-2}\ \text{and}\ -\frac{1}{2}\sqrt{-2}.$$

For a second example take $x^5-2x^4+3x^3-7x^2+8x-3=0$;
the first derived polynomial is $5x^4-8x^3+9x^2-14x+8$,
and the common divisor x^2-2x+1, or $(x-1)^2$,
hence the proposed equation has *three* roots equal to 1.

Dividing its first member by $(x-1)^3$ or by x^3-3x^2+3x-1, the quotient is

$$x^2+x+3=0\ ;\ \text{whence}\ x=\frac{-1\pm\sqrt{-11}}{2}\ ;$$

thus the equation is completely resolved.

Again, take the equation

$$x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0\ ;$$

the derived polynomial is

$$7x^6+30x^5+30x^4-24x^3-45x^2-6x+8\ ;$$

and the common divisor is

$$x^4+3x^3+x^2-3x-2.$$

The equation $x^4+3x^3+x^2-3x-2$ cannot be resolved directly, but by applying the method of equal roots to it, that is, by seeking for a common divisor between its first member and its derived polynomial, $4x^3+9x^2+2x-3$, we find a common divisor, $x+1$; which proves that the *square* of $x+1$ is a factor of $x^4+3x^3+x^2-3x-2$, and the *cube* of $x+1$ a factor of the first member of the proposed equation.

Dividing $x^4+3x^3+x^2-3x-2$ by $(x+1)^2$ or x^2+2x+1, we have x^2+x-2, which placed equal to zero, gives the two roots $x=1$, $x=-2$, or the two factors $x-1$ and $x+2$. Hence we have

$$x^4+3x^3+x^2-3x-2=(x+1)^2(x-1)(x+2).$$

Therefore the first member of the proposed equation is equal to

$$(x+1)^3(x-1)^2(x+2)^2\,;$$

or, the proposed equation has *three* roots equal to -1, *two* equal to 1, and *two* equal to -2.

Take the examples,

1st. $x^7-7x^6+10x^5+22x^4-43x^3-35x^2+48x+36=0$,

$$(x-2)^2(x-3)^2(x+1)^3=0.$$

2d. $x^7-3x^6+9x^5-19x^4+27x^3-33x^2+27x-9=0$,

$$(x-1)^2(x^2+3)^3=0.$$

295. When, in the application of the above method, we obtain an equation $D=0$, of a degree superior to the second, since this equation may itself be subjected to the method, we may often be able to decompose D into its factors, and would in this way know the different species of equal roots contained in the equation $X=0$, and the number of roots of each species. As to the simple roots of $X=0$, we begin by freeing this equation from the equal factors contained in it, and the resulting equation, $X'=0$, will make known the simple roots.

There is a case in which the equal roots of $X=0$ cannot be discovered immediately, viz. when the equation $D=0$ contains only unequal roots; in which case all the roots are double (that is, enter twice,) in the proposed equation, and they cannot be determined without resolving the equation $D=0$ by methods which will be explained hereafter.

296. That nothing may be wanting upon this subject, we are going to show, that *whatever may be the equation, if it contains equal roots, we can always reduce the resolution of it to that of a series of equations, of which the first contains only the simple roots of the proposed equation, the second the double roots, the third the triple roots, &c.*

For, let $X=0$ be the proposed equation, and denote the product of the factors of the first degree which correspond to the simple roots by X', the product of the factors of the first degree which correspond to the double roots by X'', the factors corresponding to the triple, quadruple, ... roots by X''', X^{iv}.... ; so that we will have

$$X=X'.X''^2.X'''^3.X^{iv4}.X^{v5}.$$

It follows from No. 292, that the greatest common divisor between X and its derived polynomial Y, is of the form

$$D=X''.X'''^{2}.X^{\text{iv}\,3}. \; . \; . \; . \; .,$$

since the equal factors of the proposed equation must be found in D, raised to a power less by unity than in the proposed equation.

This being the case, we operate upon D in the same manner we operated upon X, and denoting the greatest common divisor of D, and its derived polynomial by D'. We have

$$D'=X'''.X^{\text{iv}\,2}.X^{\text{v}\,3}.$$

By operating upon D' as upon D and X, we find

$$D''=X^{\text{iv}}.X^{\text{v}\,2}.$$

$$D'''=X^{\text{v}}.$$

(We suppose that 5 is the greatest number of times that the same root can enter in the proposed equation, that is, that $D'''=0$ contains only simple roots).

If we now divide, successively, X by D, D by D', D' by D'', D'' by D''', and denote the quotients by Q, Q', Q'', Q''', we have

$$Q=\frac{X}{D}=X'.X''.X'''.X^{\text{iv}}.X^{\text{v}},$$

$$Q'=\frac{D}{D'}=X''.X'''.Y^{\text{iv}}.X^{\text{v}},$$

$$Q''=\frac{D'}{D''}=X'''.X^{\text{iv}}.X^{\text{v}}.$$

$$Q'''=\frac{D''}{D'''}=X^{\text{iv}}.X^{\text{v}}.$$

Then dividing Q by Q', Q' by Q'', Q'' by Q''', and Q''' by D''', we find

$$\frac{Q}{Q'}=X', \quad \frac{Q'}{Q''}=X'', \quad \frac{Q''}{Q'''}=X''', \quad \frac{Q'''}{D'''}=X^{\text{iv}} \text{ and } D'''=X^{\text{v}}.$$

From which we see, that by means of three systems of operations, viz. one for the common divisor, and two series of divisions, we can isolate successively the factors X', X'', X''', X^{iv}, X^{v}, which, placed equal to zero, give, the first the simple roots, the second the double roots, &c.

See the following table of these different operations.

$X = X'X''^{2}X'''^{3}X^{iv\,4}X^{v\,5}$	$Q = X'X''X'''X^{iv}X^{v}$	$\frac{Q}{Q'} = X' = 0$
$D = X''X'''^{2}X^{iv\,3}X^{v\,4}$	$Q' = X''X'''X^{iv}X^{v}$	$\frac{Q'}{Q''} = X'' = 0$
$D' = X'''X^{iv}X^{v}$	$Q'' = X'''X^{iv}X^{v}$	$\frac{Q''}{Q'''} = X''' = 0$
$D'' = X^{iv}X^{v}$	$Q''' = X^{iv}X^{v}$	$\frac{Q'''}{D'''} = X^{iv} = 0.$
$D''' = X^{v} = 0$		

It should be remarked, that the degree of $X'=0$ expresses the number of simple roots of the proposed equation, the degree of $X''=0$ the number of double roots, that of $X'''=0$ the number of triple roots, &c.; and the complete resolution of these equations makes known the different species of double, triple, quadruple, &c. roots.

Thus, the method of equal roots is not, in general, a method for the complete resolution of an equation, but only a *method of reduction*. It is only in the case in which $X'=0, X''=0, X'''=0$, are of the first or second degree, that all the roots of the proposed equation can be immediately obtained.

297. The theory of equal roots can be applied to finding the relations which must exist between the coefficients of a polynomial of the second, third, ... degree, with reference to x, in order that this polynomial may be a perfect square, cube...... For this, it will be sufficient to form the derived of the proposed polynomial, then express (239) the condition to be fulfilled, in order that this derived polynomial may be a relative divisor of the proposed polynomial.

Take, for example, the trinomial of the second degree ax^2+bx+c, the derived polynomial of which is $2ax+b$,

$$\begin{array}{r|l} 2ax^2+2bx+2c & 2ax+b \\ \hline bx+2c & x+b \\ 2abx+4ac & \\ \hline 4ac-b^2. & \end{array}$$

By applying this process for the common divisor, with its modifications, to these polynomials, we find $4ac-b^2$, for a remainder, and if we suppose $4ac-b^2=0$, or $b^2-4ac=0$, $2ax+b$ *will be the* greatest common divisor of ax^2+bx+c and its derived polynomial, which is $2ax+b$ itself; therefore ax^2+bx+c may be regarded as the square of $2ax+b$ multiplied by a factor independent of x.

In fact, we have seen (No. 112), that $b^2-4ac=0$, is the condition that a trinomial of the second degree should be a perfect square.

Again, take the polynomial - - - - - - - ax^3+bx^2+cx+d,
of which the derived polynomial is - - - $3ax^2+2bx+c$;
in seeking for the greatest common divisor, we obtain the remainder $(6ac-2b^2)x+9ad-bc$. Now, by placing this remainder equal to nothing, we establish the condition that $3ax^2+2bx+c$ is the common divisor between the given polynomial and its derived polynomial; but this remainder must be nothing, whatever may be the value of x; therefore (No. 188) we have $6ac-2b^2=0$, and $9ad-bc=0$. Indeed, the first of these two conditions gives $c=\frac{b^2}{3a}$, and the second $d=\frac{bc}{9a}=\frac{b^3}{27a^2}$; whence, substituting in the proposed polynomial, we have

$$ax^3+bx^2+cx+d=a\left(x^3+\frac{b}{a}x^2+\frac{b^2}{3a^2}x+\frac{b^3}{27a^3}\right)=a\left(x+\frac{b}{3a}\right)^3$$

The same reasoning applies to polynomials of the 4th, 5th.... degrees.

298. The process for finding the common divisor also serves in some other cases to reduce the degree of an equation; for instance, when we know a certain relation between two of the roots of the proposed equation.

To illustrate this, take the general equation

$$x^m+Px^{m-1}+Q^{m-2}+....+Tx+U=0....(1),$$

and suppose that we have the relation $b=ka+h$ between two of the roots a and b (k and h being numbers which are known and given *à priori*).

Since the equation (1) must be satisfied by the two quantities a and $ka+h$, it follows that, if we put $kx+h$, in the place of x, which gives the new equation

$$(kx+h)^m+P(kx+h)^{m-1}+....+T(kx+h)+U=0....(2),$$

the two equations (1) and (2) must be satisfied by the same value a; there must therefore be a relative common divisor between their two first members.

Consequently, by applying the process for finding the greatest common divisor to these two polynomials, and placing this divisor equal to 0, we will obtain from it the value of a. This value being substituted in the relation $b=ka+h$, will make known the corresponding value of b.

If this common divisor is of the first degree with reference to x, we may conclude that *two* roots only of the equation satisfy the given relation. If it is of the second degree, there will be *two pair* of roots which will satisfy this relation, and their determination does not present any difficulty.

In general, let D be the common divisor obtained. The resolution of the proposed equation will then depend only upon that of the equation $D=0$, and the equation obtained by dividing the first member of the proposed equation by each of the factors of the first degree corresponding to the roots of $D=0$, and those deduced from the relation $b=ka+h$.

Let there be, for example, the equation

$$x^4-12x^3+48x^2-71x+30=0 \ldots (1)$$

we will suppose that two of the roots a and b, have with each other the relation $b=2a+1$.

By substituting $2x+1$ for x in the proposed equation, and reducing, we obtain

$$8x^4-32x^3+36x^2-7x-2=0.$$

Applying the rule for the greatest common divisor to the first members of this and the proposed equations, we find the relative divisor $x-2$; which gives

$$x-2=0, \text{ whence } x \text{ or } a=2.$$

Substituting this value of a in the relation $b=2a+1$, we have $b=5$.

Hence the first member of the proposed equation is divisible by

$$(x-2)(x-5) \text{ or } x^2-7x+10;$$

performing this division, we have for the quotient

$$x^2-5x+3=0, \text{ whence } x=\frac{5}{2}\pm\frac{1}{2}\sqrt{13}.$$

Thus the proposed equation is completely resolved.

299. The fundamental principle of the theory of equal roots can be deduced from the above analysis.

Making $h=0$ in the relation $b=ak+h$; it becomes $b=ka$, and the equation (2) of No. 298, will reduce to

$$k^m.x^m+Pk^{m-1}.x^{m-1}+....+Tk.x+U=0.$$

But since we must have, for the same value of x,

$$x^m+Px^{m-1}+Qx^{m-2}+.....+Tx+U=0,$$

we may substitute for one of these equations, the result obtained by subtracting one from the other, which gives

$$(k^m-1)x^m+P(k^{m-1}-1)x^{m-1}+.....+T(k-1)x=0,$$

or dividing each term by $(k-1)x$,

$$(k^{m-1}+k^{m-2}+...+k+1)x^{m-1}+P(k^{m-2}+k^{m-3}+...+k+1)x^{m-2}$$
$$+Q(k^{m-3}+k^{m-4}+....+k+1)x^{m-3}+...+T=0.$$

If now, besides the hypothesis $h=0$, we suppose $k=1$, which amounts to making the two roots a and b equal, the preceding equation becomes

$$mx^{m-1}+P(m-1)x^{m-2}+Q(m-2)x^{m-3}+....+T=0,$$

an equation of which the first member must have a common relative divisor with that of the proposed equation.

But $mx^{m-1}+P(m-1)x^{m-2}+....+T$ is nothing more than the derived polynomial of the first member of the proposed equation $x^m+Px^{m-1}+Qx^{m-2}+...+Tx+U=0$.

Therefore, in the case in which this last equation contains equal roots, *there must be a common divisor between the first member of it and its derived polynomial.*

Of Reciprocal Equations.

300. There is a remarkable class of equations susceptible of reduction; viz. those in which *the coefficients at equal distances from the extremes are equal to each other.* Such are the equations,

$$x^4+px^3+qx^2+px+1=0,\quad x^5+px^4+qx^3+qx^2+px+1=0.$$

They are called reciprocal equations, because if a is a root of the equation, then will $\frac{1}{a}$ be another root of it.

For, in the first of the above equations, since a is supposed *to be a root,* we have

$$a^4+pa^3+qa^2+pa+1=0;$$

but, by substituting $\frac{1}{a}$ in place of x, it becomes

$$\frac{1}{a^4}+p.\frac{1}{a^3}+q.\frac{1}{a^2}+p.\frac{1}{a}+1=0;$$

or $$1+pa+qa^2+pa^3+a^4=0;$$

which is nothing more than the preceding equality, written in an inverse order.

Hence we see that the roots of these equations, taken *two and two*, are the *inverse* or *reciprocals* of each other; whence it follows, that when half of the roots are determined, the other half can be obtained by dividing unity by each of the first. We will now show that the resolution of every reciprocal equation can be reduced to that of an equation of a subduple degree.

Take the equation of the 6th degree,

$$x^6+px^5+qx^4+rx^3+qx^2+px+1=0;$$

dividing it by x^3 (3 being half of the degree of the equation), it takes the form

$$x^3+\frac{1}{x^3}+p\left(x^2+\frac{1}{x^2}\right)+q\left(x+\frac{1}{x}\right)+r=0.$$

Now, making $x+\frac{1}{x}=z$; there will result

$$x^2-zx+1=0;$$

an equation which will give two values of x corresponding to the same value of z; we can therefore obtain the values of x when those of z are known.

From the equation $x+\frac{1}{x}=z$, we deduce successively

1st. by squaring and transposing, $$x^2+\frac{1}{x^2}=z^2-2;$$

2d. by multiplying the two new equations together,

$$x^3+x+\frac{1}{x}+\frac{1}{x^3}=z^3-2z; \quad \text{whence} \quad x^3+\frac{1}{x^3}=z^3-3z.$$

Substituting these expressions for $x+\frac{1}{x}$, $x^2+\frac{1}{x^2}$, $x^3+\frac{1}{x^3}$, in the above equation; it becomes

$$z^3-3z+p(z^2-2)+qz+r=0\,;$$

or reducing, $z^3+pz^2+(q-3)z+r-2p=0$;
an equation of the 3d degree, whilst the proposed equation is of the 6th. In like manner we might reduce an equation of the 4th, 8th, 10th, 12th....degree, to one of the 2d, 4th, 5th, 6th.... degree.

It yet remains to consider *reciprocal equations of an uneven degree.*

Take, for example, the equation

$$x^5+px^4+qx^3+qx^2+px+1=0.$$

We see at once that -1 is a root of this equation, for if we substitute -1 for x, we will obtain for the result - - - - - - - $-1+p-q+q-p+1$, an expression in which all the terms destroy each other. (It will be the same for every reciprocal equation of an uneven degree.)

Hence (No. 252) the first member is divisible by $x+1$; and by performing this division we obtain (No. 253),

$$\begin{array}{r|r|r|l} x^4-1 & x^3+1 & x^2-1 & x+1=0, \\ +p & -p & +p & \\ & +q & & \end{array}$$

a *reciprocal* equation of an even degree, upon which we can operate as in the case treated of above.

301. N. B. The roots of an equation of an uneven degree are also the *reciprocals* of each other, taken *two and two*, when the coefficients of the terms at equal distances from the extremes, are *equal and affected with contrary signs.*

Take, for example, the equation

$$x^5+px^4+qx^3-qx^2-px-1=0:$$

substituting $\frac{1}{x}$ for x, it becomes

$$\frac{1}{x^5}+p.\frac{1}{x^4}+q.\frac{1}{x^3}-q.\frac{1}{x^2}-p.\frac{1}{x}-1=0\,;$$

or, clearing the fraction and changing the signs,

$$x^5+px^4+qx^3-qx^2-px-1=0.$$

Hence, a being one of the roots of this equation, $\frac{1}{a}$ is necessarily another.

Moreover, it is visible that 1 will verify the equation; and if we divide by $x-1$, it becomes

$$\left.\begin{array}{r}x^4+1\\+p\end{array}\right|\begin{array}{r}x^3+1\\+p\\+q\end{array}\left|\begin{array}{r}x^2+1\\+p\end{array}\right|x+1=0,$$

an equation in which the coefficients at equal distances from the extremes *are equal and affected with the same sign.*

302. *Applications.* Let there be the general equation involving two terms; it is evident that 1 is a root of it; and dividing by $x-1$, we find (No. 31)

$$x^{m-1}+x^{m-2}+x^{m-3}+\dots+x^2+x+1=0,$$

a reciprocal equation, the resolution of which can, by means of the preceding principles, be reduced to that of an equation of a subduple degree.

Take, for example, the equation $x^5-1=0$;
by dividing by $x-1$, we have

$$x^4+x^3+x^2+x+1=0,$$

which can be put under the form

$$x^2+\frac{1}{x^2}+x+\frac{1}{x}+1=0\dots(1).$$

Making $x+\frac{1}{x}=z$, whence $x^2-zx+1=0$; there will result

$$x^2+2+\frac{1}{x^2}=z^2, \text{ or } x^2+\frac{1}{x^2}=z^2-2.$$

Substituting the values of $x+\frac{1}{x}$, and $x^2+\frac{1}{x^2}$ in the equation

(1) we obtain $\quad z^2-2+z+1=0,$

or reducing, $\quad z^2+z-1=0;$

hence $\quad z=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5};$

but the equation $x^2-xz+1=0$ gives

$$x=\frac{z}{2}\pm\frac{1}{2}\sqrt{z^2-4};$$

hence, by substituting for z its two values, and reducing, we have

$$x=-\frac{1}{4}\pm\frac{1}{4}\sqrt{5}\pm\frac{1}{4}\sqrt{10\pm2\sqrt{5}}.\sqrt{-1}.$$

The equation $x^{10}-1=0$, can also be completely resolved by the same method.

For, we have

$$x^{10}-1=(x^5-1)(x^5+1)=0.$$

We already know the roots of the equation $x^5-1=0$; as to those of the equation $x^5+1=0$, since, by changing x into $-x$, it becomes $x^5-1=0$, it is only necessary to take the roots of this equation with contrary signs.

CHAPTER VII.

Resolution of Numerical Equations, involving one or more Unknown Quantities.

THE principles established in the preceding chapter, are applicable to all equations, whether their coefficients are numerical or algebraic, and these principles should be regarded as the elements which have been employed in the resolution of equations of the higher degrees.

It has been said already, that analysts have hitherto been able to resolve only the general equations of the third and fourth degree. The formulas they have obtained for the values of the unknown quantities are so complicated and inconvenient, when they can be applied, (which is not always possible,) that the problem of the resolution of algebraic equations, of any degree whatever, may be regarded as more curious than useful. Therefore analysts have principally directed their researches to the resolution of *numerical equations*, that is, to those which arise from the algebraic translation of a problem in which the given quantities are particular numbers; and methods have been found, by means of which we can always determine the roots of a *numerical equation of any given degree.*

It is proposed to develope these methods in the first part of this chapter.

The object of the second part is the supplement of *elimination*, or the resolution of numerical equations involving two or more unknown quantities.

To render the reasoning general, we will represent the proposed equation by $x^m+Px^{m-1}+Qx^{m-2}+\ldots\ldots=0$, in which P, Q.....denote particular numbers, real, positive, or negative.

§ I. *Fundamental Principles. Limits of Roots.*

318. Fundamental principle. *When two numbers* p *and* q, *substituted in the place of* x *in a numerical equation, give two results, affected with contrary signs, the proposed equation contains a real root, comprehended between these two numbers.*

Let the proposed equation be

$$x^m+Px^{m-1}+Qx^{m-2}+\text{- - - -}\ Tx+U=0.$$

The first member will, in general, contain both positive and negative terms; denote the sum of the positive terms by A, and the sum of the negative terms by B, the equation will then take the form

$$A-B=0.$$

Suppose $p<q$, and that p substituted for x gives a *negative* result, and q a *positive* result.

Since the first member becomes negative by the substitution of p, and positive by the substitution of q, it follows that we have in the first case $A<B$, and in the second $A>B$. Now it results from the nature of the quantities A and B, that they both increase as x increases, since they contain only absolute numbers, and positive and entire powers of x; therefore, by making x augment by insensible degrees, from p to q, the quantities A and B will also increase by insensible degrees. Now since A, by hypothesis, from being less than B, afterwards becomes greater than it, A must necessarily have a more rapid increment than B, *which insensibly destroys the excess that* B *had over* A, *and finally produces an excess of* A *over* B. From this, we conceive that in the passage from $A<B$ to $A>B$, there must be an intermediate value for which A becomes equal to B, and this value substituted in the member corresponding to this circumstance is a root of the equation, since it verifies $A-B=0$, or the proposed equation. *Hence, when two numbers, &c.*

In the preceding demonstration, p and q have been supposed to be absolute numbers; but the proposition is not less true, whatever may be the signs with which p and q are affected. *For we will remark,* in the first place, that the above reasoning applies equally to the case in which one of the numbers p and q, p for example, is 0; that is, it could be proved, in this case, that there was at least one real root between 0 and q.

Let both p and q be *negative*, and represent them by $-p'$ and $-q'$.

If, in the equation $x^m+Px^{m-1}+Qx^{m-2}+ - - - - Tx+U=0$, we change x into $-y$, which gives the transformation

$$(-y)^m+P(-y)^{m-1}+Q(-y)^{m-2}+ - - - - T(-y)+U=0,$$

it is evident that substituting $-p'$ and $-q'$ in the proposed equation, amounts to the same thing as substituting p' and q' in the transformation, for the results of these substitutions are in both cases

$$(-p')^m+P(-p')^{m-1}+Q(-p')^{m-2}+ - - - \quad T(-p')+U,$$

and $(-q')^m+P(-q')^{m-1}+Q(-q')^{m-2}+ - - - - T(-q')+U$;

Now, since p and q or $-p'$ and $-q'$, substituted in the proposed equation, give results with contrary signs, it follows that the numbers p' and q', substituted in the transformation, also give results with contrary signs; therefore, by the first part of the proposition, there is at least one real root of the transformation contained between p' and q'; and in consequence of the relation $x=-y$, there is at least one value of x comprehended between $-p'$ and $-q'$ or p and q. This demonstration applies to cases in which $p=0$ or $q=0$.

Lastly, suppose p *positive* and q *negative* or equal to $-q'$ by making $x=0$ in the equation, the first member will reduce to its last term, which is necessarily affected with a sign contrary to that of p, or that of $-q$; whence we may conclude that there is a root comprehended between 0 and p, or between 0 and $-q'$, and consequently between p and $-q$ (No. 63).

319. *Second Principle.* When the two numbers, substituted in place of x, in an equation, give results affected with contrary signs, we may conclude that there is at least one real root comprehended between them, but we are not certain that there are no more, and *there may be any odd number of roots comprised between them.* This is the result of the following demonstration. *When an uneven number* $(2n+1)$ *of the real roots of an equation, are comprehended between two numbers, the results obtained by substituting* them for x, are affected with *contrary signs, and if they* comprehend an even number 2n, the results

obtained by their substitution are necessarily affected with the same sign.

To make this proposition as clear as possible, denote those roots of the proposed equation, $X=0$, which are supposed to be comprehended between p and q, by $a, b, c, ---$, and the product of the factor of the first degree, with reference to x, corresponding to those real roots which are not comprised between them, and to the imaginary roots, by Y, the signs of p and q being arbitrary.

The first member, X, can be put under the form

$$(x-a)(x-b)(x-c) ---- \times Y.$$

Now substitute in X, or the preceding product, p and q in place of x; we will obtain the two results.

$$(p-a)(p-b)(p-c) ---- \times Y',$$
$$(q-a)(q-b)(q-c) ---- \times Y'',$$

Y' and Y'' representing what Y becomes, when we replace x by p and q; these two quantities are necessarily affected with the same sign, for if they were not, by the first principle $Y=0$ would have at least one real root comprised between p and q, which is contrary to the hypothesis.

To determine the signs of the above results more easily, divide the first by the second, we obtain

$$\frac{(p-a)(p-b)(p-c)\ldots\ldots\times Y'}{(q-a)(q-b)(q-c)\ldots\ldots\times Y''}$$ which can be written thus; $\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\ldots\frac{Y'}{Y''}$.

Now since the roots $a, b, c, ---$ are comprised between p and q,

we have $p \gtrless a, b, c, d ----$,

but $q \lessgtr a, b, c, d ----$;

whence we deduce $p-a, p-b, p-c, \ldots \gtrless 0$,

and $x-a, x-b, x-c, \ldots \lessgtr 0$.

hence, since $p-a$ and $q-a$ are affected with contrary signs, as well as $p-b$ and $q-b, p-c$ and $q-c$ ----, the partial quotients are all *negative;* moreover $\frac{Y'}{Y''}$ is essentially positive, since Y′ and Y″ are affected with the same sign; therefore the product $\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times$ - - - - $\frac{Y'}{Y''}$, will be *negative*, when the number of roots, a, b, c - - - -, comprehended between p and q, is uneven, and *positive* when the number is even.

Consequently, the two results $(p-a)\ (p-b)\ (p-c)\ldots\times Y'$, and $(q-a)\ (q-b)\ (q-c)\ \ldots\ldots\times Y''$, will have contrary or the same signs, according as the number of roots comprised between p and q is *uneven* or *even.*

Limits of the real Roots of Equations.

320. The different methods for resolving numerical equations, consist generally in substituting particular numbers in the proposed equation, in order to discover if these numbers verify it, or whether there are roots comprised between these numbers. But by reflecting a little upon the composition of the first member, the first term being positive, and affected with the highest power of x, which is greater with respect to that of the inferior degree, in proportion to the magnitude of x, we are sensible that there are certain numbers, above which it would be useless to substitute, because all of these numbers would give positive results.

Every number which exceeds the greatest of the positive roots of an equation, is called a *superior limit* of the positive roots.

From this definition, it follows that the limit is susceptible of an infinite number of values; for when a number is found to exceed the greatest positive root, every number greater than this, is, for a still stronger reason, a superior limit. But it may be proposed to determine the simplest possible limit. Now we are sure of having one of the limits, when we obtain *a number, which, substituted in place of* x, *renders the first member positive, and which, at the same time, is such, that every greater number will also give a positive result.*

We will determine such a number.

321. Before resolving this question, we will propose a more simple one, viz.: *determine a number, which, substituted in place of* x *in an equation, will render the first term* x^m *greater than the* arithmetical sum *of all the others.*

Suppose that all the terms of the equation are negative, except the first, so that

$$x^m - Px^{m-1} - Qx^{m-2} - \ldots\ldots - Tx - U = 0.$$

It is required to find a number for x which will render

$$x^m > Px^{m-1} + Qx^{m-2} + \text{---} + Tx + U.$$

Let k denote the greatest coefficient, and substitute it in place of the coefficients; the inequality will become

$$x^m > kx^{m-1} + kx^{m-2} + \text{----} + kx + k.$$

It is evident that every number substituted for x which will satisfy this condition, will, for a stronger reason, satisfy the preceding. Now, dividing this inequality by x^m, it becomes

$$1 > \frac{k}{x} + \frac{k}{x^2} + \frac{k}{x^3} + \text{-----} + \frac{k}{x^{m-1}} + \frac{k}{x^m}.$$

Making $x = k$, the second member becomes $\frac{k}{k}$, or 1 plus a series of positive fractions; then the number k will not satisfy the inequality; but by supposing $x = k+1$, we obtain for the second member the series of fractions

$$\frac{k}{k+1} + \frac{k}{(k+1)^2} + \frac{k}{(k+1)^3} + \text{----} + \frac{k}{(k+1)^{m-1}} + \frac{k}{(k+1)^m},$$

which, considered in an inverse order, is an increasing geometrical progression, the first term of which is $\frac{k}{(k+1)^m}$, the ratio $k+1$, and the last term $\frac{k}{k+1}$; hence the expression for the sum of all the terms is, (No. 201),

$$\frac{\frac{k}{k+1}\cdot(k+1) - \frac{k}{(k+1)^m}}{k+1-1}, \text{ or } 1 - \frac{1}{(k+1)^m},$$

which is evidently less than unity.

Any number $> k+1$, put in place of x, will render the sum *of the fractions* $\frac{k}{x} + \frac{k}{x^2} + \text{----}$ still less.

Therefore, *the greatest coefficient of the equation plus unity, or any greater number, will render the first term* x^m *greater than the arithmetical sum of all the others.*

322. *Ordinary limit of the positive Roots.* The number obtained above may be considered a prime limit, since this number, or any greater number, rendering the first term superior to the sum of all the others, the results of the substitution of these numbers for x must be constantly positive; but this limit is commonly much too great, because, in general, the equation contains several positive terms. We will, therefore, seek for a limit suitable for all equations.

Let x^{m-n} denote the power of x, corresponding to the first negative term which follows x^m, and we will consider the most unfavourable case, viz. that in which all of the succeeding terms are negative, and affected with the greatest of the negative coefficients in the equation.

Let S be this coefficient, and try to satisfy the condition

$$x^m > Sx^{m-n} + Sx^{m-n-1} + \text{- - - - -} Sx + S;$$

or, dividing both members of this inequality by x^m,

$$1 > \frac{S}{x^n} + \frac{S}{x^{n+1}} + \frac{S}{x^{n+2}} + \text{- - - - -} + \frac{S}{x^{m-1}} + \frac{S}{x^m}.$$

Now by supposing $x^n = S$ or $x = \sqrt[n]{S}$, the second member becomes $\frac{S}{S}$, or 1, plus a series of positive fractions; but by making $x = \sqrt[n]{S} + 1$, or (supposing, for simplicity, $\sqrt[n]{S} = S'$, whence $S = S'^n$,) $x = S' + 1$, the second member becomes

$$\frac{S'^n}{(S'+n)^n} + \frac{S'^n}{(S'+1)^{n+1}} + \text{- - - - - -} + \frac{S'^n}{(S'+1)^{m-1}} + \frac{S'^n}{(S'+1)^m},$$

which is a progression by quotients, $\frac{S'^n}{(S'+1)^m}$ being the first term, $S'+1$ the ratio, and $\frac{S'^n}{(S'+1)^n}$ the last term. Hence the expression for the sum of all these fractions is

$$\frac{\frac{S'^n}{(S'+1)^n}\cdot(S'+1) - \frac{S'^n}{(S'+1)^m}}{S'+1-1} = \frac{S'^{n-1}}{(S'+1)^{n-1}} - \frac{S'^{n-1}}{(S'+1)^m}.$$

which is evidently greater than 1.

Moreover, every number $> S'+1$ or $\sqrt[n]{S}+1$, will when substituted for x, render the sum of the fractions - - - - - - - - $\frac{S}{x^n}+\frac{S}{x^{n+1}}+$ - - - - still smaller, since the numerators remaining the same, the denominator will increase. Hence $\sqrt[n]{S}+1$, and any greater number, will render the first term x^m greater than the arithmetical sum of all the negative terms of the equation, and will consequently give a positive result for the first member.

Therefore $\sqrt[n]{S}+1$, or *unity increased by that root of the greatest negative coefficient which is indicated by the number of terms which precede the first negative term, is a superior limit of the positive roots of the equation.*

Make $n=1$, in which case the first negative term is the second term of the equation; the limit becomes $\sqrt[1]{S}+1$, or $S+1$; that is, *the greatest negative coefficient plus unity.*

Let $n=2$, then the two first terms are positive, or the term x^{m-1} is wanting in the equation; the limit is then $\sqrt[2]{S}+1$.

When $n=3$ the limit is $\sqrt[3]{S}+1$ - - - -

Find the superior limits for the positive roots in the following examples:

$x^4-5x^3+37x^2-3x+39=0$; $\sqrt[n]{S}+1=\sqrt[1]{5}+1=6$;

$x^5+7x^4-12x^3-49x^2+52x-13=0$; $\sqrt[n]{S}+1=\sqrt{49}+1=8$,

$x^4+11x^2-25x-67=0$; $\sqrt[n]{S}+1=\sqrt[3]{67}+1=6$;

$3x^3-2x^2-11x+4=0$; $\sqrt[n]{S}+1=\frac{11}{3},+1=5$.

323. By transforming the given equation into another, we may often obtain a smaller limit than $\sqrt[n]{S}+1$.

For example, the first of the above equations can be put under the form

$$x^3(x-5)+37x\left(x-\frac{3}{37}\right)+39=0.$$

Now, it is evident that 5, or any greater number, substituted for x, will give a positive result; hence 5 is a limit, whereas the formula $\sqrt[n]{S}+1$ would give 6.

In like manner the second equation becomes

$$x^2(x^2-49)+7\left(x-\frac{12}{7}\right)+52\left(x-\frac{1}{4}\right)=0,$$

From which it is evident that 4, or any greater number, will give a positive result.

This method, which can only be applied to certain equations, consists in *decomposing the first member into several parts, each composed of two factors, the first of which is a positive monomial, and the other a binomial involving* x, *having its second term numerical and negative, then determining* x *in such a manner, that the factor within the parenthesis may be positive.*

It is rarely applicable to equations containing two or more consecutive terms affected with the sign —.

For example, it cannot be applied to the equation

$$x^5-5x^4-13x^3+17x^2-69=0.$$

324. *Newton's method for determining the smallest limit in entire numbers.*

Let $X=0$, be the proposed equation; if in this equation we make $x=x'+u$, x' being indeterminate, we will obtain (No. 278.)

$$X'+Y'u+\frac{Z'}{2}u^2+ \text{ - - - } +u^m=0. \text{ (1)}$$

Conceive that after successive trials we have determined a number for x, which, substituted in X', Y', $\frac{Z'}{2}$, - - -, renders all these coefficients positive at the same time; this number will be greater than the greatest positive root of the equation $X=0$.

For, the coefficients of the equation (1) being all positive, no absolute number can verify it; therefore *all* of the real values of u must be *negative;* but from the equation $x=x'+u$, we have $u=x-x'$; and in order that the values of u corresponding to each of the values of x and x' (already determined) may be negative, it is absolutely necessary that the greatest positive value of x should be less than the value of x',

Example.

$$x^4-5x^3-6x^2-19x+7=0.$$

As x' is indeterminate, the letter x may be retained in the formation of the derived polynomials, and we have

$$X = x^4 - 5x^3 - 6x^2 - 19x + 7,$$

$$Y = 4x^3 - 15x^2 - 12x - 19,$$

$$\frac{Z}{2} = 6x^2 - 15x - 6,$$

$$\frac{V}{2.3} = 4x - 5.$$

The question is, as stated above, reduced to finding the smallest number which, substituted in place of x, will render all of these polynomials positive.

It is plain that 2 and every number >2, will render the polynomial of the first degree positive.

2, substituted in the polynomial of the second degree, gives a negative result; but 3, or any number >3, gives a positive result.

3 and 4, substituted in the polynomial of the third degree, give a negative result; but 5 and any greater number give a positive result.

Lastly, 5 substituted in X, gives a negative result, and so does 6; for the three first terms $x^4 - 5x^3 - 6x^2$ are equivalent to the expression $x^3(x-5) - 6x^2$, which is reduced to 0 when $x=6$; but $x=7$ evidently gives a positive result. Hence 7 *is a superior limit of the positive roots of the proposed equation;* and since it has been shown that 6 gives a negative result, it follows that there is at least one real root between 6 and 7.

Applying this method to the equation

$$x^5 - 3x^4 - 8x^3 - 25x^2 + 4x - 39 = 0,$$

the superior limit will be found to be 6.

We would find 7, for the superior limit of the positive roots of the equation

$$x^5 - 5x^4 - 13x^3 + 17x^2 - 69 = 0.$$

This method is scarcely ever used, except in finding incommensurable roots.

325. It remains to determine the *superior limit* of the negative roots, and the *inferior limits* of the positive and negative roots.

Hereafter we will designate the superior limit of the positive roots of an equation by the letter L.

1st. If in the equation $X=0$, we make $x=-y$, which gives the equation $Y=0$, it is clear that the positive roots of this new equation, taken with the sign —, will give the negative roots of the proposed equation; therefore, determining, by the known methods, the superior limit L′ of the positive roots of the equation $Y=0$, we will have $-L'$ for the *superior limit* (numerically) *of the negative roots of the proposed equation.*

2d. If in the equation $X=0$, we make $x=\frac{1}{y}$, which gives the equation $Y=0$, it follows from the relation $x=\frac{1}{y}$ that the greatest positive values of y correspond to the smallest of x; hence, designating the superior limit of the positive roots of the equation $Y=0$ by L″, we will have $\frac{1}{L''}$ for *the inferior limit of the positive roots of the proposed equation.*

3d. Finally, if we replace x, in the proposed equation, by $-\frac{1}{y}$, and find the superior limit L‴ of the transformed equation $Y=0$, $-\frac{1}{L'''}$ will be *the inferior limit* (numerically) *of the negative roots of the proposed equation.*

326. *Every equation in which there are no variations in the signs,* that is, *in which all the terms are positive, must have all of its real roots negative;* for every positive number substituted for x will render the first member essentially positive.

Every complete equation, having its terms alternately positive and negative, must have its real roots all positive; for every negative number substituted for x in the proposed equation, would render all the terms positive, if the equation was of an even degree, and all of them negative if it was of an odd degree. Hence the sum would not be equal to zero in either case.

This is also true for *every incomplete equation, from which there results, by substituting* —y *for* x, *an equation having all of its terms affected with the same sign.*

Consequences deduced from the preceding Principles.

327. First. *Every equation of an odd degree the coefficients*

of which are real, ***has at least one real root affected with a sign contrary to that of its last term.***

For, let $x^m+Px^{m-1}+----Tx\pm U=0$, be the proposed equation; and first consider the case in which the last term is *negative*.

By making $x=0$ the first member becomes $-U$. But by substituting for x, the greatest negative coefficient plus unity, (321), or $(K+1)$, the first term x^m will become greater than the arithmetical sum of all the others, the result of this substitution will therefore be *positive;* hence, ***there is at least one real root comprehended between*** 0 ***and*** $K+1$, which root is positive, and consequently affected with a sign *contrary* to that of the last term.

Suppose now that the last term is *positive*.

Making $x=0$, we obtain $+U$ for the result; but by putting $-(K+1)$ in place of x, we will obtain a *negative* result, since the first term becomes negative by this substitution; hence the equation has at least one real root comprehended between 0 and $-(K+1)$ that is negative or *affected with a sign contrary* to that of the last term.

Second. *Every equation of an even degree*, involving only real coefficients, ***of which the last term is negative, has at least two real roots, one positive and the other negative.*** For, let $-U$ be the last term; making $x=0$, there results $-U$. Now substitute either $K+1$, or $-(K+1)$, K being the greatest negative coefficient of the equation (321): as m is an even number, the first term x^m will remain positive; besides, by these substitutions, it becomes greater than the sum of all the others; therefore the results obtained by these substitutions are both *positive*, or affected with a sign contrary to that given by the hypothesis $x=0$; hence the equation ***has at least two real roots***, one comprehended between 0 and $K+1$, or *positive*, and the other between 0 and $-(K+1)$, or *negative*.

328. Third. *If an equation*, involving only real coefficients, ***contains imaginary roots, the number of these roots must be even.***

For, conceive that the first member has been divided by all of the simple factors corresponding to the real roots; the coefficients of the *quotient* will be real (No. 253); ***it must also be of an even degree***; for if it was uneven, by placing it equal to zero, we would obtain an equation which would contain at least one real root, which, from the nature of the equation, it cannot have.

Remark. There is a property of the above polynomial quotient which belongs exclusively to equations containing only imaginary roots; viz. *it remains always positive for any real value substituted for* x.

For, if it could become negative, since we could also obtain a positive result, by substituting K+1 or the greatest negative coefficient plus unity for x, it would follow that this polynomial placed equal to zero, would have at least one real root comprehended between K+1 and the number which would give a negative result.

It also follows, that the last term of this polynomial must be *positive*, otherwise $x=0$ would give a negative result.

329. Fourth. *When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative this number is uneven.*

For, first suppose that the last term is +U, or *positive*. Since by making $x=0$, there will result +U, and by making $x=\mathrm{K}+1$, the result will also be positive, it follows that 0 and K+1 give two results affected with the same sign, and consequently (No. 319), the number of real roots (if any) comprehended between them is even.

When the last term is −U, then 0 and K+1 give two results affected with contrary signs, and consequently comprehend either a *single real root*, or *an odd number of them.*

The *reciprocal* of this proposition is evident.

Descartes' Rule.

330. *An equation of any degree whatever cannot have a greater number of positive roots than there are* variations *in the signs of its terms, nor a greater number of negative roots than there are* permanences *of these signs.*

The proposition would evidently be demonstrated, if it was shown that the multiplication of the first member by a factor $x-a$ corresponding to a *positive* root, would introduce *at least one variation*, and that the multiplication by a factor $x+a$, would introduce *at least* one permanence.

Let there be the equation

$$x^m \pm Ax^{m-1} \pm Bx^{m-2} \pm Cx^{m-3} \pm \ldots\ldots \pm Tx \pm U = 0,$$

in which the signs succeed each other in any manner whatever; by multiplying it by $x-a$, we have

$$\begin{array}{r|r|r|r|r} x^{m+1}\pm A & x^{m}\pm B & x^{m-1}\pm C & x^{m-2}\pm \ldots\ldots \pm U & x \\ -a & \mp Aa & \mp Ba & \mp Ta & \mp Ua \end{array}$$

The coefficients which form the first horizontal line of this product, are those of the proposed equation, taken with the same sign; and the coefficients of the second line are formed from those of the first, multiplied by a, taken with contrary signs and advanced one rank towards the right.

Now, so long as each coefficient of the upper line is greater than the corresponding one in the lower, it will determine the sign of the total coefficient; hence, in this case there will be from the first term to that preceding the last inclusively, the same variations and the same permanences as in the proposed equation; but the last term $\mp Ua$ having a sign contrary to that which immediately precedes it, there must be one more variation than there was in the proposed equation.

When a coefficient in the lower line is affected with a sign contrary to the one corresponding to it in the upper, and is also greater than this last, there is a change from a permanence of sign to a variation; for the sign of the term in which this happens, being the same as that of the inferior coefficient, must be contrary to that of the preceding term, which has been supposed to be the same as that of its superior coefficient. Hence, each time we descend from the upper to the lower line, in order to determine the sign, there is a variation which is not found in the proposed equation; and if, after passing into the lower line, we continue in it throughout, we will find the same variations and the same permanences as in the proposed equation, since the coefficients of this line are all affected with signs contrary to those of the primitive coefficients. When we ascend from the lower to the upper line, there may be either a variation or a permanence. But by even supposing that this passage produces permanences in all cases, since the last term $\pm Ua$ forms a part of the lower line, it will be necessary to go once more from the upper line to the lower, than from the lower to the upper. Hence the new equation must have at least one more *variation than the proposed;* and it will be the same for each positive root introduced into it.

It may be demonstrated in an analogous manner, that *the multiplication by a factor $x+a$*, corresponding to a negative root, *would introduce one permanence more.* Hence, in any equation, the number of positive roots cannot be greater than the number of VARIATIONS of sign, nor the number of negative roots greater than the number of PERMANENCES.

331. *Consequence.* When the roots of an equation are all real, *the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences.*

For, let m denote the degree of the equation, n the number of variations of the signs, p the number of permanences; we will have $m=n+p$. Moreover, let n' denote the number of positive roots, and p' the number of negative roots, we will have $m=n'+p'$; whence

$$n+p=n'+p'.$$

Now, we have just seen that n' cannot be $>n$, and p' cannot be $>p$; therefore we must have $n'=n$, and $p'=p$.

332. *Remark.* When an equation wants some of its terms, we can often discover the presence of imaginary roots, by means of the above rule.

For example, take the equation

$$x^3+px+q=0,$$

p and q being essentially positive; introducing the term which is wanting, by affecting it with the coefficient ± 0, it becomes

$$x^3\pm 0.x^2+px+q=0.$$

By considering only the superior signs we would only obtain permanences, whereas the inferior sign would give two variations. This proves that the equation has some imaginary roots; for if they were all three real, it would be necessary by virtue of the superior sign, that they should be all negative, and, by virtue of the inferior sign, that two of them should be positive and one negative, which are *contradictory results.*

We can conclude nothing from an equation of the form

$$x^3-px+q=0;$$

for introducing the term $\pm 0.x^2$, it becomes

$$x^3 \pm 0.x^2 - px + q = 0,$$

which contains one permanence and two variations, whether we take the superior or inferior sign. Therefore this equation may have its three roots real, viz, two positive and one negative; or, two of its roots may be imaginary and one negative, since its last term is negative. (No. 329).

§ II. *Of the Commensurable Roots of Numerical Equations.*

333. Every equation in which the coefficients are whole numbers, that of the first term being unity, can only have whole numbers for its commensurable roots.

For, let there be the equation

$$x^m + Px^{m-1} + Qx^{m-2} + \text{ - - - } + Tx + U = 0;$$

in which P, Q - - - T, U, are whole numbers, and suppose that it may have a commensurable fraction $\frac{a}{b}$ for a root, substituting this fraction for x, the equation becomes

$$\frac{a^m}{b^m} + P\frac{a^{m-1}}{b^{m-1}} + Q\frac{a^{m-2}}{b^{m-2}} + \text{ - - - - } + T\frac{a}{b} + U = 0;$$

whence, multiplying the whole equation by b^{m-1}, and transposing,

$$\frac{a^m}{b} = -Pa^{m-1} - Qa^{m-2}b - \text{ - - - } - Tab^{m-2} - Ub^{m-1};$$

but the second member of this equality is composed of a series of entire numbers, whilst the first is essentially fractional, for a and b being prime with each other, a^m and b will also be prime with each other; hence this equality cannot exist.

Therefore it is impossible for any commensurable fraction to satisfy the equation. Now it has been shown (280) that an equation containing rational, but fractional, coefficients, can be transformed into another in which the coefficients are whole numbers, that of the first term being unity. Hence *the research of the commensurable roots* (entire or fractional) *can always be reduced to that of the entire roots.*

334. This being the case, take the general equation.

$$x^m + Px^{m-1} + Qx^{m-2} + \text{ - - - - } + Rx^3 + Sx^2 + Tx + U = 0.$$

and let a denote any entire number, positive or negative, which *will verify it.*

Since a is a root, we will have the equality

$$a^m+Pa^{m-1}+ - - - - - +Ra^3+Sa^2+Ta+U=0 - - - (1);$$

replacing a by all the entire positive and negative numbers between 1 and the limit $+L$, and between -1 and $-L'$, those which verify the above equality will be the roots of the equation. But these trials being long and troublesome, we will deduce from the equality (1), (which is *a necessary and sufficient condition*), other conditions equivalent to this, and easier verified.

Transposing all the terms except the last, and dividing by a, the equality (1) becomes

$$\frac{U}{a}=-a^{m-1}-Pa^{m-2}- - - - - - -Ra^2-Sa-T - - - (2);$$

now, the second member of this equality is an entire number, hence $\frac{U}{a}$ must be an entire number; therefore *the entire roots of the equation are comprised among the divisors of the last term.*

Transposing $-T$ in the equality (2) and dividing by a, and making $\frac{U}{a}+T=T'$; it becomes

$$\frac{T'}{a}=-a^{m-2}-Pa^{m-3} - - - - -Ra-S - - - - (3);$$

the second member of this equality being an entire number, $\frac{T'}{a}$ or, *the quotient of the division of* $\frac{U'}{a}+T$ *by* a *is an entire number.*

Transposing the term $-S$ and dividing by a, it becomes, (by supposing - - - - - - - - - - - - - - - - - - $\frac{T'}{a}+S=S'$,)

$$\frac{S'}{a}=-a^{m-3}-Pa^{m-4}- - - - - - - - -R - - - - - - (4),$$

the second member of this equality being an entire number, $\frac{S'}{a}$ or, *or the quotient of the division of* $\frac{T'}{a}+S$ *by* a *is an entire number.*

By continuing to transfer the terms of the second member into the first, we will, after $m-1$ transformations obtain an equality of the form - - - - - - - - - - - - - - - $\frac{Q'}{a}=-a-P,$

Then, transposing the term $-P$, dividing by a, and making $\frac{Q'}{a}+P=P'$, we will find $\frac{P'}{a}=-1$, or $\frac{P'}{a}+1=0$.

This equality, which is only a transformation of the equality (1), is the *last condition which it is necessary and sufficient* that the entire number a should satisfy, in order that it may be known to be a root.

From the preceding conditions we may conclude that, in order that an entire number a (positive or negative) may be a root of the proposed equation, it is necessary

That *the quotient of the last term divided by* a *should be an entire number;*

Adding to this quotient the coefficient of x^1 (taken with its sign), *the quotient of this sum divided by* a *must be entire;*

Adding the coefficient of x^2 to this quotient, *the quotient of this new sum by* a *must be entire;* and so on.

Finally, adding the coefficient of the second term, or of x^{m-1}, to the preceding quotient, *the quotient of this sum divided by* a, *must be entire and equal to* -1; or, *the result of the addition of unity, or the coefficient of* x^m, *to the preceding quotient, must be equal to* 0.

Every number which will satisfy these conditions will be a root, and those which do not satisfy them should be rejected.

All of the entire roots may be determined at the same time, as follows.

After having determined all the divisors of the last term, write those which are comprehended between the limits $+L$ *and* $-L'$ *upon the same horizontal line; then underneath these divisors write the quotients of the last term by each of them.*

Add the coefficient of x^1 *to each of these quotients, and write the sums underneath the quotients which correspond to them; then divide these sums by each of the divisors, and write the quotients underneath the corresponding sums;* taking care to reject the fractional quotients and the divisors which produce them; and so on.

When there are terms wanting in the proposed equation, their coefficients (which are to be regarded as equal to 0) must be taken into consideration.

Example.

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0.$$

The superior limit of the positive roots of this equation is $13+1$ or 14. The coefficient 48 is not considered, since the two last terms can be put under the form $16(x-3)$; hence when $x>3$ this part is essentially positive.

The superior limit of the negative roots is (No. 325) $-(1+\sqrt{48})$, or -8.

Therefore the divisors are 1, 2, 3, 4, 6, 8, 12; moreover, neither $+1$, nor -1, will satisfy the equation, because the coefficient -48 is itself greater than the sum of all the others; we should therefore try only the *positive divisors* from 2 to 12, and the *negative divisors* from -2 to -6 inclusively.

By observing the rule given above, we have

12,	8,	6,	4,	3,	2,	—2,	—3,	—4,	—6
—4,	—6,	—8,	—12,	—16,	—24,	+24,	+16,	+12,	+8
+12,	+10,	+8,	+4,	0,	—8,	+40,	+32,	+28,	+24
+1,	+..,	..,	+1,	0,	—4,	—20,	..,	—7,	—4
—12,	..,	..,	—12,	—13,	—17,	—33,	..,	—20,	—17
—1,	..,	..,	—3,	..,	..,	..,	..,	+5,	..
—2,	..,	..,	—4,	..,	..,	..,	..,	+4,	..
..,	..,	..,	—1,	..,	..,	..,	..,	—1,	..

The *first* line contains the divisors, the *second* contains the quotients of the division of the last term -48, by each of the divisors. The *third* line contains these quotients augmented by the coefficient $+16$, and the *fourth* the quotients of these sums by each of the divisors; this second condition excludes the divisors $+8$, $+6$, and -3.

The *fifth* is the preceding line of quotients, augmented by the coefficient -13, and the *sixth* is the quotients of these sums by each of the divisors; this third condition excludes the divisors 3, 2, -2 and -6.

Finally, the *seventh* is the third line of quotients, augmented by the coefficient -1, and the *eighth* is the quotients of these sums by each of the divisors. The divisors $+4$ and -4 are the only ones which give —1; hence +4 and —4 are the only entire roots of the equation.

In fact, if we divide $x^4-x^3-13x^2+16x-48$, by the product $(x-4)(x+4)$, or x^2-16, the quotient will be x^2-x+3, which placed equal to zero, gives

$$x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-11};$$

therefore the four roots are 4, —4, and $\frac{1}{2}\pm\frac{1}{2}\sqrt{-11}$.

Examples.

1st. $x^4-5x^3+25x-21=0.$

2d. $15x^5-19x^4+6x^3+15x^2-19x+6=0.$

3d. $9x^6+30x^5+22x^4+10x^3+17x^2-20x+4=0.$

Of Real Incommensurable Roots.

340. When an equation has been formed from all the divisors of the first degree which correspond to its commensurable roots, the resulting equation contains the *incommensurable roots* of the proposed equation, *either real or imaginary.*

The true form of the real incommensurable roots of an equation will remain unknown, so long as there is not a general method for resolving equations of the higher degrees. Although this problem has not been resolved, yet there are methods for approximating as near as we please to the numerical values of these roots.

For greater simplicity, we will divide this theory into two parts.

In the first, we will suppose that the difference between any two roots of the proposed equation is greater than unity; and, *in the second,* that any of these differences may be less than unity.

First Part. When the difference between any two real roots is greater than unity.

[We will suppose, in what follows, that we have obtained the narrowest limits $+L$ and $-L'$, either by the method of decomposition (No. 323), or by Newton's method (No. 324).]

341. Each of the incommensurable roots being necessarily composed of *an entire part* and *a part lest than unity*, we will first determine the entire part of each root.

For this purpose, it is necessary to substitute, in the equation, for x, the series of natural numbers 0, 1, 2, 3...... and —1, —2, —3......, comprised between +L and —L′. Since there must be a real root between two numbers, which, by their substitution produce results affected with different signs (318), it follows *that each pair of numbers giving results affected with contrary signs, will comprehend a real root, and but one*, since by hypothesis the difference between any two of the roots is greater than unity. The entire part of the root will be the smallest of the two numbers substituted.

There are two cases which may occur; viz. by these different substitutions there may be *as many changes of* sign as there are units in the degree of the equation; in which case we may conclude that *all the roots are real. Or*, the number of changes of the sign will be less than the degree of the equation, and, in this case, it will have as many real roots as there are changes of sign; the other roots will be imaginary. In both cases, this method makes known the *entire part* of each of the real roots.

It now remains to determine *the part which is less than unity.*

342. *Lagrange's Method of Approximation.*

Let X=0 be an equation, all the roots of which have a different entire part, and suppose this part determined.

Let a and $a+1$ denote the two entire numbers between which one of them is found; a will express the entire part of this root.

To obtain the fractional part, make in the equation X=0, or

$$x + Px^{m-1} + Qx^{m-2} + \ldots . Tx + U = 0, \quad x = a + \frac{1}{y};$$

there will result the equation

$$X'y + Y'y^{m-1} + \frac{Z'}{2}y^{m-2} + \ldots . + 1 = 0.$$

For more simplicity, we will denote this equation by Y=0; it is of the same degree as the proposed equation, and consequently has m roots. Since the relation $x = a + \frac{1}{y}$, should give *all the values of* x, knowing those of y, and that, there is a real root, and but one, between a and $a+1$, one of the m values of $\frac{1}{y}$ must be positive and less than unity; this requires that one of the m values of y should be positive and greater than 1; farther, *there can be but one of these values greater than* 1, for this would amount to supposing more than one value of x comprised between a and $a+1$.

If then, in the equation $Y=0$, we substitute the series of entire numbers 1, 2, 3......, we are certain of obtaining a change of sign, and the two numbers which produce this change will comprehend the required value of y.

Let b and $b+1$ be these two numbers, and make $y=b+\frac{1}{y'}$ in the equation $Y=0$, this will give a new equation $Y_1=0$; and this will also have *one real positive root, greater than* 1, which will be discovered by the substitution of the entire numbers, 1, 2, 3..... in $Y_1=0$.

Let c and $c+1$ be the consecutive numbers which, having produced a change of sign, comprehend the value of y'; then make in the equation $Y_1=0$, $y'=c+\frac{1}{y''}$; this will give an equation $Y_2=0$ having *a single real positive root greater than unity.*

Let d and $d+1$ be the two numbers which comprehend it, and make $y''=d+\frac{1}{y'''}$, and continue this series of transformation as far as is desired.

Combining the relations

$$x=a+\frac{1}{y},\quad y=b+\frac{1}{y'},\quad y'=c+\frac{1}{y''},\quad y''=d+\frac{1}{y'''}\quad ------;$$

There will result

$$x=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\dots\dots}}}$$

343. Application of the above theory to the equation

$$x^3-5x-3=0\dots\dots(1).$$

The superior limits of the positive and negative roots being, $+3$ *and* -2, we make

$$x=-2,\ -1,\ 0,\ 1,\ 2,\ 3;$$

when $x=-2$ the result is -1,
$x=-1$ - - - - - - $+1$,
$x=\ 0$ - - - - - - -3,
$x=\ 1$ - - - - - - -7,
$x=\ 2$ - - - - - - -5.
$x=\ 3$ - - - - - - $+9$.

As there are three changes of sign, it follows that the three roots of the equation are real; viz. *one positive* contained between 2 and 3, *two negative*, one of which is contained between 0 and -1, the other between -1 and -2.

We will first consider the positive value between 2 and 3.

In the equation (1) make $x=2+\frac{1}{y}$, there will result an equation of the form

$$X'y^3+Y'y^2+\frac{Z'}{2}y+1=0,$$

in which we have, (278)

$$X'=(2)^3-5(2)^1-3=-5,$$
$$Y'=3(2)^2-5=7,$$
$$\frac{Z'}{2}=3(2)=6,$$
$$\frac{V'}{2.3}=1;$$

which by substituting, gives

$$5y^3-7y^2-6y-1 \text{ - - - - } (2).$$

In this equation, making $y=1,\ 2,\ 3$ - - - -

For $y=1$ there results -9,
$y=2$ - - - - - - -1,
$y=3$ - - - - - - $+53$,

hence the required value of y is between 2 and 3.

In the equation (2), make $y=2+\frac{1}{y}$; there will result the equation

$$X''y'^3+Y''y'^2+\frac{Z''}{2}y+\frac{V''}{2.3}=0,$$

in which $X''= 5(2)^3-7(2)^2-6(2)-1=-1,$
$Y''=15(2)^2-14(2)-6=26$
$\frac{Z''}{2}=15(2)-7=23,$
$\frac{V''}{2.3}=5;$

which, by substituting, gives

$$y'^3-26y'^2-23y'-5=0 \text{ - - - - } (3).$$

As this equation can be put under the form

$$y'^2(y'-26)-23y'-5=0,$$

it is visible that any value less than 26, substituted for y, will give a negative result.

But, making $y'=26$, there results -603,
$y'=27$ - - - - - - - - $+103$;

hence y' is between 26 and 27.

Making $y'=26+\frac{1}{y''}$, in equation 3, we obtain

$$X'''y''^3+Y'''y''^2+\frac{Z'''}{2}y''+\frac{V'''}{2.3}=0,$$

in which $X''' = (26)^3-26(26)^2-23(26)-5=-603,$
$Y''' =3(26)^2-52(26)-23=653,$
$\frac{Z'''}{2}=3(26)-26=52,$
$\frac{V'''}{2.3}=1;$

which gives

$$603y''^3-653y''^2-52y''-1=0 \text{ - - - - } (4),$$

or $$y''^2(603y''-653)-52y''-1=0.$$

Since $y''=1$, gives for a result -103,

and $y''=2$, - - - - - - - - - - $+2107$,

it follows that y'' is comprised between 1 and 2.

Again, making in the equation (4), $y''=1+\frac{1}{y'''}$, we obtain

$$X^{iv}y'''^3+Y^{iv}y'''^2+\frac{Z^{iv}}{2}y'''+\frac{V^{iv}}{2.3}=0,$$

in which $\quad X^{\text{iv}}=603(1)^3-653(1)^2-52(1)-1=-103,$

$Y^{\text{iv}}=1809(1)^2-1306(1)-52=451,$

$\frac{Z^{\text{iv}}}{2}=1809(1)-653=1156,$

$\frac{V^{\text{iv}}}{2.3}=603;$

which gives

$$103y'''^3-451y'''^2-1156y'''-613=0,$$

or $\quad y'''^2(103y'''-451)-1156y'''-603=0.$

It is easily perceived that the value of y''' is between 5 and 6, so that if we wished to continue the operation, we would suppose $y'''=5+\frac{1}{y^{\text{iv}}}$. But stopping at the results already obtained, the equations

$$x=2+\frac{1}{y},\quad y=2+\frac{1}{y'},\quad y'=26+\frac{1}{y''},\quad y''=1+\frac{1}{y'''},\quad y'''=5+\frac{1}{y^{\text{iv}}},$$

give for the value of x,

$$x=2+\cfrac{1}{2+\cfrac{1}{26+\cfrac{1}{1+\cfrac{1}{5}}}}.$$

The consecutive approximations of x being

$$\frac{2}{1},\ \frac{5}{2},\ \frac{132}{53},\ \frac{137}{55},\ \frac{817}{328},$$

in the last of which the error is less than $\frac{1}{(328)^2}$, or $\frac{1}{107584}$. (Arith. 174).

344. We will now proceed with the determination of the negative values.

For more simplicity, we will change x into $-x$, in the proposed equation, which then becomes

$$-x^3+5x-3=0,\ \text{or}\ x^3-5x+3=0,$$

in which the positive roots taken with the sign $-$, are the negative roots of the proposed equation. Thus the question is reduced to determining the positive roots of this new equation.

One of these roots is comprised between 1 and 2, the other between 0 and 1. See the following table for the determination of them.

1st. The root comprised between 1 and 2.

$$x^3-5x+3=0.$$

$$x=1+\frac{1}{y}.$$

1st Transformation, $y^3+2y^2-3y-1=0;$
$y=1$ gives - - - - - -1,
$y=2$ - - - - - - - - $+9$,

$$y=1+\frac{1}{y'}.$$

2d Transformation, $y'^3-4y'^2-5y'-1=0;$
$y'=4$, gives - - - -21,
$y'=5$, - - - - - - - - -1,
$y'=6$, - - - - - - - $+41$.

$$y'=5+\frac{1}{y''}.$$

3d Transformation, $y''^3-30y''^2-7y''-1=0;$
$y''=30$ gives - - - -211,
$y''=31$ - - - - - - - $+753$,

$$y''=30+\frac{1}{y'''}.$$

Hence $$x=1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{30}}}.$$

Consecutive approximations, $\frac{1}{1}, \frac{2}{1}, \frac{11}{6}, \frac{332}{181};$

$\frac{332}{181}=1{,}8342$ to within 0,0001.

Therefore that value of x which is contained between 1 and 2, is $x=1{,}8342$ to within 0,0001.

This value can easily be verified by substituting the two numbers 1,8342 and 1,8343 in the equation.

2d. The value contained between 0 and 1.

$$x^3-5x+3=0.$$

$$x=\frac{1}{y}.$$

1st Transformation, $3y^3-5y^2+1=0$;
$y=1$ gives - - - - - -1,
$y=2$ - - - - - - - $+5$,

$$y=1+\frac{1}{y'}.$$

2d Transformation, $y'^3+y'^2-4y'-3=0$;
$y'=1$ gives - - - - -5,
$y'=2$ - - - - - - - - $+1$,

$$y'=1+\frac{1}{y''}.$$

3d Transformation. $5y''^3-y''^2-4y''-1=0$;
$y''=1$ gives - - - - - -1,
$y''=2$ - - - - - - - $+27$,

$$y''=1+\frac{1}{y'''}.$$

4th Transformation, $y'''^3-9y'''^2-14y'''-5=0$;
$y'''=9$ gives - - - - - -131,
$y'''=10$ - - - - - - - - -45,
$y'''=11$ - - - - - - - $+83$,

$$y'''=10+\frac{1}{y^{\text{iv}}}.$$

5th Transformation, $45y^{\text{iv}3}-106y^{\text{iv}2}-21y^{\text{iv}}-1=0$,
or $y^{\text{iv}2}(45y^{\text{iv}}-106)-21y^{\text{iv}}-1=0$;
$y^{\text{iv}}=2$ gives - - - - -107,
$y^{\text{iv}}=3$ - - - - - - - $+197$,

$$y^{\text{iv}}=2+\frac{1}{y^{\text{v}}}.$$

Hence $x=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{10+\cfrac{1}{2}}}}}$

Consecutive approximations, $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{21}{32}, \frac{44}{67}$;

$\frac{44}{67}=0,6566$ to within 0,0001.

Therefore that value of x which is comprised between 0 and 1 is 0,6566 to within 0,0001.

Hence the two negative roots of the proposed equation - - - - $x^3-5x-3=0$, are

$$x=-1,8342 \text{ and } x=-0,6566.$$

Newton's Method of Approximation.

345. In order that this method may be more easily comprehended, we will resume the equation $x^3-5x-3=0$, and determine the root contained between 2 and 3.

The required root being between 2 and 3, we will try to contract these limits, by taking the mean $2\frac{1}{2}$, or 2, 5, and substituting it in the equation $x^3-5x-3=0$; the result of which is $+0.125$. Now 2 has already given -5 for a result, therefore the root is between 2 and 2, 5.

We will now consider another number, between 2 and 2, 5; but as, from the results given from 2 and 2, 5, it is to be presumed that the root is nearer 2, 5 than 2, suppose $x=2,4$; we will obtain $-1,176$; whereas 2, 5 has given $+0,125$. Therefore the root is between 2, 4 and 2, 5.

By continuing to take the means, we would be able to contract the two limits of the roots more and more. But when we have once obtained, as in the above case, the value of x to at least 0, 1, we may approximate nearer in another way, and it is in this that Newton's method principally consists.

In the equation $x^3-5x-3=0$, make $x=2,4+u$.

There will result (No. 278) the transformation

$$X'+Y'u+\frac{Z'}{2}u^2+u^3=0;$$

in which
$$X'=(2,4)^3-5(2,4)-3=-1,176,$$
$$Y'=3(2,4)^2-5=12,28,$$
$$\frac{Z'}{2}=3(2,4)=7,2.$$

The equation involving u, being of the third degree, cannot be resolved directly, but by transposing all the terms except $Y'u$, and dividing both members by Y', it can be put under the form

$$u=-\frac{X'}{Y'}-\frac{Z'}{2.Y'}u^2-\frac{1}{Y'}u^3.$$

This being the case, since one of the three roots of this equation must be less than $\frac{1}{10}$, from the relation $x=2,4+u$, the corresponding values of u^2 and u^3 are less than $\frac{1}{100}$ and $\frac{1}{1000}$. Moreover, the inspection of the numerical values of Y' and Z', proves that $\frac{Z'}{2.Y'}$ is <1; therefore the value of u only differing numerically from $-\frac{X'}{Y'}$ by the quantity $\frac{Z'}{2.Y'}u^2+\frac{1}{Y'}u^3$, (which most frequently is less than $\frac{1}{100}$), is expressed by $-\frac{X'}{Y'}$ to within 1,01.

As, in this example,

$$-\frac{X'}{Y'}=\frac{+1,176}{12,28}=\frac{1176}{12280}=0,09\ \ldots,$$

there will result $u=0,09$, to within $\frac{1}{100}$, and consequently

$$x=2.4+0,09=2,49, \text{ to within } \frac{1}{100}.$$

In fact, 2,49 substituted in the first member of the proposed equation, gives $-0,011751$;
whilst 2,50 has given $+0,125$.

To obtain a new approximation, make $x=2,49+u'$ in the proposed equation, and we have

$$X''+Y''u'+\frac{Z''}{2}u'^2+u'^3=0;$$

in which $X''=(2,49)^3-5(2,49)-3=-0,011751,$

$Y''=3(2,49)^2-5=13,6003,$

$\frac{Z''}{2}=3(2,49)=7,47.$

But the equation involving u' may be written thus:

$$u'=-\frac{X''}{Y''}-\frac{Z''}{2.Y''}u'^2-\frac{1}{Y''}u'^3.$$

And since one of the values of u' must be less than $\frac{1}{100}$, the corresponding values of u'^2, u'^3, are less than $\frac{1}{10000}$, $\frac{1}{1000000}$; hence $-\frac{X''}{Y''}$ will represent the value of u' to within $\frac{1}{10000}$.

Since we have

$$-\frac{X''}{Y''}=\frac{0{,}011751}{13{,}6003}=\frac{11751}{13600300}=0{,}0008\ldots,$$

it follows that $u'=0{,}0008$, to within $\frac{1}{10000}$, and consequently

$$x=2{,}49+0{,}0008=2{,}4908, \text{ to within } \frac{1}{10000}.$$

Again, by supposing $x=2{,}4908+u''$, we would obtain a value of x to within $\frac{1}{100000000}$.

Each operation commonly gives the root to twice as many places of decimals as the previous operation.

346. Generally, let p and $p+1$ be two numbers between which one of the roots of the equation $X=0$ is comprised.

First determine the value of this root to within $\frac{1}{10}$, by substituting a series of numbers comprised between p and $p+1$, until two numbers are obtained which do not differ from each other by more than $\frac{1}{10}$.

Then, calling x' the value of x *obtained to within* $\frac{1}{10}$, *suppose* $x=x'+u$ *in the equation* $X=0$;
which gives

$$X'+Y'u+\frac{Z'}{2}u^2 \cdots\cdots +u^m=0;$$

which can be put under the form

$$u = -\frac{X'}{Y'} - \frac{Z'}{2.Y'}u^2 - - - - \frac{1}{Y'}u^m,$$

X', Y', Z' - - - - being easily calculated. (No. 278).

Since the sum of the terms, which follow $-\frac{X'}{Y'}$ in the second member of this equation is, commonly, less than $\frac{1}{100}$, they can be neglected, and calculating $-\frac{X'}{Y'}$ to within $\frac{1}{100}$, we add the result to x', which gives a new value x'' approximating *to within* $\frac{1}{100}$ of the exact value.

To obtain a 3d approximation, *we suppose* $x = x'' + u'$ *in the proposed equation, which gives*

$$X'' + Y''u' + \frac{Z''}{2}u'^2 + - - - - + u'^m = 0;$$

whence $$u' = -\frac{X''}{Y''} - \frac{Z''}{2.Y''}u'^2 - - - - - - \frac{1}{Y''}u'^m.$$

Neglecting the terms $-\frac{Z''}{2.Y''} - - - - - - \frac{1}{Y''}$ which are supposed to be less than 0.0001, we calculate the value of $-\frac{X''}{Y''}$, continuing the operation to the $\frac{1}{10000}$ place of decimals, and add the result to x''; this gives a third approximation x''', exact to within $\frac{1}{10000}$.

Repeat this series of operations for each of the positive roots. As for the negative roots, they are found in the same way as the positive roots, by changing x into $-x$ in the proposed equation.

Second Part. *Case in which the difference between two roots of an equation may be less than unity.* (No. 340).

650. When, by substituting consecutive entire numbers, comprised between the limits $+L$ and $-L'$, we obtain as many changes of sign as there are units in the degree of the equation, it is evident that *all the roots of it are real*, and the entire part is different for each of them.

But if the number of changes of sign is less than the degree of the proposed equation, it will show that some of the *roots are real*, and some *imaginary*, or that there are *incommensurable roots* comprised between consecutive entire numbers. In fact, we have seen (No. 319) that two numbers which, substituted in the first member of an equation, give results with contrary signs, may comprehend any uneven number of real roots, and that two numbers which give results with the same sign, may comprehend any even number of real roots.

For example, if an equation contains the roots $\sqrt{2}$ and $\sqrt{3}$, comprised between 1 and 2, these numbers, when substituted in the proposed equation, might give results affected with the same sign.

In like manner, an equation containing the roots $\sqrt{11}$, $\sqrt{13}$, $\sqrt{15}$, comprised between 3 and 4, might, when these last numbers were substituted, give results affected with different signs.

Hence we see that the substitution of consecutive entire numbers would not be sufficient, in this case, to detect all the real roots of the proposed equation.

We would avoid this difficulty, if we could determine, *à priori*, a quantity δ, numerically *less than the least difference* between any two of the real roots of a given equation. For by making δ the common difference between the numbers substituted, it is evident that any consecutive two of these numbers, which, substituted in the proposed equation, give results with contrary signs, will comprehend a root, and but one; if they give results affected with the same sign, they will not comprehend a root; that is, *the number of the real roots of the equation will be equal to the number of changes of sign.*

351. The quantity δ can easily be determined from the equation, of which the roots are the squares of the differences of the roots of the proposed equation.

For, denote the proposed equation by $X=0$, and the equation involving the squares of the differences (Nos. 290 and 312) by $Z=0$.

We will remark, in the first place, that the square of the difference between any two real roots of the proposed equation being positive, we must seek for these squares only amongst the positive roots of the equation $Z=0$. (Its negative roots correspond to differences between imaginary roots, or between real and imaginary roots.) Hence, finding the inferior limit of the positive roots of the equation $Z=0$, and extracting the square root of it, we will have a quantity less than the least difference between the real roots of the proposed equation; that is to say, the required quantity δ.

To obtain this limit, we must (No. 325) suppose $z=\frac{1}{v}$ in the equation $Z=0$, which will give the equation $V=0$. Let l be the superior limit of the positive roots of the equation $V=0$, $\frac{1}{l}$ will be the inferior limit of $Z=0$; therefore $\frac{1}{\sqrt{l}}$ is the quantity δ, which it was required to determine.

When l is found <1, there will result $\frac{1}{\sqrt{l}}$, or $\delta>1$; and this will indicate that *the difference between any two real roots is greater than unity*. We will then be certain that the number of real roots is equal to the number of changes of sign obtained by substituting consecutive entire numbers.

But we commonly find $l>1$, whence $\frac{1}{\sqrt{l}}$, or $\delta<1$. In this case, it will be more convenient to replace $\sqrt{l}$ by the next greater entire number, k, which will give $\frac{1}{k}<\frac{1}{\sqrt{l}}$, and for a still stronger reason, $\frac{1}{k}$ less than the least of the differences between the real roots. We may then make $\frac{1}{k}$ the common difference between the numbers substituted; that is, by substituting in $X=0$ the series of numbers

$$0, \frac{1}{k}, \frac{2}{k} - - - - 1,\ 1+\frac{1}{k},\ 1+\frac{2}{k} - - - - 2,\ 2+\frac{1}{k} - - - - \text{to } L,$$

$$\text{and } 0, -\frac{1}{k}, -\frac{2}{k} - - - - -1 - - - - - - - - - - - - - \text{to } L',$$

we will obtain as many changes of sign as there are real roots in the equation.

We may avoid the substitution of fractions by the following transformation: In the given equation, make $x=\frac{y}{k}$; there will result an equation, the roots of which will be k times greater than those of the given equation. Consequently, the differences between these new roots will be k times greater than the differences between the roots of the proposed equation; so that if $a-b$ denotes the smallest of the differences in $X=0$, since we have $a-b>\frac{1}{k}$, there will result $ka-kb>1$. Hence the new equation $Y=0$ is such, that the differences between all its real roots, considered two and two, are greater than unity.

Consequently, by substituting in this equation the series of natural numbers 0, 1, 2, 3 - - - -, and -1, -2, -3 - - - -, comprised between the two limits, we obtain as many changes of sign as there are real roots in the equation $Y=0$. (The two limits of $Y=0$ are $+kL$ and $-kL'$, when $+L$ and $-L'$ denote the limits of $X=0$).

352. From what has been said, it will be seen, that in order to detect all of the real incommensurable roots of an equation $X=0$, it is necessary

1st. *To form the equation involving the squares of the differences* $Z=0$. 2d. *To determine the inferior limit* $\frac{1}{l}$ *of the positive roots of this last equation.* If the quantity $\frac{1}{l}$ is greater than unity, it is a proof that the difference between any two roots of the proposed equation is also greater than unity; consequently, the substitution of consecutive entire numbers in the proposed equation will be sufficient to detect all of the real roots. 3d. But if $\frac{1}{l}$, or $\frac{1}{\sqrt{l}}<1$, *we replace* $\frac{1}{\sqrt{l}}$ *by* $\frac{1}{k}$, k being the entire number immediately superior to $\sqrt{l}$, *and we make* $x=\frac{y}{k}$ *in the proposed equation,* which gives an equation $Y=0$, *of which we find all the real roots* by the method exposed in the *first part* of this paragraph.

4th. Finally. *In the relation* $x=\frac{y}{k}$ *we substitute for* y *all its values, and we thus obtain all the real roots of the proposed equation.*

353. *Examples.* Take the equation

$$x^3-6x-7=0.$$

It is easily ascertained that the superior limits of the positive and negative roots are, $+3$, -2.

Now, by substituting the numbers $+3, +2, +1, 0, -1, -2$, we obtain for results, $+2, -11, -12, -7, -2, -3$; and we see that $+3$ and $+2$ are the only numbers which give results affected with contrary signs. Whence we may conclude *that one of the roots of the equation is real and the other two imaginary*, or, that *its three roots are real, but such that their differences taken two and two are less than unity.*

To avoid any uncertainty, form the equation *involving the squares of the differences*. It has been found (No. 289) to be

$$z^3-36z^2+324z+459=0.$$

In this equation make $z=\frac{1}{v}$; there will result

$$v^3+\frac{324}{459}v^2-\frac{36}{459}v+\frac{1}{459}=0,$$

which can be written thus:

$$v_3+\frac{1}{459}+\frac{36}{459}v(9v-1)=0.$$

Now, it is visible that $v=$ or $>\frac{1}{9}$ gives a positive result; therefore, the superior limit of the positive roots of this equation being less than unity, the corresponding quantity $\frac{1}{l}$ is >1; and consequently, *the differences between the real roots of the proposed equation are greater than unity.* Moreover, the substitution of the natural numbers in the proposed equation, produce *but one* change of sign; hence the proposed equation has *but one real root*, which is comprised between 2 and 3.

We will calculate the part of this root which is less than unity, by Lagrange's method.

$$x^3-6x-7=0,$$

$$x=2+\frac{1}{y}.$$

1st Transformation $11y^3-6y^2-6y-1=0$;

$y=1$ gives -2,

$y=2$ - - - $+51$,

$y=1+\frac{1}{y'}$.

2d Transformation $2y'^3-15y'^2-27y'-11=0$,

or $y'^2(2y'-15)-27y'-11=0$;

$y'=8$ gives -163,

$y'=9$.........-11,

$y'=10$.........$+219$.

$y'=9+\frac{1}{y''}$.

3d Transformation $11y''^3-189y''^2-39y''-2=0$,

or $y''^2(11y''-189)-39y''-2=0$;

$y''=17$ gives -1243,

$y''=18$.........$+2212$,

$y''=17+\frac{1}{y'''}$.

Hence $x=2+\frac{1}{y}$, $y=1+\frac{1}{y'}$, $y'=9+\frac{1}{y''}$, $y''=17+\frac{1}{y'''}$;
from which we obtain the continued fraction

$$x=2+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{17}}},$$

of which the reductions are $\frac{2}{1}$, $\frac{3}{1}$, $\frac{29}{10}$, $\frac{496}{171}$,

and the last $\frac{496}{171}$, converted into a decimal fraction, gives 2.9005 for the value of x to within $\frac{1}{10000}$.

(See this equation No. 348.)

354. Again, take the equation

$$8x^3-6x-1=0 \text{ - - - - } (1).$$

The superior limits of the positive and negative roots are *+1 and —1.*

Making $x=+1,\ 0,\ -1$, we obtain the results . . . . . . $+1,\ -1,\ -3$; their substitution produces but *one* change of sign; we will, therefore, have recourse to the *equation involving the squares of the differences.*

This equation, found by the method of elimination, (No. 290), is

$$64z^3-288z^2+324z-81=0.$$

Making $z=\frac{1}{v}$, we will have

$$81v^3-324v^2+288v-64=0,$$

which can be put under the form

$$81v^2(v-4)+32(9v-2)=0;$$

and it is easily ascertained that 3 is, in whole numbers, the narrowest superior limit of the positive roots; therefore, $l=3$; whence $\frac{1}{\sqrt{l}}=\frac{1}{\sqrt{3}}$.

Replacing $\sqrt{3}$ by 2, we find that $\frac{1}{2}$ is less than the least difference between the real roots of the proposed equation.

Making (No. 352) $x=\frac{y}{2}$ in the proposed equation, we obtain

$$y^3-3y-1=0 \text{ - - - - } (3),$$

in which the difference between any two roots is greater than unity.

The superior limits of the positive and negative roots being $+2$ and -2, it will be sufficient to make, in the equation (3),

$$y=+2,\ +1,\ 0,\ -1,\ -2,$$

which give the results $\qquad +1,\ -3,\ -1,\ +1,\ -3.$

By these substitutions we evidently obtain *three changes* of sign.

Therefore, the equation (3) has *three* real roots, one between 1 and 2, another between 0 and -1, and the third between -1 and -2.

Consequently, the equation (1) has *three* real roots, one between $\frac{1}{2}$ and 1, the second between 0 and $-\frac{1}{2}$, the third between $-\frac{1}{2}$ and -1.

To approximate nearer to the value of these roots, first apply one of the methods of approximation to the equation (3); after which, substitute the values of y obtained, in the relation $x=\frac{y}{2}$, and we will obtain the corresponding value of x.

In this way we will find

for the equation $y^3-3y-1=0 \begin{cases} y= & 1.8794, \\ y= & -0.3474, \\ y= & -1.5320, \end{cases}$

and for the equation $8x^3-6x-1=0 \begin{cases} x= & 0.9397, \\ x= & -1.1737, \\ x= & -0.7660. \end{cases}$

These values are exact to within 0.0001.

§ IV. *Of Elimination.*

361. Having explained the different methods for resolving equations of any degree, involving but one unknown quantity, (or, rather, for determining their *real roots*,) we will now proceed with the resolution of equations involving two or more unknown quantities.

When the number of equations given is equal to the number of unknown quantities, they admit, in general, of only *a limited number of systems of values* for these unknown quantities. Now the complete determination of all these systems constitutes the general problem of *elimination*.

A part of this problem has already been explained in No. 282, ... , viz. that of which the object is *to form the final equation*, that is, an equation which, being a function of but one of the unknown quantities, will give all the values of this unknown quantity that are required for the verification of the proposed equations, at the same time that certain values of the other unknown quantities verify them.

The second part consists *in resolving the final equation, and determining the values of the other unknown quantities corresponding to the values of the unknown quantity involved in this equation.*

We will here consider more particularly *the case in which the two equations involve but two unknown quantities.*

362. Before entering into all the details relative to the second part of elimination, we will reconsider the manner of forming the final equation.

When the method for finding the common divisor is used to eliminate one of the unknown quantities, an equation is obtained containing all the *compatible* values of the other unknown quantity; but this equation generally contains, also, values foreign to the question. It is proposed to show that this is the case.

(Every system of values of x and y, which, substituted in the two equations, will satisfy them, is, for the sake of brevity, called *a solution.*)

363. Let $A=0$ and $B=0$ be two equations of any degree whatever; and after having arranged them with reference to one of the letters, x, for example, apply to them the process for finding the greatest common divisor, with the modifications for rendering the quotients and remainders entire. (We will, moreover, suppose, for the present, that no factor which is a function of y is suppressed in the course of the operation; because we will see that these factors, placed equal to 0, may give compatible values.)

Let $a, a', a'' \ldots.$ denote the different factors introduced, these factors being, in general, functions of y; and let $Q, Q', Q'', \ldots., R, R', R'', \ldots..$ denote the successive quotients and remainders.

The series of operations will give rise to the following identities:

$$a\,A=B\,Q\ +R \quad - - - - - (1),$$
$$a'B=R\,Q'+R' \quad - - - - - (2),$$
$$a''R=R'Q''+R'' \quad - - - - - (3),$$
$$- - - - - - - - - - -$$
$$- - - - - - - - - - -$$
$$a^{(n)}R^{(n-2)}=R^{(n-1)}Q^{(n)}+R^{(n)} \quad - - - - (n).$$

($R^{(n)}$ is supposed to be the remainder independent of x, and a function of y only).

This being the case, the identity (1) shows that every solution

(362) of the system [$A=0$, $B=0$] must satisfy $R=0$; for a and Q being entire polynomials, they cannot become infinite in consequence of particular values attributed to x and y. Therefore, the solutions of the system [$A=0$, $B=0$] answer to the system [$B=0$, $R=0$].

The identity (2) also proves that *every solution* of the system [$B=0$, $R=0$] will satisfy $R'=0$; whence we may conclude that the solutions of the system [$A=0$, $B=0$] also answer for the system [$R=0$, $R'=0$].

The identity (3) proves that *every solution* of the system [$R=0$, $R'=0$] satisfies $R''=0$. Hence the solutions of [$A=0$, $B=0$] answer to the new system [$R'=0$, $R''=0$]. And so on.

Whence it follows, that *all the solutions of the proposed system* [$A=0$, $B=0$] *are necessarily comprehended in the last system* [$R^{(n-1)}=0$, $R^{(n)}=0$].

Therefore the equation $R^{(n)}=0$ contains all the compatible values of y.

But it may contain values which are foreign to the question.

To show this, take the identities (1), (2), (3), - - - - (n).

From the first, we see that every solution of the system [$B=0$, $R=0$] satisfies $aA=0$. Now this last equation can be divided into two others, $a=0$ and $A=0$; from which it follows that the solutions of [$B=0$, $R=0$] comprehend not only those of the proposed system [$A=0$, $B=0$], but also the solutions of the system [$a=0$, $B=0$].

In like manner, in consequence of the identity (2), every solution of [$R=0$, $R'=0$] satisfies $a'B=0$, which can be decomposed into the two equations $a'=0$ and $B=0$; hence the solutions of [$R=0$, $R'=0$] comprehend those of the system [$B=0$, $R=0$], and those of the system [$a'=0$, $R=0$]. But we have just seen that the system [$B=0$, $R=0$] already contains all the solutions of [$A=0$, $B=0$] and [$a=0$, $B=0$]. Consequently, the system [$R=0$, $R'=0$] contains all the solutions of the three systems

$$[A=0,\ B=0],\ [a=0,\ B=0],\ [a'=0,\ R=0].$$

By continuing this course of reasoning, we will see that the last *system of equations* [$R^{(n-1)}=0$, $R^{(n)}=0$] comprehends not only those of the proposed system [A=0, B=0], but also all the solutions of the systems

$$[a=0, B=0], \; [a'=0, R=0] \; - - - - \; [a^{(n)}=0, R^{(n-1)}=0].$$

364. If we were certain that by substituting in the equation B=0 each of the values of y, which give $a=0$, we could find one or more values of x from the result of this substitution, all the *solutions* thus obtained would be found in .... $[R^{(n-1)}=0, R^{(n)}=0]$, and would be foreign to [A=0, B=0]. Whence, in order to free $R^{(n)}=0$ from the values of y, corresponding to these *foreign solutions*, it will be sufficient to divide $R^{(n)}$ by a, which is a function of y only.

The same reasoning applies to the factors a', a'' - - - - ,

Hence it follows that, if $R^{(n)}$ is divided by the product - - - - $a \times a' \times a'' \times$ - - - - $\times a^{(n)}$, *the resulting quotients will, when placed equal to* 0, *form the true final equation.*

But we are not always sure that the value of y obtained from $a=0$ will, when substituted in B, give a result from which a value of x can be found, for this value of y, which, in all cases, must render the coefficient of the first term of B equal to 0, may also destroy the coefficients of all the inferior powers of x to the first inclusively; and in this case there would not be any corresponding value of x.

365. The following example will illustrate what has been said above.

Take the two equations

$$y^3x^2-3y^3x-y^2+2=0 \; - - - - - - - - - - - \; (1).$$

$$(y^2-3y+2)x^2+(y-1)x-3y+1=0 \; - - - \; (2).$$

To eliminate x between these two equations, if we take the polynomial (1) for a dividend, and (2) for a divisor, we must multiply (1) by the factor y^2-3y+2, or the coefficient of the first term of the divisor.

After this preparation, the division is performed, and we obtain for a remainder,

$$(-3y^5+8y^4-5y^3)x+2y^4+2y^3-6y+4 \; - - - - - \; (3).$$

We now take the polynomial (2) for a dividend, and the remainder (3) for a divisor.

Since the coefficient of the first term of this remainder, viz.

$$-3y^5+8y^4-5y^3,$$

can be reduced to $-\ y^3(3y^2-8y+5)$,

or $-\ y^3(y-1)(3y-5)$,

and as the coefficient of the first term of the polynomial (2), viz.

$$y^2-3y+2,$$

can be decomposed into $(y-1)(y-2)$

it follows that, in order to obtain entire quotients and remainders, it will be sufficient to multiply the polynomial (2) - - - - - by $y^6(y-1)(3y-5)^2$.

Performing this multiplication, then the division of the polynomial (2) thus prepared, by the remainder (3), we obtain a certain quotient which it is not necessary to write here; and the last remainder, that is, the polynomial independent of x, placed equal to 0, will give

$$\left.\begin{array}{l}27y^{10}-136y^9+214y^8-112y^7+65y^6-100y^5\\+30y^4-24y^3+120y^2-112y+32\ \text{- - - - - - -}\end{array}\right\}=0\ \text{- - -}(4).$$

We will now see what foreign factors may be contained in this equation.

1st. Since the first member of the equation (1) has been multiplied by the factor y^2-3y+2, which, being decomposed, becomes $(y-1)(y-2)$, it follows that the equation (4) may contain the two binomials $y-1$ and $y-2$ as foreign factors.

Now, it is easily seen that this equation is not satisfied by $y=1$, and that it is by $y=2$; the division by $y-2$ gives an exact quotient equal to

$$27y^9-82y^8+50y^7-12y^6+41y^5-18y^4-6y^3$$
$$-36y^2+48y-16\ \text{- - - - -}\ (5).$$

To understand the reason why the factor $y-1$ should not form a part of the polynomial (4), it will be sufficient to observe that, if we make $y=1$ in the equation (2), all the powers of x will disappear, and there will not result from it any corresponding value for x.

But, if we make $y=2$ in the equation (2), it will reduce to $x-5=0$, whence $x=5$; that is, the two equations

$$y^2-3y+2=0,$$

and $\qquad (y^2-3y+2)x^2+(y-1)x-3y+1=0,$

admit of a *single solution*, $x=5$ and $y=2$.

Hence the equation (4) must contain the factor $y-2$, which corresponds to this solution, but cannot contain the factor $y-1$, which *does not answer to any solution.*

2d. In the second principal operation, the factor $y^6(y-1)(3y-5)^2$ has been introduced, in order to render the division possible.

But it is easily ascertained that none of the simple factors which compose this product, can enter into the final equation; for by making $y=0$, or $y=1$, or $y=\frac{5}{3}$, in the remainder (3) placed equal to 0, the part involving x would vanish, and there would not result from it any value for x.

In general, the introduction of the factor $a^{(n)}$ (No. 363) into the remainder which precedes that of the 1st degree, *never has any influence upon the final equation*, because the values of y obtained from this factor when it is placed equal to zero, would reduce the remainder of the first degree to a *numerical quantity*; hence it cannot have any corresponding value of x.

We will now, in this same example, take the polynomial (2) for a dividend and the polynomial, (1) for a divisor.

Multiplying (2) by y^3, the coefficient of the first term of (1), and performing the division, which gives a certain quotient and the remainder of the first degree,

$$(3y^5-8y^4+5y^3)x-2y^4-2y^3+6y-4 \text{ - - - - } (6).$$

We then take the polynomial (1) for a dividend and the remainder (6) for a divisor.

Since the coefficient of the first term of this remainder can be reduced to

$$y^3(3y^2-8y+5)$$

or

$$y^3(y-1)(3y-5),$$

and as the coefficient of the first term of (1) is y^3, it will be sufficient to multiply (1) by $y^2(y-1)^2(3y-5)^2$.

After this preparation, the division is performed, and we obtain for a remainder independent of x,

$$27y^9-82y^8+50y^7-12y^6+41y^5-18y^4$$
$$-6y^3-36y^2+48y-16,$$

which is identical with the quotient (5) obtained by dividing the remainder (4) by the factor $y-2$.

Here, none of the factors introduced in the course of the operation are found in the final equation, and the reason of it is that the factor y^3, by which (2) has been multiplied, being placed equal to 0, the divisor (1) is reduced to a quantity *independent of* x; consequently it has no value of x corresponding to $y=0$.

366. If the last example has been well understood, it will be easy to understand the following rule, *for freeing the final equation from any foreign factors which it may contain.*

Let $F(y)$ denote any one of the factors introduced in the course of the operation, and

$$bx^n+cx^{n-1}+dx^{n-2}+ \text{- - - -} +tx+u,$$

the remainder corresponding to it, so that the simple factors of $F(y)$ may be factors of the coefficient b.

First. When, as most frequently happens, $F(y)$ is immediately decomposable into simple factors, *place each of them equal to* 0, *and deduce the values of* y.

When none of these values of y destroy the coefficients - - - - b, c, d - - - - to t inclusively, we are certain that $F(y)$ is a factor of the final equation, and *we then divide the first member of this equation by this factor*, as being foreign to it.

When one or more values destroy at the same time $b, c, d...t$, the corresponding simple factors are not found in the final equation, which does not contain any foreign factors arising from $F(y)$, except the simple factors of which the values do not render b, c, d - - - t nothing at the same time. *We then divide the first member of the final equation by the product of these last factors.*

Second. When the equation $F(y)=0$ cannot be immediately resolved, which is the case when the coefficient b is of the 3d or of a higher degree, *we seek for the greatest common divisor between* F(y) *and all the coefficients* b, c, d - - - t. (*See* No. 262). If we do not find one we may be certain that $F(y)$ is a factor of the final equation, *and we free it from this factor.*

But if there is a common divisor $F'(y)$, divide $F(y)$ by $F'(y)$, which give a quotient $F''(y)$, which is then the only foreign factor [arising from the introduction of the factor $F(y)$] contained in the final equation. The first member of this equation must therefore be divided by $F''(y)$.

This reasoning applies to every other factor which may be introduced. Hence we see that it is always possible to free the final equation from any foreign factors that it may contain, and this, without supposing that any of the methods for resolving an equation involving a single unknown quantity are known.

367. *First Remark.* The method of elimination by the common divisor, applied to two equations of the second degree involving x and y, always gives the true final equation.

For we have seen (No. 116) that two equations of this kind can be reduced to the form

$$ax^2+bx+c, \text{ and } a'x^2+b'x+'e=0;$$

a and a' being numerical quantities.

This being the case, the first preparation to which one of the polynomials is subjected, only requiring the introduction of a numerical factor, it can have no influence upon the final equation. As to the second, which consists in multiplying one of the proposed polynomials by the coefficient of x in the remainder of the first degree $a''x+b''$, we have already proved (No. 365) that this factor cannot enter the final equation.

This is always the case when one of the equations is of the second degree with respect to x, (whatever may be the degree with respect to y), provided the coefficient of x^2 is a numerical quantity.

368. *Second Remark.* The degree of the final equation often indicates the presence of foreign factors. For it has been shown that *the degree of the final equation cannot exceed the product of the degrees of the two proposed equations.* Therefore, when an equation of a higher degree is obtained, it must contain some foreign values. But when the degree is equal to the product of the degrees of the two equations, we cannot conclude that it does not contain foreign values; for the degree of the final equation is sometimes less than the product of the degrees of the two equations. It may even happen that the degree is *nothing*, or, in other words, that we will not have a final equation. (See No. 374).

In general, the degree of the final equation is equal to the number OF SOLUTIONS *of which the equations are susceptible.*

369. We will now proceed with *the determination of all the systems of values which should verify the two given equations.*

Take, for example, the two equations

$$yx^3-3x+1=0 \text{ - - - - - } (1),$$

$$(y-1)x^2+x-2=0 \text{ - - - - - } (2).$$

After having multiplied the polynomial (1) by the factor $(y-1)^2$, we perform the division, and find the first remainder,

$$(y^2-5y+3)x-y^2+4y-1 \text{ - - - - - } (3).$$

Then multiplying the polynomial (2) by $(y^2-5y+3)^2$, and dividing the result by the remainder (3), we obtain a remainder

$$y^5-10y^4+37y^3-64y^2+52y-16 \text{ - - - - } (4),$$

which is a function of y only.

Since, in order to perform the first operation, we have been obliged to introduce the factor $(y-1)^2$, and as $y=1$, substituted in the equation (2), gives $x=2$, it follows that this factor must be involved in the polynomial (4) as a *foreign factor.*

We therefore divide this polynomial by $(y-1)^2$, and the result is

$$y^3-8y^2+20y-16=0 \text{ - - - - - } (5),$$

which is *the true final equation.* Applying the method of incommensurable roots to it, we will find, $y=2$, $y=2$, $y=4$.

In order to obtain the three corresponding values of x, we must (No. 285) substitute each of these values in the remainder of the first degree, and we will obtain $x=1$, $x=1$, $x=-1$; that is, the proposed equations admit of *three* solutions, two of which are $(x=1, y=2)$, and the third is $(x=-1, y=4)$.

In fact, suppose $y=2$ in the equations (1) and (2); they become

$$2x^3-3x+1=0, \text{ and } x^2+x-2=0,$$

which admit of the common divisor $x-1$.

In like manner, $y=4$, substituted in (1) and (2), gives

$$4x^3-3x+1=0, \text{ and } 3x^2+x-2=0,$$

which admit of the common divisor $x+1$.

370. Take the two equations

$$x^3-(3y-3)x^2+(3y^2-6y-1)x-y^3+3y^2+y-3=0 \quad - - - (1),$$

$$x^2+(2y+4)x+y^2+4y+3=0 \quad - - - - - - (2).$$

By applying the ordinary process, we obtain for the remainder of the first degree,

$$(3y^2+3y)x+y^3+6y^2+5y \quad - - - - - (3);$$

and for the remainder independent of x,

$$y^6+3y^5+y^4-3y^3-2y^2=0 \quad - - - - - (4).$$

This is *the true final equation;* for, from what has been said in No. 367, the equation (2) being of the second degree, and the coefficient of x^2 numerical, the introduction of a factor has no effect upon the final equation.

This being the case, by resolving the equation (4), we will see that it can be put under the form

$$y^2(y+1)^2(y-1)(y+2)=0.$$

We will first consider the two values of y which are involved but once in this equation.

By making $y=1$ in the remainder (3), and placing this remainder equal to 0, we find $6x+12=0$, whence $x=-2$; that is, the factor $x+2$ becomes common to the proposed polynomials,

In like manner, substituting $y=-2$ in the remainder (3), we obtain $6x+6=0$, whence $x=-1$; therefore, $x+1$ becomes a common divisor of the two equations.

But when we make $y=0$ in the remainder (3), all the terms of it vanish; that is, we find $0.x=0$, whence $x=\frac{0}{0}$.

To explain this circumstance, it must be observed that the hypothesis $y=0$, making the remainder of the first degree with reference to x equal to *nothing*, the remainder of the second degree must become the *exact divisor* of the two polynomials. In fact, by substituting $y=0$ in the remainder of the second degree, that is, in the equation (2), it becomes

$$x^2+4x+3.$$

In the same way, making $y=0$ in the first member of the equation (1), we find

$$x^3+3x^2-x-3.$$

Now, it is easily seen that this last polynomial is divisible by the preceding, and gives $x-1$ for a quotient.

From the equation $x^2+4x+3=0$, we moreover find

$$x=-1, \quad x=-3;$$

which are the two values of x corresponding to $y=0$.

The value $y=-1$ remains to be considered. The hypothesis $y=-1$, made upon the remainder (3), also destroys all its terms, that is, we find $0.x=0$, whence $x=\frac{0}{0}$.

But by substituting $y=-1$ in the divisor of the second degree, or the equation (2), it becomes

$$x^2+2x=0,$$

whence $\quad x=0, \quad x=-2.$

The same value $y=-1$ substituted in equation (1), reduces it to

$$x^3+6x^2+8x=0,$$

the first member of which is divisible by x^2+2x, and gives $x+4$ for a quotient.

Hence the *solutions* of the proposed equations are

$$\begin{array}{llllll} y= & 1, & 2, & 0, & 0, & -1, -1, \\ x= & -2, & -1, & -1, & -3, & 0, -2. \end{array}$$

371. N. B. It should be remarked here, 1st. *that the number of solutions is equal to the product of the degrees of the equations*; 2d. That each of the values of y, to which there are two corresponding values of x, is involved twice in the final equation.

This result agrees with what has been said in No. 368, viz. that the degree of the final equation is equal to the number of *solutions*.

372. When the substitution of one of the values of the final equation involving y, in the remainder of the first degree with reference to x, destroys all of its terms, it is a proof that there is more than one value of x corresponding to this value of y, or which amounts to the same thing, there is a common divisor of the proposed polynomials, of a higher degree than the first.

To obtain this common divisor, or the values of y, this last *value must be substituted in the remainder of the second degree.*

If all the terms of this remainder vanish, *we substitute this value in the remainder of the 3d degree, and so on, until one is obtained in which all the terms do not disappear*, and this will be the greatest common divisor of the two polynomials. *Placing this remainder equal to* 0, *and resolving the resulting equation*, we obtain all the values of x corresponding to this value of y.

Reciprocally, when there are two, three, - - - - -, n values of x, corresponding to the same value of y, the remainders of the 1st, 2d, - - - - $n-1^{th}$ degree must necessarily vanish; and the remainder of the n^{th} degree, placed equal to 0, will give the n values of x.

The factor of the first degree corresponding to this value of y, is involved to the n^{th} power in the final equation, which agrees with the remark of No. 371.

373. The remainder of the first degree with reference to x, being supposed of the form $Mx+N$, in which M and N are generally functions of y, if we substitute one of the values of y found from the final equation, two peculiar cases may occur which should be noticed.

1st. *In consequence of this substitution* N *may be reduced to* 0, *and* M *to any numerical quantity.*

In this case it is evident that the corresponding value of x is reduced to zero, since we would have

$$x=-\frac{N}{M}=0.$$

2d. N may be reduced to a numerical quantity, and M to 0. In this case, the equation $x=-\frac{N}{M}$ gives $x=-\frac{N}{0}$, that is, the corresponding value of x becomes *infinite*.

This result, which is sometimes obtained from formulas of the first and second degree, may agree with the question, when it is of a nature to admit of *infinite solutions* for x and *finite solutions* for y.

374. Hitherto we have supposed that the application of this method leads to a final equation involving y; but this is not always the case.

The remainder independent of x, which is necessarily obtained, may be a numerical quantity, or 0.

We will first examine the case in which *the remainder is numerical,* that is, independent of x and y.

This result evidently proves that the two equations are *incompatible,* or do not admit of *any solution*; since (No. 284) in order to form the final equation, the remainder must be placed equal to 0, which would lead to an absurdity.

In fact, since by applying the method of the common divisor to the two proposed polynomials, we find a *numerical remainder,* before any particular substitution is made for y, if we give any value whatever to this unknown quantity, and then apply the same process to the polynomials resulting from this substitution, we must find the same remainder. From which we see, that no value given to y will produce a common divisor involving x, a condition which it is absolutely necessary for the compatible values of y to fulfil.

Take, for example, the two equations

$$yx^3-(y^3-3y-1)x+y=0, \quad x^2-y^2+3=0,$$

$$\begin{array}{l|l|l}
yx^3-y^3 & x+y & x^2-y^2 \\
\quad +3y & & \quad +3 \\
\quad +1 & & \hline \quad yx \\
\hline
\end{array}$$

1st Rem. - - - $x+y$

$$\begin{array}{l|l}
x^2-y^2 & x+y \\
\quad +3 & \hline x-y \\
\hline
\end{array}$$

2d Rem. - - - - - - - $+3.$

Hence the equations are *incompatible.*

375. We will now consider the case in which *the remainder is nothing.*

This result proves that the two equations have a common divisor involving x, before the substitution of any particular value for y, (See No. 283), and *that these equations are indeterminate.*

Let D be the factor common to the two first members; the proposed equations can be put under the form

$$A'\times D=0, \quad B'\times D=0.$$

This being the case, they may be satisfied, either by supposing $D=0$, or by making $A'=0$, $B'=0$.

If now we suppress the factor D in the first members of the two equations, the resulting equations will be $A'=0$, $B'=0$, and they will admit of only a limited number of *solutions.*

Now, since it is possible that the common factor which exists between the proposed equations, and which renders them *indeterminate*, may be altogether foreign to the question which gives rise to the equations; moreover, as this factor, placed equal to 0, may only admit of *imaginary solutions*, we may conceive that it will be useful to know the solutions of the equations $A'=0$ and $B'=0$.

Take, for an example, the two equations

$$x^3-3yx^2+3y^2x-5x^2+10yx+6x-y^3-5y^2-6y=0 \;---\; (1),$$

$$x^3-5yx^2+8y^2x-x-4y^3+y=0 \;-------------\; (2).$$

Arranging the two polynomials with reference to x, and dividing (1) by (2), we obtain for the first remainder,

$$(2y-5)x^2-(5y^2-10y-7)x+3y^3-5y^2-7y \;----\; (3).$$

Taking this remainder for a divisor, the polynomial (2) for the dividend, and making the usual preparation, we find for the second remainder

$$(y^4-10y^3+35y^2-50y+24)x-y^5+10y^4-35y^3$$
$$+50y^2-24y \;----\; (4).$$

Finally, after having multiplied the polynomial (3) by the square of the coefficient of x in (4), if we divide the polynomial (3) thus prepared by (4), we will obtain *zero* for a remainder. This result would seem to indicate that (4) is a common divisor of the two polynomials (1) and (2), since the last division is exact; but this is evidently impossible, from the inspection of the remainder (4), of which the coefficients are functions of y, while the coefficient of x^3 in (1) and (2) is unity.

To explain this contradiction, it should be observed, that we have multiplied (3) by the square of the coefficient of x in (4), without previously ascertaining (No. 38) whether this coefficient is not a factor of that part of (4) involving x^0. Now, the fact is, it is a factor of it; for this part can be put under the form

$$-y(y^4-10y^3+35y^2-50y+24).$$

It follows from this remark, that (4) itself is not a common divisor of the proposed equations, but (4) freed from the factor $y^4-10y^3+35y^2-50y+24$, or $x-y$.

Suppressing then the factor $x-y$ in the two polynomials (1) and (2), we obtain

$$x^2-(2y+5)x+y^2+5y+6=0 \text{ - - - - } (5),$$
$$x^2-4yx+4y^2-1=0 \text{ - - - - - - - - - } (6).$$

Consequently, *when we operate upon two equations in which the first members, arranged with reference to* x, *are affected with coefficients which are prime with each other, if, after a certain number of operations, we obtain* ZERO FOR A REMAINDER, *and the coefficients of the preceding remainder are not prime with each other, it is not this remainder which is the common divisor of the proposed equations, but it is this remainder divided by the factor common to these coefficients.*

It now remains to apply the process for elimination to the equations (5) and (6), in order to deduce the final equation which corresponds to them, and afterwards, all the solutions of these equations. But we will show that this operation is not necessary, that is, that the first series of operations *gives the final equation relative to the equations* (5) *and* (6), *as well as the remainder of the first degree, with reference to* x, *corresponding to this final equation.*

376. In order to explain this circumstance in a general manner, we will consider the two equations $A=0$, $B=0$, which we will suppose contains a common divisor involving x and y, or x only.

By first applying the process to the two polynomials, we will find *a first series* of quotients and *a first series* of remainders, which remainders will (No. 265) all contain the common factor. But if we suppress this factor in A and B, and then operate upon the resulting polynomials A′ and B′, we must find *the same quotients* and remainders which do not differ from those of the first series, except that they do not contain the common factor; hence, the last remainder of the second series of operations, only differs from the last remainder of the first series, in not containing the common factor.

Therefore the first member of the final equation corresponding to $A'=0$, $B'=0$, *is nothing more than the last remainder of* the first series of operations, *freed from the factor common to* A, *and* B; in other words, *it is that function of* y *which is a common factor of the coefficients of this remainder.*

With respect to the remainder of the first degree with reference to x in the second series of operations, *it is equal to the remainder preceding the last* in the first series, divided by the common factor; it is therefore *the last quotient of the first series.*

In the preceding example, the final equation relative to the equations (5) and (6) is

$$y^4-10y^3+35y^2-50y+24=0 \text{ - - - - } (7);$$

and the remainder of the first degree with reference to x is nothing more than the quotient of the remainder (3) by $x-y$; or

$$(2y-5)x-3y^2+5y+7 \text{ - - - - - } (8).$$

The equation (7), resolved by the method of commensurable roots, gives

$$y=1,\ 2,\ 3,\ 4.$$

Substituting each of these values in the remainder (8), and resolving this remainder when placed equal to 0, we find the corresponding values of x, viz.

$$x=3,\ 5,\ 5,\ 7.$$

377. For another example, take the two equations

$$x^3-(3y-1)x^2+(y^2-2y)x+y^2+y=0 \text{ - - - - } (1),$$

$$x^3-(y-1)x^2-(y-1)x+1=0 \text{ - - - - - - - - } (2).$$

We first obtain for the remainder of the second degree

$$2yx^2-(y^2-y-1)x-y^2-y+1 \text{ - - - - - } (3),$$

and for the remainder of the first degree,

$$(y^4-5y^2+2y-1)x+y^4-5y^2+2y-1 \text{ - - - } (4).$$

But instead of operating as in the preceding example, that is, instead of multiplying (3) by the square of y^4-5y^2+2y-1, and dividing (3) thus prepared by (4), we observe that the remainder (4) can be put under the form

$$(y^4-5y^2+2y-1)\ (x+1).$$

Omitting, then, the factor involving y, and dividing (3) by $x+1$, we obtain an exact quotient,

$$2yx-y^2-y+1 \text{ - - - - - } (5).$$

Whence we may conclude, that $x+1$ is a common divisor of the proposed equations.

These equations, freed from the factor $x+1$, reduce to

$$x^2-3yx+y^2+y=0 \text{ - - - - - - } (6),$$

$$x^2-xy+1=0 \text{ - - - - - - - - - - } (7);$$

and the question is reduced to eliminating between these two equations. Now, from what has been said above, the final equation is

$$y^4-5y^2+2y-1=0,$$

and the equation of the first degree with reference to x, corresponding to it, is the result (5) placed equal to 0, viz.

$$2yx-y^2-y+1=0.$$

The first of these two equations not admitting of any commensurable roots, the method of incommensurable roots must be applied to it; after which, we substitute each of the values of y obtained in the second equation, which will then give the corresponding values of x.

Examples.

1st. $x^3+(3y-1)x^2+(3y^2-2y-9)x+y^3-y^2-9y+9=0,$

$x^2-2xy+y^2+4x-4y+3=0;$

equation involving y - - - - - - $y(y-2)^2(y^2-1)(y-3)=0,$

" " x - - - - - - - $x^2(x^2-1)(x+3)(x+2)=0.$

2d. $y^3+(2x+2)y^2-x^2y+7xy-5y-2x^3+9x^2-7x-6=0,$

$y^2+3x^2+4xy+y+5x-2=0;$

equation involving x - - - $(x-1)^2(4x^2-1)(x-2)=0,$

" " y - - - $(4y^2-25)(y+2)(y+3)(y+5)=0.$

3d. $x^2-2xy+y^2-4x+4y+3=0$,
$x^2-2xy+y^2-6x+6y-7=0$;

4th. $x^2+2xy+y^2-10x-10y+21=0$,
$x^2-2xy+y^2+6x-6y+5=0$;

equation involving y - - - - - - - $(y-4)^2(y-2)(y-6)=0$,

" " x - - - - - - - $(x-1)^2(x-3)(x+1)=0$.

FIGURATE NUMBERS.

Of Figurate Numbers, and the Series upon which they depend.

There is a certain class of series of which we can easily obtain the *general term*, and *the expression for the sum of a limited number of terms;* these series are deduced from a progression by differences.

430. *Determination of the general term of the series* . . . . . $a^m+b^m+c^m+d^m$ . . . . ; a, b, c, d . . . . . being the different terms of a progression by differences.

It has been shown (No. 194) that, in any progression by differences

$$\div\ a.b.c.d.e.f.g.h\ldots\ldots,$$

the expression for the general term l, is $l=a+(n-1)r$, r denoting the ratio and n the number of terms; hence the *general term* of the series of m^{th} powers of the terms of this progression is

$$l^m=[a+(n-1)r].$$

For example, let it be required to find the 15th term of the series of 5th powers of the terms of the progression - - - - - - $\div 1.3.5.7.9.11.13\ldots\ldots$; by making $n=15$, $m=5$, $a=1$, $r=2$, we will have

$$l_5=(1+14\times2)^5=29^5=20511149.$$

431. *Summation of the Series.*

$$a^m+b^m+c^m+d^m+\ldots+k^m+l^m,$$

$a, b, c, d \ldots k, l,$ being the terms of a progression by differences.

We have, by the binomial theorem,

$$b^m = (a+r)^m = a^m + mra^{m-1} + m\frac{m-1}{2}r^2a^{m-2} + \text{---------},$$

$$c^m = (b+r)^m = b^m + mrb^{m-1} + m\frac{m-1}{2}r^2b^{m-2} + \text{----------},$$

. . . .

. . . .

$$l^m = (k+r)^m = k^m + mrk^{m-1} + m\frac{m-1}{2}r^2k^{m-2} + \text{----------}.$$

Adding these equations together member to member, and denoting the sum of the m^{th}, $m-1^{th}$ ------ powers by ------ S_m, S_{m-1}, S_{m-2} . . ., S_2, S_1, we obtain

$$S_m - a^m = S_m - l^m + mr(S_{m-1} - l^{m-1}) + m\frac{m-1}{2}r^2(S_{m-2} - l^{m-2}) +$$

. . . . or reducing, we obtain the formula

$$l^m - a^m = mr(S_{m-1} - l^{m-1}) + m\frac{m-1}{2}r^2(S_{m-2} - l^{m-2}) + \text{---} \quad (A),$$

which expresses the sums of the powers, from S_{m-1}, to S_0 inclusively (S_0 being equal to $a^0 + b^0 + c^0 + d^0 + \ldots\ldots + k^0 - l^0$, which is equivalent to n).

To show the use of the formula (A), make successively $m = 1, 2, 3, 4, 5 \ldots.$

1st. Let $m=1$, we find

$$l - a = r(S_0 - l^0);\ \text{whence } S_0 = \frac{l-a}{r} + 1 = \frac{(n-1)r + r}{r} = n,$$

a result which has already been obtained.

2d. $m=2$; it becomes $l^2 - a^2 = 2r(S_1 - l) + r^2(S_0 - l^0)$,

$$\text{whence } S_1 - l = \frac{l^2 - a^2}{2r} - \frac{r(l-a)}{2r};$$

$$\text{therefore } S_1 = \frac{l^2 - a^2}{2r} + \frac{r(l+a)}{2r} = \frac{(l+a)(l-a+r)}{2r};$$

or, since $l = a + (n-1)r$, whence $l - a + r = nr$,